# Perverse Sheaves I Talk 10 (First Half)

Yining Chen

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**Abstract**

These notes about perverse $t$-structures are written for the talk 10 of the MSc. Seminar on perverse sheaves held by Dr. Stefan Dawydiak in the University of Bonn and the seminar page is in `https://www.math.uni-bonn.de/people/dawydiak/perverse.html`.

## Contents



## 1 Perverse $t$-Structures

In these notes, we always suppose our base field $k = \mathbb{C}$ is the field of complex numbers.

Let $X$ be a variety over $k$. Recall $D(X,k)$ or simply $D(X)$ is the *derived category of sheaves of $k$-vector spaces on $X^{an}$* where $X^{an}$ means it's with the analytic topology. $D^b_c(X) \subseteq D(X)$ is the full subcategory consisting of bounded complexes $\mathcal{F}$ whose cohomological sheaves $H^i(\mathcal{F})$ are *constructible* for all $i$. Next we define two full subcategories of $D^b_c(X)$.

**Definition 1.1.** The full subcategory ${}^pD^{\leq 0}_c(X)$ of $D^b_c(X)$ consists of bounded constructible complex $\mathcal{F} \in D^b_c(X)$ such that

$$\dim \operatorname{supp} H^i(\mathcal{F}) \leq -i$$

for all $i$.

And on the other hand we want to define ${}^pD^{\geq 0}_c(X)$, so that we will obtain

$$\mathrm{Perv}(X) := {}^pD^{\leq 0}_c(X) \cap {}^pD^{\geq 0}_c(X)$$

the category of *perverse sheaves*.

**Verdier Duality** For a variety $p_X : X \to \mathrm{pt}$ over $k$, the *dualizing complex* is $\omega_X = p^!_X \underline{k}_{\mathrm{pt}}$ and from [1, Cor. 2.2.10] we see if $X$ is smooth of dimension $n$, then $\omega_X \simeq k_X[2n]$ is centered at degree $-2n$.

The *Verdier duality* functor is defined to be

$$\mathbb{D} : D^b_c(X)^{op} \to D^b_c(X),\ \mathcal{F} \mapsto \mathbb{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \omega_X)$$

and the evaluation map $\mathcal{F} \to \mathbb{D}(\mathbb{D}(\mathcal{F}))$ is an equivalence.

**Definition 1.2.** The full subcategory ${}^pD^{\geq 0}_c(X)$ of $D^b_c(X)$ consists of bounded constructible complex $\mathcal{F} \in D^b_c(X)$ such that

$$\dim \operatorname{supp} H^i(\mathbb{D}\mathcal{F}) \leq -i$$

for all $i$. This means $\mathcal{F} \in {}^pD^{\geq 0}_c(X)$ if and only if $\mathbb{D}\mathcal{F} \in {}^pD^{\leq 0}_c(X)$.

**Definition 1.3.** The category of *perverse sheaves* is defined to be $\mathrm{Perv}(X) := {}^pD_c^{\leq 0}(X) \cap {}^pD_c^{\geq 0}(X)$.

**Remark 1.4.** The equivalence $\mathbb{D}\mathbb{D}\mathcal{F} \simeq \mathcal{F}$ implies the Verdier duality functor exchanges ${}^pD_c^{\leq 0}(X)$ and ${}^pD_c^{\geq 0}(X)$, so it preserves $\mathrm{Perv}(X)$.

**Remark 1.5** (bounded $t$-structure)**.** We will see later that the pair $({}^pD_c^{\leq 0}(X), {}^pD_c^{\geq 0}(X))$ defines a bounded $t$-structure in the triangulated category $D_c^b(X)$[1] in the sense of last talk and it's called the *perverse $t$-structure*.

Note that to see the perverse $t$-structure is *bounded* we need to show for any $\mathcal{F} \in D_c^b(X)$ there exists a positive integer $n > 0$ such that

$$\mathcal{F} \in {}^pD_c^{\leq n}(X) := {}^pD_c^{\leq 0}(X)[-n] \text{ and } \mathcal{F} \in {}^pD_c^{\geq -n}(X) := {}^pD_c^{\geq 0}(X)[n]$$

$\mathcal{F} \in {}^pD_c^{\leq n}(X)$ is equivalent to $\dim \mathrm{supp} H^{i+n}(\mathcal{F}) \leq -i$. Since $\mathcal{F}$ is a bounded complex, there is an integer $m > 0$ such that for all $i \geq m$, $H^i(\mathcal{F}) = 0$. Let $m' = \mathrm{Max}(\dim \mathrm{supp} \mathrm{H^i}(\mathcal{F}))$. Then for all $n \geq m + m'$, we will have $\mathcal{F} \in {}^pD_c^{\leq n}(X)$. A similar argument can be applied to $\mathbb{D}\mathcal{F}$. So this proves the perverse $t$-structure is bounded.

