TALK OUTLINE: SEMINAR ON PERVERSE SHEAVES

DERIVED CATEGORIES. (April 8)

Our aim in this lecture is to give an introduction to the derived category of an abelian category and to the associated theory of derived functors between derived categories of abelian categories.

It should be noted that working with derived categories and derived functors between them provides an extension of classical Homological Algebra at a higher level, thus giving powerful tools as well as important advantages.

The lecture is divided in 6 parts:

Part 1: is devoted to the working setting of $\mathbb{K}$-linear abelian categories and exactness conditions

Part 2: we move to the category of complexes, discussing cohomology and the need to construct the derived category, providing motivation.

Part 3: we analyze the tools we need in order to construct the derived category, namely localization of categories, emphasizing the case of chain complexes and quasi-isomorphisms.

Part 4: this leads us to the derived category and its variants, and we describe its basic structure.

Part 5: having constructed the derived category, we define derived functors at this level and discuss some of their fundamental properties.

Part 6: we apply the theory of derived functors as developed in Part 5 to one of the most important cases: that of deriving the $Hom^\bullet$ functor of complexes, and we briefly discuss the connection between the derived functor $RHom^\bullet$ and the abelian groups $Ext^*(-,-)$ of extensions.

The main example in which the theory will be applied to is the $\mathbb{C}$-linear abelian category of sheaves of $\mathbb{C}$-vector spaces over a topological space.

Part 1: We begin by describing the fundamental working setting of the lecture. We fix a commutative ring $\mathbb{K}$, and we consider

$$\boxed{\mathbb{K}\text{-linear abelian categories}}$$

Recall that a category $\mathcal{A}$ is called $\mathbb{K}$-linear if:

① There exists a zero object 0 in $\mathcal{A}$.

② The direct sum (or coproduct) of any two objects exists in $\mathcal{A}$.

③ For any two objects X, Y in $\mathcal{A}$, the Hom-space $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is a $\mathbb{K}$-module and the composition of morphisms in $\mathcal{A}$ is $\mathbb{K}$-bilinear.

Note that we may identify: $\mathbb{Z}$-linear categories $\equiv$ $\equiv$ additive categories.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-linear categories.

A functor $F: \mathcal{A} \to \mathcal{B}$ is called a $\mathbb{K}$-linear functor, if $\forall X, Y \in \mathrm{Ob}(\mathcal{A})$, the maps $F_{X,Y}: \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$ $f \mapsto F_{X,Y}(f) := F(f)$ are homomorphisms of $\mathbb{K}$-modules.

Note that we may identify: $\mathbb{Z}$-linear functor $\equiv$ $\equiv$ additive functor.

We are mainly concerned with $\mathbb{K}$-linear abelian categories:

A ($\mathbb{K}$-linear) category $\mathcal{A}$ is called an abelian category $\Leftrightarrow$:

① Any morphism in $\mathcal{A}$ has a kernel and a cokernel.

② Any monomorphism in $\mathcal{A}$ is a kernel of some morphism and any epimorphism in $\mathcal{A}$ is a cokernel of some morphism.

Example: ① The category Mod-R, where R is a ring is a $\mathbb{Z}$-linear abelian category.

② If R is a $\mathbb{K}$-algebra, where $\mathbb{K}$ is a commutative ring then Mod-R is a $\mathbb{K}$-linear abelian category.

③ If X is a topological space, then the category $Sh(X)$ of sheaves of $\mathbb{C}$-vector spaces is a $\mathbb{C}$-linear abelian category. ■

We now move on to

$$\boxed{\text{exactness conditions of } \mathbb{K}\text{-linear functors}}$$

We fix a $\mathbb{K}$-linear functor $F: \mathcal{A} \to \mathcal{B}$ between $\mathbb{K}$-linear abelian categories $\mathcal{A}$ and $\mathcal{B}$.

We also fix a short exact sequence [s.e.s.]

$$(E) \quad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \text{ in } \mathcal{A}.$$

① F is called left exact, if for any s.e.s. (E) in $\mathcal{A}$, the following diagram is exact in $\mathcal{B}$:

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

② F is called right exact, if for any s.e.s. (E) in $\mathcal{A}$ the following diagram is exact in $\mathcal{B}$:

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

③ F is called exact, if F is left exact as well as right exact, i.e., for any s.e.s. (E) in $\mathcal{A}$, the following diagram is exact in $\mathcal{B}$:

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

Note that exact ($\mathbb{K}$-linear) functors $F: \mathcal{A} \to \mathcal{B}$ preserve any exactness conditions in $\mathcal{A}$: if

$$\cdots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \longrightarrow \cdots \quad (A^\bullet, d_{A^\bullet}^\bullet)$$

is any exact complex of objects in $\mathcal{A}$ then the diagram

$$\cdots \longrightarrow F(A^{n-1}) \xrightarrow{F(d_A^{n-1})} F(A^n) \xrightarrow{F(d_A^n)} F(A^{n+1}) \longrightarrow \cdots$$

is exact in $\mathcal{B}$.

Main Idea: Study how far an additive functor is from being exact, and try to approximate $\mathbb{K}$-linear functor via exact functors. ■

The next step is to study

$\boxed{\text{objects representing exact functors}}$

The most important $\mathbb{K}$-linear functors defined on a $\mathbb{K}$-linear category $\mathcal{A}$ are the Hom-functors:

$$\mathrm{Hom}_{\mathcal{A}}(X,-): \mathcal{A} \to \text{Mod-}\mathbb{K} \text{ and } \mathrm{Hom}_{\mathcal{A}}(-,X): \mathcal{A}^{op} \longrightarrow \text{Mod-}\mathbb{K}.$$

It is easy to see and well known that both Hom-functors $\mathrm{Hom}_{\mathcal{A}}(X,-)$ and $\mathrm{Hom}_{\mathcal{A}}(-,X)$ are left exact. Thus, it makes sense to define:

- $X \in \mathrm{Ob}(\mathcal{A})$ is called projective $\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{A}}(X,-)$ is exact.
- $X \in \mathrm{Ob}(\mathcal{A})$ is called injective $\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{A}}(-,X)$ is exact.

$\mapsto$ Since $\mathrm{Hom}_{\mathcal{A}}(X,-)$ is left exact, it follows that X: projective $\Leftrightarrow$ for any epimorphism $g: B \longrightarrow C$ and any morphism $a: X \to C$, there exists a morphism $b: X \longrightarrow B$ such that $a = g \circ b$.

$$\begin{tikzcd} & X \arrow[dl, dashed, "\exists b"'] \arrow[d, "a"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

$\longmapsto$ Since $Hom_{\mathcal{A}}(-,X)$ is left exact, it follows that X: injective $\Longleftrightarrow$ for any monomorphism $f: A \to B$ and any morphism $a: A \to X$, there exists a morphism $b: B \to X$ such that $a = b \circ f$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[dl, dashed, "\exists b"] \\ & X & \end{tikzcd}$$

Main Idea: Since projective and injective objects represent exact functors, it is important to study how far an object $A \in Ob(\mathcal{A})$ is from being projective or injective, and then to "approximate" if possibile, arbitrary objects of $\mathcal{A}$ via projective or injective objects. ∎

Part 2: We fix a $\mathbb{K}$-linear abelian category $\mathcal{A}$.

