Perverse Sheaves I Talk 10 (Second Half)

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June 17 2024

We continue the discussion of section 3.1 of [Achar]. Let as always X be a variety.

The Perverse t-Structure

Proving this important theorem will take up most of this talk. In particular, it shows that perverse sheaves indeed form an abelian category.

Theorem 1. *The pair $({}^pD^b_c(X)^{\leq 0}, {}^pD^b_c(X)^{\geq 0})$ is a bounded t-structure on $D^b_c(X)$.*

Proof. Overview:
1st: axiom: immediate from the definitions
2nd axiom: content of lemma 1
boundedness axiom: corollary of lemma 2
3rd axiom: content of lemma 3 (much harder). □

Lemma 1. *Let $\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$ and $\mathcal{G} \in {}^pD^b_c(X)^{\geq 1}$. We have* $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

Proof. As $\mathcal{F}$ is bounded we can thus reduce as usual to the case that $\mathcal{F} \simeq \mathbf{H}^j(\mathcal{F})[-j]$.

Next, we fix a stratification of X such that $\mathbb{D}\mathcal{G}$ is constructible and $\mathrm{supp}\mathcal{F}$ is a union of strata, which we call $(X_s)_{s\in\mathscr{S}}$. We now proceed by induction over the number of strata. Let $i : X_t \hookrightarrow X$ be a closed embedding with open complement $j : U = X \setminus X_t \hookrightarrow X$.

Using the distinguished triangle $j_! j^*\mathcal{F} \to \mathcal{F} \to i_* i^*\mathcal{F}$ and the adjunctions $j_! \dashv j^*$ and $i_* \dashv i^!$ we obtain a piece from the long exact sequence

$$\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}).$$

By induction, we have $\mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}) = 0$. Thus it is enough to show that $\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$. If $i^*\mathcal{F} = 0$ we would be done. So let's assume this is not the case. In this case $X_t \subseteq \mathrm{supp}\mathbf{H}^j(\mathcal{F})$. In particular, $\dim X_t \leq \dim \mathrm{supp}\mathbf{H}^j(\mathcal{F}) \leq -j$ by assumption. By lemma 1.13(6) from the previous talk we obtain:

$$i^!\mathcal{G} \in D^b_{\mathrm{locf}}(X_t)^{\geq -\dim X_t + 1} \subseteq D^b(X)^{\geq j+1}$$

As $i^*\mathcal{F}$ is by assumption concentrated in degree j, indeed $\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$ finishing the proof. □

We next show a lemma implying the boundedness axiom as we know that the usual t-structure on $D^b_c(X)$ is bounded.

Lemma 2. *We have:*

1. $D^b_c(X)^{\leq -\mathrm{dim}X} \subseteq {}^pD^b_c(X)^{\leq 0} \subseteq D^b_c(X)^{\leq 0}$
2. $D^b_c(X)^{\geq 0} \subseteq {}^pD^b_c(X)^{\geq 0} \subseteq D^b_c(X)^{\geq -\mathrm{dim}X}$.

Proof. We will only proof the first assertion. The second is given as an exercise. The right inclusion is clear from the definitions. Thus we only need to show $D^b_c(X)^{\leq -\mathrm{dim}X} \subseteq {}^pD^b_c(X)^{\leq 0}$.

Let $\mathcal{F} \in D^b_c(X)^{\leq -\mathrm{dim}X}$. For $i \leq -\mathrm{dim}X$, we have $\mathrm{dimsupp}\mathbf{H}^i(\mathcal{F}) \leq \mathrm{dim}X$. On the other hand for $i > -\mathrm{dim}X$ we have $\mathbf{H}^i(\mathcal{F}) = 0$. In particular in this case, $\mathrm{dimsupp}\mathbf{H}^i(\mathcal{F}) = 0$. Thus, $\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$. □

We are now left to show the hardest part.