**Notation 1.6.** From [1, Thm. A.7.8], for a $t$-structure in a triangulated category, its heart is an abelian category. Therefore $\mathrm{Perv}(X)$ is actually an abelian category. And we have truncation functors

$${}^p\tau^{\leq n} : D_c^b(X) \to {}^pD_c^{\leq n}(X) = {}^pD_c^{\leq 0}(X)[-n], \quad {}^p\tau^{\geq n} : D_c^b(X) \to {}^pD_c^{\geq n}(X) = {}^pD_c^{\geq 0}(X)[-n]$$

where ${}^p\tau^{\leq n}$ is right adjoint to ${}^pD_c^{\leq n}(X) \hookrightarrow D_c^b(X)$ and ${}^p\tau^{\geq n}$ is left adjoint to ${}^pD_c^{\geq n}(X) \hookrightarrow D_c^b(X)$.

Also for every $n \in \mathbb{Z}$, we have the cohomology functor

$${}^pH^n : D_c^b(X) \to \mathrm{Perv}(X), \;\; \mathcal{F} \mapsto {}^p\tau^{\leq 0}\,{}^p\tau^{\geq 0}(\mathcal{F}[n])$$

### Good Stratifications

**Definition 1.7** (good stratification)**.** Let $X$ be a variety with a stratification $(X_s)_{s\in\mathscr{I}}$ and for every $s \in \mathscr{I}$, $j_s : X_s \hookrightarrow X$ is the inclution map. This stratification is called a *good stratification* if for every $s \in \mathscr{I}$ and every local system of finite type $\mathcal{L}$ on $X_s$, we have $j_{s*}\mathcal{L} \in D_{\mathscr{I}}^b(X)$.

**Remark 1.8.** Every stratification can be refined by a good one because from [1, Thm. 2.7.1] or Talk 7, $j_{s*}\mathcal{L} \in D_c^b(X)$.

**Example 1.9** (Normal Crossings Stratification)**.** A Weil divisor $D = \sum_i D_i \subseteq X$ on a smooth variety $X$ of dimension $n$ is a *simple normal crossing* if every component $D_i$ is smooth and for every point $p \in X$ a local equation of $D$ is $x_1 \cdots x_r$ for independent local parameters $x_i \in \mathcal{O}_{X,p}$ with $r \leq n$.

Let $Z \subseteq X$ be a simple normal crossing in a smooth variety $X$ of dimension $n$ with irreducible components $Z_1, \cdots, Z_k$. Then we can partition $X$ into subvarieties labelled by subsets $J \subseteq \{1, \cdots, k\}$ such that

$$X_J := \{x \in X | x \in Z_j \text{ if and only if } j \in J\} = \{x \in X | I(x) = J\}$$

where $I(x) = \{i | x \in Z_i\}$. For example if $J = \{1\}$, then $X_J = Z_1 \setminus (Z_2 \cup \cdots \cup Z_k)$. For a general subset $J$, we have $X_J = (\cap_{j\in J} Z_j) \setminus (\cup_{i\notin J} Z_i)$. And by definition $X_\emptyset = X \setminus Z$. The closure of $X_J$ will be $Z_J := \cap_{j\in J} Z_j$.

Non-empty subsets in $(X_J)_{J\subseteq\{1,\cdots,k\}}$ gives a stratification of $X$, which is called the *normal crossings stratification*. From [1, Lem. 2.4.2], this stratification is good.

For a good stratification $(X_s)_{s\in\mathscr{I}}$, the induced $t$-structure on $D_{\mathscr{I}}^b(X)$ is given by

$$\begin{aligned} {}^pD_{\mathscr{I}}^{\leq 0}(X) &:= {}^pD_c^{\leq 0}(X) \cap D_{\mathscr{I}}^b(X) \\ {}^pD_{\mathscr{I}}^{\geq 0}(X) &:= {}^pD_c^{\geq 0}(X) \cap D_{\mathscr{I}}^b(X) \end{aligned}$$

The heart is denoted by $\mathrm{Perv}_{\mathscr{I}}(X) = {}^pD_{\mathscr{I}}^{\leq 0}(X) \cap {}^pD_{\mathscr{I}}^{\geq 0}(X)$.