The category of (Co)Chain Complexes over $\mathcal{A}$, denoted by $Ch(\mathcal{A})$ has objects chain complexes of objects of $\mathcal{A}$:

$$\cdots \longrightarrow A^{-2} \xrightarrow{d_A^{-2}} A^{-1} \xrightarrow{d_A^{-1}} A^{0} \xrightarrow{d_A^{0}} A^{1} \xrightarrow{d_A^{1}} A^{2} \longrightarrow \cdots \quad (A^\bullet, d_{A^\bullet}^\bullet)$$

such that: $d_A^i \circ d_A^{i-1} = 0 \quad \forall i \in \mathbb{Z}$. The morphisms d_A^i are called the differentials of $(A^\bullet, d_{A^\bullet}^\bullet)$.

A morphism between two chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$, called a chain map is a family of morphisms in $\mathcal{A}$ $f = \{f^i : A^i \longrightarrow B^i\}_{i\in\mathbb{Z}}$ such that: $f^i \circ d_A^{i-1} = d_B^{i-1} \circ f^{i-1}, \ \forall i \in \mathbb{Z}$

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-2} \arrow[r] \arrow[d, "f^{-2}"] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"] & A^{0} \arrow[r] \arrow[d, "f^{0}"] & A^{1} \arrow[r] \arrow[d, "f^{1}"] & A^{2} \arrow[r] \arrow[d, "f^{2}"] & \cdots & (A^\bullet, d_A^\bullet) \arrow[d, "f^\bullet"] \\
\cdots \arrow[r] & B^{-2} \arrow[r] & B^{-1} \arrow[r] & B^{0} \arrow[r] & B^{1} \arrow[r] & B^{2} \arrow[r] & \cdots & (B^\bullet, d_{B^\bullet}^\bullet)
\end{tikzcd}$$

that is, each square is commutative.

It is very easy to see that in this way, we obtain a $\mathbb{K}$-linear category $Ch(\mathcal{A})$: the category of (co)chain complexes over $\mathcal{A}$.

Remark: The category $Ch(\mathcal{A})$ is again an abelian category: the construction of kernels and cokernels is defined degree wise. ∎

In practice there appear to be of interest, various full subcategories of $Ch(\mathcal{A})$:

Let, as before, $(A^\bullet, d_A^\bullet)$ a chain complex.

- $(A^\bullet, d_A^\bullet)$ is called bounded above $\Leftrightarrow \exists N \in \mathbb{N}: A^i = 0, \forall i > N$.

  bounded below $\Leftrightarrow \exists N \in \mathbb{N}: A^i = 0, \forall i < -N$.

  bounded $\Leftrightarrow \exists N \in \mathbb{N}: A^i = 0, \forall |i| > N$.

Then: $Ch^-(\mathcal{A})$: the full subcategory of bounded above complexes

$Ch^+(\mathcal{A})$: the full subcategory of bounded below complexes

$Ch^b(\mathcal{A})$: the full subcategory of bounded complexes

be the corresponding full subcategories of $Ch(\mathcal{A})$.

The nth cohomology object of $(A^\bullet, d_A^\bullet)$ is defined by:

$$H^n(A^\bullet) = \mathrm{Ker}(d_A^n) / \mathrm{Im}(d_A^{n-1}), \quad n \in \mathbb{Z}$$

This makes sense, because: $d_A^n \circ d_A^{n-1} = 0 \Leftrightarrow \mathrm{Im}(d_A^{n-1}) \subseteq \mathrm{Ker}(d_A^{n-1})$

If $f^\bullet: (A^\bullet, d_A^\bullet) \longrightarrow (B^\bullet, d_B^\bullet)$ is a chain map, then using that $f^n \circ d_A^{n-1} = d_B^{n-1} \circ f^{n-1}$, $\forall n \in \mathbb{Z}$, it follows directly that a morphism is induced

$$H^n(f^\cdot): H^n(A^\cdot) \longrightarrow H^n(B^\cdot),\ \forall n \in \mathbb{Z}$$

and then we obtain a sequence of $\mathbb{K}$-linear functors

$$H^n: Ch(\mathcal{A}) \longrightarrow \mathcal{A},\ n \in \mathbb{Z}$$

Definition: A chain map $f^\cdot: (A^\cdot, d_A^\cdot)$ is called a quasi-isomorphism $\Leftrightarrow H^n(f^\cdot): H^n(A^\cdot) \xrightarrow{\simeq} H^n(B^\cdot),\ \forall n \in \mathbb{Z}$.

We shall explain below why this is a crucial concept. We are interested in conditions ensuring that two chain maps induce the same maps in cohomology and in particular that a chain map is quasi-isomorphism.

Such a condition emerges from the notion of homotopy. Let $f^\cdot, g^\cdot: (A^\cdot, d_A^\cdot) \longrightarrow (B^\cdot, d_B^\cdot)$ be two chain maps:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-2} \arrow[r] \arrow[d, shift right, "f^{i-2}"'] \arrow[d, shift left, "g^{i-2}"] & A^{i-1} \arrow[r] \arrow[d, shift right, "f^{i-1}"'] \arrow[d, shift left, "g^{i-1}"] \arrow[dl, dashed, "\varphi^{i-1}"'] & A^{i} \arrow[r] \arrow[d, shift right, "f^{i}"'] \arrow[d, shift left, "g^{i}"] \arrow[dl, dashed, "\varphi^{i}"'] & A^{i+1} \arrow[r] \arrow[d, shift right, "f^{i+1}"'] \arrow[d, shift left, "g^{i+1}"] \arrow[dl, dashed, "\varphi^{i+1}"'] & \cdots & (A^\cdot, d_A^\cdot) \arrow[d, shift right, "f^\cdot"'] \arrow[d, shift left, "g^\cdot"] \\
\cdots \arrow[r] & B^{i-2} \arrow[r, "d_B^{i-2}"'] & B^{i-1} \arrow[r, "d_B^{i-1}"'] & B^{i} \arrow[r, "d_B^{i}"'] & B^{i+1} \arrow[r] & \cdots & (B^\cdot, d_B^\cdot)
\end{tikzcd}$$

A homotopy from $f^\cdot$ to $g^\cdot$ is a family of morphisms of $\mathcal{A}$: $\varphi^\cdot = \{\varphi^i: A^i \longrightarrow B^{i-1}\}_{i \in \mathbb{Z}}$ such that

$$f^i - g^i = d_B^{i-1} \circ \varphi^i + \varphi^{i+1} \circ d_A^i,\ \forall i \in \mathbb{Z}$$ and is denoted by

$$\varphi^\cdot: f^\cdot \longrightarrow g^\cdot$$

It is easy to see that defining

$f^\bullet \simeq g^\bullet$ ($f^\bullet$ is homotopic to $g^\bullet$) $\Leftrightarrow$ $\exists$ homotopy $\varphi^\bullet : f^\bullet \to g^\bullet$

is an equivalence relation on $\mathrm{Hom}_{Ch(\mathcal{A})}[(A^\bullet, d_A^\bullet), (B^\bullet, d_B^\bullet)]$

which is compatible with composition and the $\mathbb{K}$-module structure of the Hom-space:

- If $f_1^\bullet \simeq f_2^\bullet : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$, then for any chain map $g^\bullet : (C^\bullet, d_A^\bullet) \to (A^\bullet, d_A^\bullet)$ and any chain map $h^\bullet : (B^\bullet, d_B^\bullet) \to (E^\bullet, d_E^\bullet)$ we have
  $g^\bullet \circ f_1^\bullet \simeq g^\bullet \circ f_2^\bullet$ and $f_1^\bullet \circ h^\bullet \simeq f_2^\bullet \circ h^\bullet$

- $$\left.\begin{array}{l} f_1^\bullet, f_2^\bullet : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet) \\ g_1^\bullet, g_2^\bullet : (B^\bullet, d_B^\bullet) \to (C^\bullet, d_C^\bullet) \end{array}\right\} \Rightarrow \left.\begin{array}{l} f_1^\bullet \simeq f_2^\bullet \\ g_1^\bullet \simeq g_2^\bullet \end{array}\right\} \Rightarrow g_1^\bullet \circ f_1^\bullet \simeq g_2^\bullet \circ f_2^\bullet$$

- $f_1^\bullet, f_2^\bullet : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$, $g_1^\bullet, g_2^\bullet : (B^\bullet, d_B^\bullet) \to (C^\bullet, d_C^\bullet)$,
  $h_1^\bullet, h_2^\bullet : (D^\bullet, d_D^\bullet) \to (A^\bullet, d_A^\bullet)$, then:
  $(h_1^\bullet + h_2^\bullet) \circ f_1^\bullet \simeq (g_1^\bullet + h_2^\bullet) \circ f_2^\bullet$ and $(k f_1^\bullet) \simeq (k f_2^\bullet)$
  $f_1^\bullet \circ (g_1^\bullet + g_2^\bullet) \simeq f_2^\bullet \circ (g_1^\bullet + g_2^\bullet)$ $\forall k \in \mathbb{K}$.

This allows us to define the homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ as follows:

$\mapsto$ The objects of $K(\mathcal{A})$ are the objects of $Ch(\mathcal{A})$, that is chain complexes over $\mathcal{A}$.

$\mapsto$ A morphism from $(A^\bullet, d_A^\bullet)$ to $(B^\bullet, d_B^\bullet)$ in $K(\mathcal{A})$ is a homotopy class $[f^\bullet]: (A^\bullet, d_A^\bullet) \longrightarrow (B^\bullet, d_B^\bullet)$ of a chain map $f^\bullet: (A^\bullet, d_A^\bullet) \longrightarrow (B^\bullet, d_B^\bullet)$ under the homotopy relation

The composition of $[f^\bullet]$ and $[g^\bullet]$ is defined to be $[f^\bullet \circ g^\bullet]$ and $Id_{(A^\bullet, d_A^\bullet)}$ in $K(\mathcal{A})$ is $[Id_{(A^\bullet, d_A^\bullet)}]$

The category $K(\mathcal{A})$ is the homotopy category of $\mathcal{A}$, and is obtained from $Ch(\mathcal{A})$ by identifying homotopic chain maps.

Note that $K(\mathcal{A})$ is rarely abelian. However it can be proven that $K(\mathcal{A})$ has a weaker structure: that of the triangulated category.

The Cone Construction

Consider a chain map $f^\bullet : (A^\bullet, d_{A^\bullet}^\bullet) \longrightarrow (B^\bullet, d_{B^\bullet}^\bullet)$.

The Cone of $f^\bullet$ is the chain complex $(\mathrm{Cone}(f^\bullet), d_{\mathrm{Cone}(f^\bullet)}^\bullet)$ where

$$\begin{tikzcd}[column sep=huge]
\cdots \arrow[r] & A^{-1}\oplus B^{-2} \arrow[r, "{\begin{pmatrix} -d_A^{-1} & 0 \\ f^{-1} & d_B^{-2} \end{pmatrix}}", "d^{-2}_{\mathrm{Cone}(f^\bullet)}"'] & A^{0}\oplus B^{-1} \arrow[r, "{\begin{pmatrix} -d_A^{0} & 0 \\ f^{0} & d_B^{-1} \end{pmatrix}}", "d^{-1}_{\mathrm{Cone}(f^\bullet)}"'] & A^{1}\oplus B^{0} \arrow[r, "{\begin{pmatrix} -d_A^{1} & 0 \\ f^{1} & d_B^{0} \end{pmatrix}}", "d^{0}_{\mathrm{Cone}(f^\bullet)}"'] & A^{2}\oplus B^{1} \arrow[r, "{\begin{pmatrix} -d_A^{2} & 0 \\ f^{2} & d_B^{1} \end{pmatrix}}", "d^{1}_{\mathrm{Cone}(f^\bullet)}"'] & A^{3}\oplus B^{2} \arrow[r] & \cdots \\
& \text{degree } -2 \arrow[u] & \text{degree } -1 \arrow[u] & \text{degree } 0 \arrow[u] & \text{degree } 1 \arrow[u] & \text{degree } 2 \arrow[u] &
\end{tikzcd}$$

$$\mathrm{Cone}(f^\bullet)^i = A^{i+1} \oplus B^i, \quad d^i_{\mathrm{Cone}(f^\bullet)} = \begin{pmatrix} -d_{A^\bullet}^{i+1} & 0 \\ f^i & d_{B^\bullet}^i \end{pmatrix}, \ \forall i \in \mathbb{Z}$$

$$
\begin{tikzcd}[column sep=large]
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A^{-2} \arrow[r, "f^{-2}"] \arrow[d, "d_A^{-2}"'] & B^{-2} \arrow[r, "\begin{pmatrix} 0 \\ 1_{B^{-2}} \end{pmatrix}"] \arrow[d, "d^{-2}"'] & A^{-1} \oplus B^{-2} \arrow[r, "(1_{A^{-1}}\; 0)"] \arrow[d, "\begin{pmatrix} -d_A^{-1} & 0 \\ f^{-1} & d_B^{-2} \end{pmatrix}"] & A^{-1} \arrow[d, "-d_A^{-1}"] \\
A^{-1} \arrow[r, "f^{-1}"] \arrow[d, "d_A^{-1}"'] & B^{-1} \arrow[r, "\begin{pmatrix} 0 \\ 1_{B^{-1}} \end{pmatrix}"] \arrow[d, "d_B^{-1}"'] & A^{0} \oplus B^{-1} \arrow[r, "(1_{A^{0}}\; 0)"] \arrow[d, "\begin{pmatrix} -d_A^{0} & 0 \\ f^{0} & d_B^{-1} \end{pmatrix}"] & A^{0} \arrow[d, "-d_A^{0}"] \\
A^{0} \arrow[r, "f^{0}"] \arrow[d, "d_A^{0}"'] & B^{0} \arrow[r, "\begin{pmatrix} 0 \\ 1_{B^{0}} \end{pmatrix}"] \arrow[d, "d_B^{0}"'] & A^{1} \oplus B^{0} \arrow[r, "(1_{A^{1}}\; 0)"] \arrow[d, "\begin{pmatrix} -d_A^{1} & 0 \\ f^{1} & d_B^{0} \end{pmatrix}"] & A^{1} \arrow[d, "-d_A^{1}"] \\
A^{1} \arrow[r, "f^{1}"] \arrow[d, "d_A^{1}"'] & B^{1} \arrow[r, "\begin{pmatrix} 0 \\ 1_{B^{1}} \end{pmatrix}"] \arrow[d, "d_B^{1}"'] & A^{2} \oplus B^{1} \arrow[r, "(1_{A^{2}}\; 0)"] \arrow[d, "\begin{pmatrix} -d_A^{2} & 0 \\ f^{2} & d_B^{1} \end{pmatrix}"] & A^{2} \arrow[d, "-d_A^{2}"] \\
A^{2} \arrow[r, "f^{2}"] \arrow[d] & B^{2} \arrow[r, "\begin{pmatrix} 0 \\ 1_{B^{2}} \end{pmatrix}"] \arrow[d] & A^{3} \oplus B^{2} \arrow[r, "(1_{A^{3}}\; 0)"] \arrow[d] & A^{3} \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