Lemma 3. *For $\mathcal{F} \in D^b_c(X)$ there is a triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ such that $\mathcal{F}' \in {}^pD^b_c(X)^{\leq -1}$ and $\mathcal{F}'' \in {}^pD^b_c(X)^{\geq 0}$.*

Proof. We are proceeding by Noetherian induction. Thus, assume that for all closed subvarieties of X the statement alread holds. Let $\mathcal{F} \in D^b_c(X)$. Choose a stratification $(X_s)_{s\in\mathscr{S}}$ such that $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. Consider an open stratum $j_u : X_u \hookrightarrow X$ with closed complement $i : X \setminus X_u =: Z \hookrightarrow X$. In the following, we write $m := \mathrm{dim}X_u$. From the usual (not perverse!) t-structure we obtain a triangle:

$$\tau^{\leq -m-1} j_u^*\mathcal{F} \to j_u^*\mathcal{F} \to \tau^{\geq -m} j_u^*\mathcal{F} \to \dots$$

Applying $j_{u!}$ this gives the triangle

$$j_{u!}\tau^{\leq -m-1} j_u^*\mathcal{F} \to j_{u!}j_u^*\mathcal{F} \to j_{u!}\tau^{\geq -m} j_u^*\mathcal{F} \to \dots$$

Recall the natural map $j_{u!}j_u^*\mathcal{F} \to \mathcal{F}$. Composing the first map of the above triangle with this and then taking the cone we obtain a triangle

$$j_{u!}\tau^{\leq -m-1} j_u^*\mathcal{F} \to \mathcal{F} \to \mathcal{G} \to \dots \tag{1}$$

Applying j_u^* and using that for an open embedding (as for any locally closed embedding) $j_u^*j_{u!}\mathcal{F} \simeq \mathcal{F}$ we get a triangle

$$\tau^{\leq -m-1} j_u^*\mathcal{F} \to j_u^*\mathcal{F} \to j_u^*\mathcal{G} \to \dots$$

where the first map is the truncation map. In particular, we obtain:

$$j_u^*\mathcal{G} \simeq \tau^{\geq -m} j_u^*\mathcal{F} \tag{2}$$

Analogously, we proceed for the closed complement Z of X_u. In this case, we already have by induction hypothesis that the perverse t-structure is indeed a t-structure on Z. We thus have the distinguished triangle:

$${}^p\tau^{\leq -1} i^!\mathcal{G} \to i^!\mathcal{G} \to {}^p\tau^{\geq 0} i^!\mathcal{G} \to \dots$$

Applying i_* we get:

$$i_* {}^p\tau^{\leq -1} i^! \mathcal{G} \to i_* i^! \mathcal{G} \to i_* {}^p\tau^{\geq 0} i^! \mathcal{G} \to \dots$$

Composing the first map with the natural map $i_* i^! \mathcal{G} \to \mathcal{G}$ and then taking the cone (which we call $\mathcal{F}''$) we obtain the triangle:

$$i_* {}^p\tau^{\leq -1} i^! \mathcal{G} \to \mathcal{G} \to \mathcal{F}'' \to \dots \tag{3}$$

After applying $i^!$ this yields as before:

$$i^! \mathcal{F}'' \simeq {}^p\tau^{\geq 0} i^! \mathcal{G} \tag{4}$$

We can fit the triangles (1) and (3) into an octahedron (where we obtain the lower inner maps in the lower diagram by TR3):

$$\begin{tikzcd}
j_{u!}\tau^{\leq -m-1} j_u^* \mathcal{F} \arrow[dd] & & i_* {}^p\tau^{\leq -1} i^! \mathcal{G} \arrow[ll, dotted] \arrow[dl] \\
& \mathcal{G} \arrow[ul, dotted] \arrow[dr] & \\
\mathcal{F} \arrow[ur] \arrow[rr] & & \mathcal{F}'' \arrow[uu, dotted]
\end{tikzcd}$$

$$\begin{tikzcd}
j_{u!}\tau^{\leq -m-1} j_u^* \mathcal{F} \arrow[dd] \arrow[dr] & & i_* {}^p\tau^{\leq -1} i^! \mathcal{G} \arrow[ll, dotted] \\
& \mathcal{F}' \arrow[ur] \arrow[dl] & \\
\mathcal{F} \arrow[rr] & & \mathcal{F}'' \arrow[uu, dotted] \arrow[ul, dotted]
\end{tikzcd}$$

where the dotted line are the maps of degree 1 from the triangle. In particular, the left and right triangles in the above diagram are distinguished triangles and the upper and lower triangle commute. In the lower diagram, the upper triangle is distinguished and the left and right commute. By the octahedral axiom the lower triangle in the lower diagram is thus also a distinguished triangle:

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to \dots$$

To show that the existence of this triangle proves the lemma we need that $\mathcal{F}' \in {}^pD_c^b(X)^{\leq -1}$ and $\mathcal{F}'' \in {}^pD_c^b(X)^{\geq 0}$. We begin with $\mathcal{F}'$. Consider the distinguished triangle from the lower diagram:

$$j_{u!}\tau^{\leq -m-1} j_u^* \mathcal{F} \to \mathcal{F}' \to i_* {}^p\tau^{\leq -1} i^! \mathcal{G} \to \dots$$

By previous results (lemmata 1.13 and 1.14 from the first half), we know that $j_{u!}\tau^{\leq -m-1}j_u^*\mathcal{F} \in {}^pD_c^b(X)^{\leq -1}$ and $i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G} \in {}^pD_c^b(X)^{\leq -1}$. Thus $\mathcal{F} \in {}^pD_c^b(X)^{\leq -1}$.

We now turn to $\mathcal{F}''$. Using lemma 1.14(2) it suffices to show

$$j_s^!\mathcal{F}'' \in D_{\mathrm{locf}}^b(X_s)^{\geq -\dim X_s}$$

for all strata X_s. In the case where $s = u$ we can apply $j_u^!$ to the triangle (3). As the left term vanishes, we obtain an isomorphism $j_u^!\mathcal{F}'' \simeq j_u^!\mathcal{G}$. As j is an open embedding together with the isomorphism (2) this gives $j_u^!\mathcal{F}'' \simeq \tau^{\geq -m}j_u^*\mathcal{F} \in D_{\mathrm{locf}}^b(X_u)^{\geq -\dim X_u}$. Let now $s \neq u$. We can factor j_s as $j_s' : X_s \to Z$ and then the closed embedding $i : Z \to X$. By (contravariant) functoriality of $(-)^!$ this gives $j_s^!\mathcal{F}'' \simeq (j_s')^!i^!\mathcal{F}''$. Similarly to the other case, we can now apply $(j_s')^!$ to (4). Again using Noetherian induction we are finished, as we can assume the statement to already hold for Z. □

Perverse Sheaves on Closed Subvarieties

We now want to relate the perverse sheaves on a closed subvariety Z to those on the whole variety X. Recall the notion of a Serre subcategory:

Definition 1. Let C be an abelian category. A Serre subcategory S of C is a nontrivial full subcategory such that for any short exact sequence in C of the form $0 \to A \to A' \to A'' \to 0$, we have $A, A'' \in C$ if and only if $A' \in C$.

Proposition 1. *Let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety. The functor i_* induces an equivalence of categories*

$$\mathrm{Perv}(Z) \xrightarrow{\sim} \{\mathcal{F} \in \mathrm{Perv}(X) | \mathrm{supp}\mathcal{F} \subseteq Z\}.$$

Moreover, the right-hand side is a Serre subcategory.

Proof. The first assertion follows from lemma 1.13(4) and a general result on fully-faithfulness of i_* for closed embeddings and its essential image from chapter 1.

Now assume that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of perverse sheaves on X. If the supports of $\mathcal{F}'$ and $\mathcal{F}''$ are contained in Z, it is immediate that so is the support of $\mathcal{F}$. Let us now assume that $\mathrm{supp}\mathcal{F} \subseteq Z$. We write U for the complementary open. Using lemma 1.13(1) from the talk before we obtain a short exact sequence:

$$0 \to \mathcal{F}'_{|U} \to \mathcal{F}_{|U} \to \mathcal{F}''_{|U} \to 0$$

By assumption, the middle term vanishes. So, the outer terms also vanish, proving the proposition. □

References

[Achar] Pramod N. Achar. *Perverse Sheaves and Applications to Representation Theory*. Mathematical surveys and monographs 258. Providence, Rhode Island: American Mathematical Society, 2021. ISBN: 978-1-4704-5597-2.