[1] [1, Thm. 3.1.9]

**Remark 1.10.** [1, Ex. 2.8.2] says if $\mathscr{I}$ is a good stratification, the Verdier duality functor has a restriction $\mathbb{D}: D^b_{\mathscr{I}}(X)^{op} \to D^b_{\mathscr{I}}(X)$. So in this case $\mathbb{D}$ preserves $\mathrm{Perv}_{\mathscr{I}}(X)$.

**Theorem 1.11.** *Let $X$ be a variety over $k$.*

- *The perverse t-structure $({}^pD_c^{\leq 0}(X), {}^pD_c^{\geq 0}(X))$ is a t-structure in the triangulated category $D_c^b(X)$.*
- *Moreover if $(X_s)_{s\in\mathscr{I}}$ is a good stratification for $X$, then $({}^pD_{\mathscr{I}}^{\leq 0}(X), {}^pD_{\mathscr{I}}^{\geq 0}(X))$ gives a t-structure in $D^b_{\mathscr{I}}(X)$.*

*Sketch of the proof.* We need to check three conditions for $({}^pD_c^{\leq 0}(X), {}^pD_c^{\geq 0}(X))$ to be a $t$-structure in $D_c^b(X)$.

(1). ${}^pD_c^{\leq -1}(X) \subseteq {}^pD_c^{\leq 0}(X)$ and ${}^pD_c^{\geq 0}(X) \subseteq {}^pD_c^{\geq -1}(X)$.

(2). If $\mathcal{F} \in {}^pD_c^{\leq -1}(X)$ and $\mathcal{G} \in {}^pD_c^{\geq 0}(X)$, then $\mathrm{Hom}_{D_c^b(X)}(\mathcal{F}, \mathcal{G}) = 0$.

(3.) For any $\mathcal{F} \in D_c^b(X)$, there is a distinguished triangle

$$\mathcal{H} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$$

with $\mathcal{H} \in {}^pD_c^{\leq -1}(X)$ and $\mathcal{G} \in {}^pD_c^{\geq 0}(X)$.

The first condition is just by definition, but note that $\mathbb{D}(\mathcal{F}[-1]) = (\mathbb{D}\mathcal{F})[1]$. Conditions of (2) and (3) will be checked in the second half part of this talk.

□

**Shifted Local Systems** Now let's discuss relations between perverse sheaves and local systems.

If we choose the stratification $(Xs)_{s\in\mathscr{I}}$ to be the trivial stratification i.e. $X_s = X$, then $D^b_{\mathscr{I}}(X)$ will be $D^b_{\mathrm{locf}}(X)$ the derived category consisting of complexes whose cohomologies are local systems of finite type.

**Proposition 1.12.** *Let $X$ be a smooth connected variety of dimension $n$. Then we have*

$$ {}^pD^{\leq 0}_{\mathrm{locf}}(X) = D^{\leq -n}_{\mathrm{locf}}(X) $$
$$ {}^pD^{\geq 0}_{\mathrm{locf}}(X) = D^{\geq -n}_{\mathrm{locf}}(X) $$

*So that if we let $\mathscr{I}$ be the trivial stratification on $X$, then*

$$\mathrm{Perv}_{\mathscr{I}}(X) = \mathrm{Loc}^{\mathrm{ft}}(X)[n]$$

*In particular, if $X = \mathrm{pt}$, $\mathrm{Perv}_{\mathscr{I}}(\mathrm{pt}) = \mathrm{Vect}_k^{ft}$.*

In this case, objects in $\mathrm{Loc}^{\mathrm{ft}}(X)[n]$ are called *shifted local systems*.