$$(A^\bullet, d_A^\bullet) \xrightarrow{f^\bullet} (B^\bullet, d_B^\bullet) \xrightarrow{g^\bullet} (\mathrm{Cone}(f^\bullet), d^\bullet_{\mathrm{Cone}(f^\bullet)}) \xrightarrow{h^\bullet} (A^\bullet, d_A^\bullet)[1] \quad ⊛$$

⊙ Hence there are chain maps $g^\bullet : (B^\bullet, d_B^\bullet) \longrightarrow (\mathrm{Cone}(f^\bullet), d^\bullet_{\mathrm{Cone}(f^\bullet)})$, $h^\bullet : (\mathrm{Cone}(f^\bullet), d^\bullet_{\mathrm{Cone}(f^\bullet)}) \longrightarrow (A^\bullet, d_{A^\bullet}^\bullet)_{[1]}$ and clearly $g^\bullet \circ f^\bullet \simeq 0$, $h^\bullet \circ g^\bullet \simeq 0$.

The final piece of information concerning the $\mathbb{K}$-linear categories $Ch(\mathcal{A})$ and $K(\mathcal{A})$ are related to the behaviour of the cohomology functors $H^n: Ch(\mathcal{A}) \longrightarrow \mathcal{A}$, $\forall n \in \mathbb{Z}$.

- If $f^\cdot, g^\cdot: (A^\cdot, d_A^\cdot) \longrightarrow (B^\cdot, d_B^\cdot)$ is a chain map, then:

$$f^\cdot \simeq g^\cdot \Longrightarrow H^n(f^\cdot) = H^n(g^\cdot), \ \forall n \in \mathbb{Z}$$

This implies that the cohomology functors H^n can be defined on the homotopy category $K(\mathcal{A})$ as follows:

$$H^n: K(\mathcal{A}) \longrightarrow \mathcal{A}: H^n(A^\cdot, d_A^\cdot) = \operatorname{Ker}(d_A^n) / \operatorname{Im}(d_A^{n-1})$$

If $[f^\cdot]: (A^\cdot, d_A^\cdot) \longrightarrow (B^\cdot, d_B^\cdot)$ ($[f^\cdot]$: homotopy class of a chain map) then

$H^n([f^\cdot]) = H^n(f^\cdot)$, $\forall n \in \mathbb{Z}$ is a morphism in $K(\mathcal{A})$, and we usually write $f^\cdot$ instead of $[f^\cdot]$.

- If $(A^\bullet, d_A^\bullet), (B^\bullet, d_B^\bullet), (C^\bullet, d_C^\bullet) \in Ch(\mathcal{A})$ then a short exact sequence of complexes in $Ch(\mathcal{A})$ is a diagram

$$0 \longrightarrow (A^\bullet, d_A^\bullet) \xrightarrow{f^\bullet} (B^\bullet, d_B^\bullet) \xrightarrow{g^\bullet} (C^\bullet, d_C^\bullet) \longrightarrow 0$$

of chain maps $f^\bullet, g^\bullet$ such that, $\forall n \in \mathbb{Z}$, the diagram

$$0 \longrightarrow A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \longrightarrow 0$$ is a s.e.s. in $\mathcal{A}$.

Diagrammatically:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A^{n-1} \arrow[r, "f^{n-1}"] \arrow[d, "d_A^{n-1}"] & B^{n-1} \arrow[r, "g^{n-1}"] \arrow[d, "d_B^{n-1}"] & C^{n-1} \arrow[r] \arrow[d, "d_C^{n-1}"] & 0 \\
0 \arrow[r] & A^{n} \arrow[r, "f^{n}"] \arrow[d, "d_A^{n}"] & B^{n} \arrow[r, "g^{n}"] \arrow[d, "d_B^{n}"] & C^{n} \arrow[r] \arrow[d, "d_C^{n}"] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r, "f^{n+1}"] \arrow[d] & B^{n+1} \arrow[r, "g^{n+1}"] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

- The rows are short exact sequences in $\mathcal{A}$
- The columns are chain complexes
- Any square is commutative.

Define $\mathbb{K}$-linear functors

$[1] : Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{A})$ and $[-1] : Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{A})$

- by $(A^\bullet, d_A^\bullet)[1] = (A^\bullet[1], d_A^\bullet[1])$, where $(A^\bullet[1])^{n+1} = A^{n+1}$, $(d_A^\bullet[1])^n = -d_A^{n+1}$ and if $f^\bullet : (A^\bullet, d_A^\bullet) \longrightarrow (B^\bullet, d_B^\bullet)$ is a chain map, then: $f^\bullet[1] : (A^\bullet, d_A^\bullet)[1] \longrightarrow (B^\bullet, d_B^\bullet)[1]$; $(f^\bullet[1])^n = f^{n+1}$.
- and $(A^\bullet, d_A^\bullet)[-1] = (A^\bullet_{[-1]}, d_A^\bullet[-1])$, where $(A^\bullet[-1])^n = A^{n-1}$, $(d_A^\bullet[-1])^n = -d^{n-1}$ and if $f^\bullet : (A^\bullet, d_A^\bullet) \longrightarrow (B^\bullet, d_B^\bullet)$ is a chain map then: $f^\bullet[-1] : (A^\bullet, d_A^\bullet)[-1] \longrightarrow (B^\bullet, d_B^\bullet)[-1]$; $(f^\bullet[-1])^n = f^{n-1}$

Clearly $[1]$ is a $\mathbb{K}$-linear equivalence with quasi-inverse $[-1]$ and it induces a $\mathbb{K}$-linear equivalence $[1] : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$ with quasi inverse $[-1]$.

Using the equivalences $[1]$ and $[-1]$, the diagram can be extended to a diagram

$$\begin{tikzcd}
(Cone(f^\bullet), d^\bullet_{Cone(f^\bullet)})[-1] \arrow[r, "h^\bullet{[-1]}"] & (A^\bullet, d_A^\bullet) \arrow[r, "f^\bullet"] & (B^\bullet, d_B^\bullet) \arrow[r, "g^\bullet"] & (Cone(f^\bullet), d^\bullet_{Cone(f^\bullet)}) \arrow[d, "h^\bullet"] \\
(B^\bullet, d_B)[-1] \arrow[u, "g^\bullet{[-1]}"] & & & (A^\bullet, d_A^\bullet)[1]
\end{tikzcd}$$

which is induced in $K(\mathcal{A})$ by any chain map $f^{\cdot}: (A^{\cdot}, d_A^{\cdot}) \longrightarrow (B^{\cdot}, d_B^{\cdot})$. The morphism $h^{\cdot}[-1]$ plays the role of a (weak) kernel of $f^{\cdot}$ in $K(\mathcal{A})$ and the morphism $g^{\cdot}$ plays the role of a (weak) cokernel of $f^{\cdot}$ in $K(\mathcal{A})$, and the above diagram is the basis of the construction of a triangulated structure on $K(\mathcal{A})$ which, in some sense, covers the lack of the abelian structure of $K(\mathcal{A})$.