*Proof.* At first it's easy to see $D^{\leq -n}_{\mathrm{locf}}(X) \subseteq {}^pD^{\leq 0}_{\mathrm{locf}}(X)$ because $\mathrm{supp}H^i(\mathcal{F})$ has at most dimension $n$. For the other side, it's clear for an object $\mathcal{F} \in D^b_{\mathrm{locf}}(X)$, its cohomology sheaf $H^i(\mathcal{F})$, if it's non-zero, is a local system of finite type which means $\dim\,\mathrm{supp}H^i(\mathcal{F}) = \dim\,X = n$ because $X$ is connected. So if $\mathcal{F} \in {}^pD^{\leq 0}_{\mathrm{locf}}(X)$, $\dim\,\mathrm{supp}H^i(\mathcal{F}) \leq -i$ and then when $H^i(\mathcal{F}) \neq 0$, $n \leq -i \Leftrightarrow i \leq -n$. This proves ${}^pD^{\leq 0}_{\mathrm{locf}}(X) = D^{\leq -n}_{\mathrm{locf}}(X)$.

As for ${}^pD^{\geq 0}_{\mathrm{locf}}(X) = D^{\geq -n}_{\mathrm{locf}}(X)$, firstly notice that

$$\mathcal{F} \in D^{\geq -n}_{\mathrm{locf}}(X) \text{ iff } \mathcal{F}_x \in D^{\geq -n}(\mathrm{Vect}_k) \text{ for all } x \in X$$

Let $i_x : x \hookrightarrow X$ be the embedding. Then $\mathcal{F}_x = i_x^*\mathcal{F}$. The $t$-exactness of $i_x^*$ implies $H^i(i_x^*\mathcal{F}) \simeq i_x^*H^i(\mathcal{F})$. So if $H^i(\mathcal{F}) = 0$, then $H^i(\mathcal{F}_x) = 0$. Conversely if $H^i(\mathcal{F}) \neq 0$, since $H^i(\mathcal{F})$ is a local system, there will exist some $x \in X$ such that $i_x^*H^i(\mathcal{F}) \neq 0$. This proves the equivalence above.

Next for $\mathcal{F} \in D_c^b(X)$, observe

$$\mathrm{supp}H^j(\mathbb{D}\mathcal{F}) = \{x \in X | H^{-j}(i_x^!\mathcal{F}) \neq 0\}$$

This can be obtained by Verdier duality

$$i_x^! \mathcal{F} \simeq i_x^! \mathbb{D}\mathbb{D}\mathcal{F} \simeq \mathbb{D} i_x^* \mathbb{D}\mathcal{F}$$

Note that on $\{x\}$, the Verdier duality functor $\mathbb{D}_{\{x\}}$ is just the usual functor $\mathrm{Hom}(-, k)$. So above equivalences compute

$$H^{-j}(i_x^! \mathcal{F}) \simeq H^j(\mathbb{D}\mathcal{F})_x^\vee$$

Finally for $\mathcal{F} \in D^b_{\mathrm{loc}}(X)$, Lemma 0.7 in Talk 5 or [1, Thm. 2.2.13] yields $i_x^! \mathcal{F} \simeq i_x^* \mathcal{F}[-2n]$. So if $\mathcal{F} \in D^{\geq -n}_{\mathrm{locf}}(X)$, $i_x^! \mathcal{F} \in D^{\geq n}(\mathrm{Vect}_k)$ for all $x \in X$. In this case for $-j \geq n$ such that $H^j(\mathbb{D}\mathcal{F})$ is a non-zero local system, we will have $\dim \mathrm{supp} H^j(\mathbb{D}\mathcal{F}) = n \leq -j$ which means $\mathcal{F} \in {}^p D^{\geq 0}_{\mathrm{locf}}(X)$.

On the other hand if $\mathcal{F} \in {}^p D^{\geq 0}_{\mathrm{locf}}(X)$, we have

$$\dim\{x \in X | H^{-j}(i_x^! \mathcal{F}) \neq 0\} \leq -j$$

Because if the support is non-empty then it will be $X$ and has dimension $n$, for some $j$ such that $H^j(\mathcal{F}) \neq 0$, we see $H^j(i_x^* \mathcal{F}) = H^j(i_x^! \mathcal{F}[2n]) = H^{j+2n}(i_x^! \mathcal{F}) \neq 0$. So that $n \leq j + 2n \Rightarrow j \geq -n$. So that $\mathcal{F} \in D^{\geq -n}_{\mathrm{locf}}(X)$. □

**Perverse Sheaves along Immersions** The following lemma describes how perverse $t$-structures behave along open and closed embeddings.

**Lemma 1.13.** *Let $j : U \hookrightarrow X$ be an open embedding and $i : Z \hookrightarrow X$ be a closed embedding.*

*(1).* $j^*({}^p D^{\leq 0}_c(X)) \subseteq {}^p D^{\leq 0}_c(U)$ *and* $j^*({}^p D^{\geq 0}_c(X)) \subseteq {}^p D^{\geq 0}_c(U)$

*(2).* $j_!({}^p D^{\leq 0}_c(U)) \subseteq {}^p D^{\leq 0}_c(X)$

*(3).* $j_*({}^p D^{\geq 0}_c(U)) \subseteq {}^p D^{\geq 0}_c(X)$

*(4).* $i_*({}^p D^{\leq 0}_c(Z)) \subseteq {}^p D^{\leq 0}_c(X)$ *and* $i^*({}^p D^{\geq 0}_c(Z)) \subseteq {}^p D^{\geq 0}_c(X)$

*(5).* $i^*({}^p D^{\leq 0}_c(X)) \subseteq {}^p D^{\leq 0}_c(Z)$

*(6).* $i^!({}^p D^{\geq 0}_c(X)) \subseteq {}^p D^{\geq 0}_c(Z)$

*Proof.* We prove this lemma using Verdier duality discussed in Talk 8.

At first for $(1)$, notice that the restriction functor i.e. $j^*$ is $t$-exact for the usual $t$-structure and will not increase the dimension of the support. This means for $\mathcal{F} \in {}^p D^{\leq 0}_c(X)$,

$$\dim \mathrm{supp} H^i(j^* \mathcal{F}) = \dim \mathrm{supp} j^* H^i(\mathcal{F}) \leq \dim \mathrm{supp} H^i(\mathcal{F}) \leq -i$$

so $j^* \mathcal{F} \in {}^p D^{\leq 0}_c(U)$. Next since $j^! \simeq j^*$ for an open immersion $j$, $j^*$ will commute with $\mathbb{D}$. Therefore for $\mathcal{G} \in {}^p D^{\geq 0}_c(X)$,

$$\dim \mathrm{supp} H^i(\mathbb{D}(j^* \mathcal{G})) = \dim \mathrm{supp} H^i(j^*(\mathbb{D}\mathcal{G})) \leq \dim \mathrm{supp} H^i(\mathbb{D}\mathcal{G}) \leq -i$$

which means $j^* \mathcal{G} \in {}^p D^{\geq 0}_c(U)$.

For $(2)$, the argument is similar to the above for $j^*$, because $j_!$ is $t$-exact as well and will not increase the dimension of the support by definition (see [1, Lem. 1.3.1]).

For $(3)$, let $\mathcal{F} \in {}^p D^{\geq 0}_c(U)$. By Verdier duality, we have $\mathbb{D}(j_* \mathcal{F}) \simeq j_!(\mathbb{D}\mathcal{F})$. And $\mathbb{D}\mathcal{F} \in {}^p D^{\leq 0}_c(U)$ implies $j_!(\mathbb{D}\mathcal{F}) \in {}^p D^{\leq 0}_c(X)$ by part $(2)$.

For $(4)$, since $i$ is a closed embedding, $i_* \simeq i_!$, the argument is the same as part $(1)$. It says $i_*$ is $t$-exact, will not increase the dimension of the support and commutes with $\mathbb{D}$.

For $(5)$, it's similar to $(2)$.

For $(6)$, it's similar to $(3)$. □

The next lemma says a perverse sheaf is "locally" a shifted local system.

**Lemma 1.14.** *Let $X$ be a variety and $(X_s)_{s\in\mathscr{I}}$ be a stratification on $X$. For every $s \in \mathscr{I}$, $j_s : X_s \to X$ is the inclusion map. Let $\mathcal{F} \in D^b_c(X)$.*

*(1). Suppose $\mathcal{F}$ is constructible with respect to $\mathscr{I}$. We have $\mathcal{F} \in {}^pD^{\leq 0}_c(X)$ if and only if $j_s^*\mathcal{F} \in D^{\leq -\dim X_s}_{\text{locf}}(X_s)$ for all $s \in \mathscr{I}$.*

*(2). Suppose $\mathbb{D}\mathcal{F}$ is constructible with respect to $\mathscr{I}$. We have $\mathcal{F} \in {}^pD^{\geq 0}_c(X)$ if and only if $j_s^!\mathcal{F} \in D^{\geq -\dim X_s}_{\text{locf}}(X_s)$ for all $s \in \mathscr{I}$.*