◉ A fundamental property of the cohomology functors H^n is that they induce a long exact sequence, applied to a s.e.s. of chain complexes:

If

$$(E)\quad 0 \longrightarrow (A^{\bullet}, d_A^{\bullet}) \xrightarrow{f^{\bullet}} (B^{\bullet}, d_B^{\bullet}) \xrightarrow{g^{\bullet}} (C^{\bullet}, d_C^{\bullet}) \longrightarrow 0$$

is a s.e.s. of chain complexes, then there exist morphisms

$\partial_E^n : H^n(C^{\bullet}) \to H^{n+1}(A^{\bullet})$, $\forall n \in \mathbb{Z}$, s.t. the sequence

$$\cdots \longrightarrow H^{n-1}(C^{\bullet}) \xrightarrow{\partial_E^{n-1}} H^n(A^{\bullet}) \xrightarrow{H^n(f^{\bullet})} H^n(B^{\bullet}) \xrightarrow{H^n(g^{\bullet})} H^n(C^{\bullet}) \xrightarrow{\partial_E^n} H^{n+1}(A^{\bullet}) \longrightarrow \cdots$$

is exact.

A chain complex $(A^{\bullet}, d_A^{\bullet})$ is acyclic (exact) $\Leftrightarrow H^n(A^{\bullet}, d_A^{\bullet}) = 0$, $\forall n \in \mathbb{Z}$

For instance in the above s.e.s. (E):

↦ $f^{\bullet}$: quasi-isomorphism $\Leftrightarrow (C^{\bullet}, d_C^{\bullet})$: acyclic

↦ $g^{\bullet}$: quasi-isomorphism $\Leftrightarrow (A^{\bullet}, d_A^{\bullet})$: acyclic

↦ $(A^{\bullet}, d_A^{\bullet}), (C^{\bullet}, d_C^{\bullet})$: acyclic $\Rightarrow (B^{\bullet}, d_B^{\bullet})$: acyclic

We need the passage from the (abelian) category of chain complexes to the (non-abelian, but triangulated) homotopy category of chain complexes, because quasi-isomorphisms behave better in the homotopy category.

Motivating the Introduction of the Derived Category

Key Point ~ Grothendieck's Idea: Classical Homological Algebra, say over an abelian category, is concerned with cohomologies of chain complexes arising in various situations.

However chain complexes carry more information than their cohomologies, and therefore they should be studied as objects of another category in which chain complexes with the same cohomology should be treated as isomorphic.

This led Grothendieck to the idea of constructing a new category with objects chain complexes, in which quasi-isomorphisms should be converted to isomorphisms.

This reminds us of the localization of a ring R at a set of non-zero elements $S \subseteq R$. Then, under certain conditions, there is a ring $R[S^{-1}]$ and a ring homomorphism $\varphi: R \to R[S^{-1}]$ such that

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "f"'] & R[S^{-1}] \arrow[d, dashed, "\exists! f^*"] \\ & R' \end{tikzcd}$$

$\varphi(s)$: invertible $\forall s \in S$ and for any ring R' and ring homomorphism $f: R \to R'$ such that $f(S)$: invertible, there is a unique ring homomorphism $f^*: R[S^{-1}] \longrightarrow R'$ such that $f^* \circ \varphi = f$.

So Grothendieck's idea is, starting with the category of chain complexes $Ch(\mathcal{A})$ over an abelian category $\mathcal{A}$, and considering the class Q_{is} of quasi-isomorphisms of $Ch(\mathcal{A})$, to construct a new category $D(\mathcal{A}) = Ch(\mathcal{A})[Q_{is}^{-1}]$ and a functor $Q: Ch(\mathcal{A}) \to D(\mathcal{A})$ such that $Q(f^{\cdot})$: isomorphism, and for any functor $F: Ch(\mathcal{A}) \to \mathcal{B}$ such that $F(f^{\cdot})$: isomorphism $\forall f^{\cdot} \in Q_{is}$, there exists a unique functor F^*: $F^* \circ Q = F$.

$$\begin{tikzcd} Ch(\mathcal{A}) \arrow[r, "Q"] \arrow[dr, "F"'] & D(\mathcal{A}) \arrow[d, dashed, "\exists! F^*"] \\ & \mathcal{B} \end{tikzcd}$$

This can be done via localization theory as developed in this setting by Verdier (1963), and in general by Gabriel-Zisman (1967), who, for any class of morphisms S in a category $\mathcal{A}$ constructed the localization of $\mathcal{A}$ at S as a category $\mathcal{A}[S^{-1}]$ together with a functor $Q:\mathcal{A}\to\mathcal{A}[S^{-1}]$ such that:

1. $\forall s\in S$: $Q(s)$: isomorphism
2. for any functor $F:\mathcal{A}\to\mathcal{B}$ such that $\forall s\in S$: $F(s)$: isomorphism, there exists a unique functor $F^*:\mathcal{A}[S^{-1}]\to\mathcal{B}$ such that $F^*\circ Q=F$. Then

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "Q"] \arrow[d, "F"'] & \mathcal{A}[S^{-1}] \arrow[dl, dashed, "\exists!\, F^*"] \\
\mathcal{B} &
\end{tikzcd}$$

$(\mathcal{A}[S^{-1}], Q)$ is the localization of $\mathcal{A}$ at S and $Q:\mathcal{A}\to\mathcal{A}[S^{-1}]$ is the localization functor.

However in order to have a computabile description of the localization category it is necessary, as in the case of rings (Ore conditions), that the class of maps S to be inverted and to satisfy some reasonabile properties.

As we shall see in the next part the class of quasi-isomorphisms in $Ch(\mathcal{A})$ fails to satisfy these properties, so we have to pass to the Homotopy category $K(\mathcal{A})$ in which the quasi-isomorphisms satisfy the required properties, and fortunately:

$$Ch(\mathcal{A})[Q_{is}^{-1}] \xrightarrow{\sim} K(\mathcal{A})[Q_{is}^{-1}].$$

Part 3: We briefily describe the Gabriel-Zisman locali-zation of a category $\mathcal{A}$ at a localizing class of morphism S of $\mathcal{C}$:

A class of morphisms S of a category $\mathcal{A}$ is called loca-lizing $\Leftrightarrow$ the following conditions are satisfied:

① $\forall X \in Ob(\mathcal{A})$: $Id_X \in S$

② If $f, g \in S$ and $f \circ g$ is defined, then $f \circ g \in S$.