Let's look at cohomologies of a perverse sheaf $\mathcal{F} \in \text{Perv}(X)$ where $X$ has dimension $n$. The fact $\mathcal{F} \in {}^pD^{\leq 0}_c(X)$ obviously implies $H^i(\mathcal{F}) = 0$ for all $i > 0$. Next since $\mathcal{F} \in {}^pD^{\geq 0}_c(X)$, for all $\mathcal{G} \in {}^pD^{\leq -1}_c(X)$ we will have $\text{Hom}_{D^b_c(X)}(\mathcal{G}, \mathcal{F}) = 0$[2]. Note that $D^{\leq -n}_c(X) \subseteq {}^pD^{\leq 0}_c(X)$ and therefore $D^{\leq -n-1}_c(X) \subseteq {}^pD^{\leq -1}_c(X)$. Then for all $\mathcal{G} \in D^{\leq -n-1}_c(X)$, $\text{Hom}_{D^b_c(X)}(\mathcal{G}, \mathcal{F}) = 0$. From [1, Lem. A.7.3] this means $\mathcal{F} \in D^{\geq -n}_c(X)$. So for all $i < -n$, $H^i(\mathcal{F}) = 0$.

As for $-n \leq i \leq 0$, $\mathcal{F} \in {}^pD^{\leq 0}_c(X)$ means $\dim \text{supp} H^i(\mathcal{F}) \leq -i$. Therefore $H^0(\mathcal{F})$ is supported at most on points, $H^{-1}(\mathcal{F})$ on curves, $H^{-n+1}(\mathcal{F})$ on divisors and $H^{-n}(\mathcal{F})$ everywhere.

**Example 1.15.** Let $\mathbb{C} = \mathbb{C}^\times \coprod \{0\}$ be a stratification for $\mathbb{C}$ with inclusions $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$. In the Talk 2 (Example 1.27), we have computed the cohomology of $j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}$, whose stable of stalks are as follows.

| | $\{0\}$ | $\mathbb{C}^\times$ |
|---|---|---|
| 1 | $\underline{\mathbb{C}}$ | |
| 0 | $\underline{\mathbb{C}}$ | $\underline{\mathbb{C}}$ |

So for $j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$,

| | $\{0\}$ | $\mathbb{C}^\times$ |
|---|---|---|
| 0 | $\underline{\mathbb{C}}$ | |
| -1 | $\underline{\mathbb{C}}$ | $\underline{\mathbb{C}}$ |

its $0$-th cohomology is supported on the point $0$ while its $(-1)$-th cohomology is supported everywhere. This diagram justifies the "lower triangularity" condition of perversity. Informally this means it will be a perverse sheaf. To see it formally, firstly it's clear the above computations imply $j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD^{\leq 0}_c(\mathbb{C})$ directly. Next since $\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ is a perverse sheaf by Proposition 1.12, $j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD^{\geq 0}_c(\mathbb{C})$ by (3) of Lemma 1.13.

Actually we can also compute $\mathbb{D}j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ concretely. By Verdier duality we have

$$\begin{aligned}\mathbb{D}j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] &= j_!\mathbb{D}\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \\ &= j_!\mathbb{R}\mathcal{H}\text{om}(\underline{\mathbb{C}}_{\mathbb{C}^\times}[1], \underline{\mathbb{C}}_{\mathbb{C}^\times}[2]), \text{ since } \omega_{\mathbb{C}^\times} = \underline{\mathbb{C}}_{\mathbb{C}^\times}[2] \\ &= j_!\mathcal{H}\text{om}(\underline{\mathbb{C}}_{\mathbb{C}^\times}[1], \underline{\mathbb{C}}_{\mathbb{C}^\times}[2]) \\ &= j_!\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]\end{aligned}$$

Note that by definition $j_!\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ will vanish at $0$. So its table of stalks is

| | $\{0\}$ | $\mathbb{C}^\times$ |
|---|---|---|
| 0 | | |
| -1 | | $\underline{\mathbb{C}}$ |

Then $\mathbb{D}j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD^{\leq 0}_c(\mathbb{C}) \Rightarrow j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \in {}^pD^{\geq 0}_c(\mathbb{C})$.

[2] The second condition of $t$-structure

## References

[1] Pramod N. Achar. *Perverse Sheaves and Applications to Representation Theory*, volume 258. American Mathematical Soc., 2021.