③

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "\exists t \in S"'] & Z \arrow[d, "s \in S"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s \in S"'] & Z \arrow[d, "\exists t \in S"] \\ Y \arrow[r, "g"'] & W \end{tikzcd}$$

For any two morphisms f and s as above, with $s \in S$, there exists a commutative diagram as above with $t \in S$

For any two morphisms f, s as above with $s \in S$, there exists a commutative diagram as above, with $t \in S$

④ Let $f, g: X \to Y$ morphisms in $\mathcal{A}$:

$$\begin{tikzcd} Z \arrow[r, "\exists t \in S"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\exists s \in S"] & W \end{tikzcd}$$ Then:

$\exists s \in S$ such that $s \circ f = s \circ g \Leftrightarrow \exists t \in S: f \circ t = g \circ t$

If $\mathcal{A}$ is $\mathbb{K}$-linear, then clearly ④ is equivalent to

④i $Z \xrightarrow{\exists t\in S} X \xrightarrow{f} Y \xrightarrow{\exists s\in S} W$

For any morphism $f: X \to Y$:

$\exists s\in S : s\circ f = 0 \iff \exists t \in S : f\circ t = 0.$

Let S be a localizing class of morphisms in $\mathcal{A}$. We construct a new category $\mathcal{A}[S^{-1}]$ as follows:

- $\mathrm{ob}(\mathcal{A})[S^{-1}] = \mathrm{ob}(\mathcal{A})$
- let $X, Y \in \mathrm{Ob}(\mathcal{A})[S^{-1}] = \mathrm{Ob}(\mathcal{A})$. We consider diagrams in $\mathcal{A}$

left roof

$$\begin{tikzcd} & X' \arrow[dl] \arrow[dr] & \\ X & & {} \end{tikzcd}$$

where $s \in S$

Call two such diagrams equivalent:

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \sim \begin{tikzcd} & X'' \arrow[dl, "ts"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$\iff$

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X''' \arrow[l, "s\circ p = t\circ q"'] \arrow[u, "\exists p"'] \arrow[d, dashed, "\exists q"] & Y \\ & X'' \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

$\exists$ morphisms $p: X''' \to X'$
$q: X''' \to X''$ in $\mathcal{A}$

such that $s\circ p = t\circ q \in S$.

Facts: ① "$\sim$" is an equivalence relation on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$.

② We may work with (right) roofs, i.e. diagrams of the form

right roofs

$$\begin{tikzcd} & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

where $s \in S$

and the equivalence relation

$$\begin{tikzcd} & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd} \sim \begin{tikzcd} & Y'' & \\ X \arrow[ur, "g"] & & Y \arrow[ul, "t"'] \end{tikzcd} \iff \begin{tikzcd} & Y' \arrow[d, "\exists p"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & Y''' & Y \arrow[ul, "s"'] \arrow[dl, "t"] \arrow[l, "p \circ s = q \circ t \in S"'] \\ & Y'' \arrow[u, "\exists q"'] & \end{tikzcd}$$

$\exists$ morphisms $p : Y' \to Y'''$ such that
$q : Y'' \to Y'''$
$p \circ s = q \circ t \in S$

Composition of Left Roofs: Take representatives of equivalence classes of left roofs of the form:

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$

By property 3 of the localizing class S, there exist
a commutative square

$$\begin{tikzcd} & W \arrow[dl, "\exists r"'] \arrow[dr, "h"] & \\ X' \arrow[dr, "f"'] & & Y' \arrow[dl, "t"] \\ & Y & \end{tikzcd}$$

s.t $r \in S$

Hence we have a diagram

$$\begin{tikzcd} & & W \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

and by property 2 of the localizing class S we have $s \circ r \in S$.

Hence we have a left roof

$$\begin{tikzcd} & W \arrow[dl, "s \circ r"'] \arrow[dr, "g \circ h"] & \\ X & & Z \end{tikzcd}$$

A morphism in $\mathcal{A}[S^{-1}]$ from X to Y is an equivalen-ce class of roofs
$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
and is denoted $(s,f): X \to Y$

It can be proven that the composition of the left roof $(s,f): X \to Y$ with the left roof $(t,g): Y \longrightarrow Z$

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$

i.e. the left roof $(s\circ r, g\circ h): X \to Z$
$$\begin{tikzcd}[column sep=small] & W \arrow[dl, "s\circ r"'] \arrow[dr, "g\circ h"] & \\ X & & Z \end{tikzcd}$$

is independent of the representations and induces a well defined composition $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(X,Y)\times\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(Y,Z) \to \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(X,Z)$, $\big((s,f)\circ(t,g) \longmapsto (s\circ r, g\circ h)\big)$

and the representative of the left roof
$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "Id_X"'] \arrow[dr, "Id_X"] & \\ X & & X \end{tikzcd}$$
acts as the identity morphism of $X \in Ob(\mathcal{A}[S^{-1}])$

and then: $\mathcal{A}[S^{-1}]$ is a category, which is $\mathbb{K}$-linear if $\mathcal{A}$ is $\mathbb{K}$-linear.

Remark: There are some set theoretical issues with respect to whether the class of morphisms $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(X,Y)$ is actually a set: we shall ignore them. ∎

Then we consider the functor $Q: \mathcal{A} \to \mathcal{A}[S^{-1}]$, $Q(X) = X$ and if $f: X \to Y$ is a morphism in $\mathcal{A}$ then set $Q(f) =$ eq. class of the roof
$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "Id_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

Then we have a well defined functor $Q: \mathcal{A} \to \mathcal{A}[S^{-1}]$, which is $\mathbb{K}$-linear, if $\mathcal{A}$ is $\mathbb{K}$-linear, and observe that

$\forall s \in S$: $Q(s) =$
$$\begin{tikzcd} & X \arrow[dl, "Id_X"'] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}$$
which is invertible because

$s: X \to Y$

$$\begin{tikzcd} & & X \arrow[dl, "Id_X"'] \arrow[dr, "Id_X"] & & \\ & X \arrow[dl, "1_X"'] \arrow[dr, "s"] & & X \arrow[dl, "s"'] \arrow[dr, "1_X"] & \\ X & & Y & & X \end{tikzcd} = \begin{tikzcd} & X \arrow[dl, "Id_X"'] \arrow[dr, "Id_X"] & \\ X & & X \end{tikzcd}$$

Hence: $(s, Id_X) \circ (Id_X, s) = Id_X$

$(Id_X, s) \circ (s, Id_X) = Id_Y$

So Q sends elements of S to isomorphisms in $\mathcal{A}[S^{-1}]$.

If F is a functor s.t. $F(s)$: isomorphism in category $\mathcal{B}$ $\forall s \in S$, then define: $F^*(A) = F(A)$

$$F\Bigg(\underbrace{\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}}_{\substack{\text{representative} \\ \text{of a morphism} \\ X \to Y \text{ in } \mathcal{A}[S^{-1}]}}\Bigg) = Ff \circ F(s)^{-1} : F(X) \longrightarrow F(Y)$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{A}[S^{-1}] \arrow[d, dashed, "\exists ! F^*"] \\ & \mathcal{B} \end{tikzcd}$$

The construction of the localization category $\mathcal{A}[S^{-1}]$ can be done with right roofs: one obtains an equivalent category, since both constructions satisfy the same universal property.

Then F^* is a well defined functor, satisfies $F^* \circ Q = F$ and is unique, up to isomorphism of functors with this property.

Part 4: Consider the category $Ch(\mathcal{A})$ of chain complexes over a $\mathbb{K}$-linear abelian category $\mathcal{A}$ and the category $K(\mathcal{A})$: the homotopy category of $Ch(\mathcal{A})$.

There is a natural $\mathbb{K}$-linear functor $\pi: Ch(\mathcal{A}) \to K(\mathcal{A})$ where $\pi(A^\cdot, d_A^\cdot) = (A^\cdot, d_A^\cdot)$ and $\pi(f^\cdot) = [f^\cdot]$ or f: the homotopy class of $f^\cdot$. We consider the class Q_{is} of quasi-isomorphisms in either category $Ch(\mathcal{A})$ and $K(\mathcal{A})$.

Using the diagram (**) it is not difficult to see that the class of quasi-isomorphisms is localizing in $K(\mathcal{A})$.

Remark: The class Q_{is} of quasi-isomorphisms is not localizing in $Ch(\mathcal{A})$:

Let $\mathcal{A} = Ab$, the category of abelian groups.

Then consider the non-split short exact sequence (p: prime)

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

so that the have a non zero element of $Ext^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$.

Consider the diagram in the category of complexes where we consider $\mathbb{Z}/p\mathbb{Z}$ as stalk complexes concentrated in degree 0.

$$\begin{tikzcd} L \arrow[r, dashed, "\exists s^\cdot"] \arrow[d, dashed, "\exists h^\cdot"'] & \mathbb{Z}/p\mathbb{Z}[0] \arrow[d, "f^\cdot"] \\ \mathbb{Z}/p\mathbb{Z}[1] \arrow[r, "g^\cdot[1]"'] & I^\cdot[1] \end{tikzcd}$$

$$\mathbb{Z}/p\mathbb{Z} \rightarrowtail I^0 \longrightarrow I^1 \longrightarrow 0$$

is an injective resolution of $\mathbb{Z}/p\mathbb{Z}$, so

$$\begin{tikzcd} & & 0 \arrow[d] \\ g^\cdot : \mathbb{Z}/p\mathbb{Z} \arrow[r] & I^\cdot = & I^0 \arrow[d] \\ & & I^1 \arrow[d] \\ & & 0 \end{tikzcd}$$

is a quasi isomorphism hence $g^\cdot[1]$ is also an quasi-isomorphism.

If the diagram can be completed to a commutative square where $s^\cdot$ quasi-isomorphism in $Ch(\mathcal{A})$, then we have: $f^\cdot \circ s^\cdot = g^\cdot[1] \circ h^\cdot$. This however leads to a contradiction, because $f^\cdot \circ s^\cdot \neq 0$ (it is non zero in the derived category) and this can happen only in degree 0; and on the other hand $g^\cdot[1] \circ h^\cdot$ can be non-zero only in degree 1. Hence property 3 in the definition of a localizing class does not hold. ∎

We define the derived category of $\mathcal{A}$ by:

$$D(\mathcal{A}) = K(\mathcal{A})[Q_{is}^{-1}]$$

Remark: As noted before we may define the localization $Ch(\mathcal{A})[Q_{is}^{-1}] := D'(\mathcal{A})$, although the description of $D'(\mathcal{A})$ is much more difficult. However one can show that the categories $D(\mathcal{A})$ and $D'(\mathcal{A})$ are equivalent:

Consider the corresponding localization functors $Q: K(\mathcal{A}) \longrightarrow D(\mathcal{A})$ and $Q': Ch(\mathcal{A}) \longrightarrow D'(\mathcal{A})$

$$\begin{tikzcd} Ch(\mathcal{A}) \arrow[r, "Q'"] \arrow[d, "\pi"'] & D'(\mathcal{A}) \arrow[d, dashed, "\exists!\, \pi^*"] \\ K(\mathcal{A}) \arrow[r, "Q"] & D(\mathcal{A}) \end{tikzcd}$$

Clearly the composition $Q \circ \pi$ sends quasi-isomorphisms in $Ch(\mathcal{A})$ to isomorphisms in $D(\mathcal{A})$. Hence by the localization property of $D'(\mathcal{A})$, there exists a unique, up to isomorphism, functor $\pi^*: D'(\mathcal{A}) \longrightarrow D(\mathcal{A})$ such that $\pi^* \circ Q' = Q \circ \pi$. This functor π^* is an equivalence. One can argue as follows:

Let F be a functor which sends quasi-isomorphisms to isomorphisms in $\mathcal{B}$.

$$\begin{tikzcd} Ch(\mathcal{A}) \arrow[r, "Q \circ \pi"] \arrow[dr, "F"'] & D(\mathcal{A}) \arrow[d, dashed, "\exists!\, F^*"] \\ & \mathcal{B} \end{tikzcd}$$

Since homotopic maps in $Ch(\mathcal{A})$ induce the same maps in cohomology, it follows that F factors uniquely via π, through $K(\mathcal{A})$: there exists a unique functor $F': K(\mathcal{A}) \longrightarrow \mathcal{B}$ such that $F' \circ \pi = F$ and F' sends quasi-isomorphism to isomorphism in $\mathcal{B}$.

$$\begin{tikzcd}
Ch(\mathcal{A}) \arrow[r, "\pi"] \arrow[d, "F"'] & K(\mathcal{A}) \arrow[dl, dashed, "\exists!\, F'"'] \arrow[d, "Q"] \\
\mathcal{B} & D(\mathcal{A}) \arrow[l, "\exists!\, F^*"]
\end{tikzcd}$$

By the localization property of $D(\mathcal{A})$ there exists a unique functor $F^*: D(\mathcal{A}) \longrightarrow \mathcal{B}$ such that $F^* \circ Q = F'$. Then: $F^* \circ Q \circ \pi = F' \circ \pi = F$.

$$\begin{tikzcd}
Ch(\mathcal{A}) \arrow[rr, "Q\circ\pi"] \arrow[dr, "F"'] & & D(\mathcal{A}) \arrow[dl, dashed, "\exists!\, F^*"] \\
& \mathcal{B} &
\end{tikzcd}$$

This means that $Q \circ \pi$ is the localization functor for the localization $Ch(\mathcal{A}) \longrightarrow D'(\mathcal{A})$.

Hence there exists a unique equivalence $D: D'(\mathcal{A}) \xrightarrow{\approx} D(\mathcal{A})$, which can easily be seen to be π^*.

$$\begin{tikzcd}
& D'(\mathcal{A}) \arrow[d, "\exists!\ \wr\ D"] \\
Ch(\mathcal{A}) \arrow[ur, "Q'"] \arrow[r, "Q\circ\pi"'] & D(\mathcal{A})
\end{tikzcd} \quad \square$$

From now on we denote by $Q: K(\mathcal{A}) \longrightarrow D(\mathcal{A})$ the localization functor which is a $\mathbb{K}$-linear functor, sends quasi-isomorphisms to isomorphisms, and is universal with this property.

Note: $Ob(D(\mathcal{A}))$: chain complexes $(A^\cdot, d_A^\cdot)$

Morphisms in $D(\mathcal{A})$ from $(A^\cdot, d_A^\cdot)$ to $(B^\cdot, d_B^\cdot)$ equivalence of roofs

$$\begin{tikzcd} & X^\cdot \arrow[dl, "s^\cdot"'] \arrow[dr, "f^\cdot"] & \\ A^\cdot & & B^\cdot \end{tikzcd}$$

where $s^\cdot: X^\cdot \to A^\cdot$ quasi-isomorphism in $K(\mathcal{A})$ and $f^\cdot: X^\cdot \to B^\cdot$ homotopy class in $K(\mathcal{A})$ of a chain map in $Ch(\mathcal{A})$.

The identity morphism is the roof

$$\begin{tikzcd} & A^\cdot \arrow[dl, "Id_A"'] \arrow[dr, "Id_A"] & \\ A^\cdot & & A^\cdot \end{tikzcd}$$

It follows easily that the morphism $(s^\cdot, f^\cdot): (A^\cdot, d_A^\cdot) \to (B^\cdot, d_B^\cdot)$ represented by the roof

$$\begin{tikzcd} & X^\cdot \arrow[dl, "s^\cdot"'] \arrow[dr, "f^\cdot"] & \\ A^\cdot & & B^\cdot \end{tikzcd}$$

is the zero morphism $\Leftrightarrow$ $\exists$ quasi-isomorphism $t^\cdot: Y^\cdot \to X^\cdot$ such that $f^\cdot t^\cdot = 0$

In particular the complex $(A^\cdot, d_A^\cdot)$ is the zero object in $D(\mathcal{A}) \Leftrightarrow A^\cdot$: acyclic: $H^n(\mathcal{A}) = 0$, $\forall n \in \mathbb{Z}$. $\square$

The Derived Categories $D^-(\mathcal{A})$, $D^+(\mathcal{A})$, $D^b(\mathcal{A})$

Instead of working with all complexes, we may work with bounded above, bounded below, or bounded complexes, and then applying localization theory we have the derived categories

$D^-(\mathcal{A})$: the derived category of bounded above complexes
$D^+(\mathcal{A})$: the derived category of bounded below complexes
$D^b(\mathcal{A})$: the derived category of bounded complexes.
and it can be proven that the natural functors

$$D^-(\mathcal{A}) \hookrightarrow D(\mathcal{A})$$

$$D^+(\mathcal{A}) \hookrightarrow D(\mathcal{A}) \quad \text{are fully faithful.}$$

$$D^b(\mathcal{A}) \hookrightarrow D(\mathcal{A})$$

Using the localization functor $Q: K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$ where $* = -, +, b,$ nothing, we can transfer most of the essential structure of $K^*(\mathcal{A})$ to $D^*(\mathcal{A})$. In particular we have shift functors $[1]: D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$ which is an equivalence with quasi-inverse $[-1]: D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$. In addition:

In $K^*(\mathcal{A})$ a diagram $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A^\bullet[1]$ is called a triangle $\Leftrightarrow$ it is isomorphic to a diagram of the form $X^\bullet \xrightarrow{a^\bullet} Y^\bullet \xrightarrow{\beta^\bullet} \mathrm{Cone}(a^\bullet) \xrightarrow{\gamma^\bullet} X^\bullet[1]$ arising from the cone construction $\circledast$: there are isomorphisms $\kappa^\bullet, \lambda^\bullet, \mu^\bullet$ and a commutative in $K^*(\mathcal{A})$ diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "f^\bullet"] \arrow[d, "\kappa^\bullet"', "\wr"] & B^\bullet \arrow[r, "g^\bullet"] \arrow[d, "\lambda^\bullet"', "\wr"] & C^\bullet \arrow[r, "h^\bullet"] \arrow[d, "\mu^\bullet"', "\wr"] & A^\bullet[1] \arrow[d, "\wr"', "\kappa^\bullet[1]"] \\
X^\bullet \arrow[r, "\alpha^\bullet"] & Y^\bullet \arrow[r, "\beta^\bullet"] & \mathrm{Cone}(\alpha^\bullet) \arrow[r, "\gamma^\bullet"] & X^\bullet[1]
\end{tikzcd}$$

The triangles are substitutes of short exact sequences in $K^*(\mathcal{A})$, and the class of triangles, together with the shift functor produces a triple $(K^*(\mathcal{A}), [1], \Delta)$ which satisfies the axioms of a triangulated category.

In the derived category we can transfer this structure via the functor $Q: K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$

A diagram $A^\bullet \xrightarrow{\bar{f}^\bullet} B^\bullet \xrightarrow{\bar{g}^\bullet} C^\bullet \xrightarrow{\bar{h}^\bullet} A^\bullet[1]$ in $D(\mathcal{A})$ is called a triangle if it is isomorphic to a diagram of the form $Q(X^\bullet) \longrightarrow Q(Y^\bullet) \longrightarrow Q(Z^\bullet) \longrightarrow Q(X^\bullet)[1]$ arising from a triangle $X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$ in $K(\mathcal{A})$

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & A^\bullet[1] \arrow[d] \\
Q(X^\bullet) \arrow[r] & Q(Y^\bullet) \arrow[r] & Q(Z^\bullet) \arrow[r] & Q(X^\bullet)[1]
\end{tikzcd}$$

and then the triple, $(D^*(\mathcal{A}), [1], \widetilde{\Delta})$ where $\widetilde{\Delta}$ is the class of triangles in $\Delta^*(A)$ defined above, is a triangulated category.

The triangles in $D^*(\mathcal{A})$ provide a substitute of a short exact sequence in $D^*(\mathcal{A})$, which is not abelian, in general.

Remark: Any triangle $A^\cdot \xrightarrow{\bar{f}^\cdot} B^\cdot \xrightarrow{\bar{g}^\cdot} C^\cdot \xrightarrow{\bar{h}^\cdot} A^\cdot[1]$ in $K^*(\mathcal{A})$ or $\Delta(\mathcal{A})$ induces long exact sequences in cohomology:

$$\cdots \longrightarrow H^{n-1}(A^\cdot) \xrightarrow{H^{n-1}(\bar{f}^\cdot)} H^{n-1}(B^\cdot) \xrightarrow{H^{n-1}(\bar{g}^\cdot)} H^{n-1}(C^\cdot) \xrightarrow{H^{n-1}(\bar{h}^\cdot)} H^n(A^\cdot) \xrightarrow{H^n(\bar{f}^\cdot)} H^n(B^\cdot) \xrightarrow{H^n(\bar{g}^\cdot)} H^n(C^\cdot) \to \cdots$$

where the connecting morphisms arise from $C^\cdot \cong \mathrm{Cone}(\bar{f}^\cdot) \to A^\cdot[1]$

Finally, there is a fully faithful functor

$$[0]: \mathcal{A} \longrightarrow \mathcal{D}(\mathcal{A}),\quad \begin{tikzcd} A \arrow[d, "f"'] \arrow[r, mapsto] & \cdots \arrow[r] & 0 \arrow[r] & \underset{\text{degree } 0}{A} \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \cdots \\ B \arrow[r, mapsto] & \cdots \arrow[r] & 0 \arrow[r] & \underset{\text{degree } 0}{B} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

$f^\cdot$

which is a ∂-functor in the sense that for any s.e.s.

(E) $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ in $\mathcal{A}$

we have a triangle $A[0] \xrightarrow{f^\cdot} B[0] \xrightarrow{g^\cdot} C[0] \xrightarrow{h^\cdot} A[1]$ in $\mathcal{D}(\mathcal{A})$ where $h^\cdot$ represents the element of $\mathrm{Ext}(C,A)$ which corresponds to the s.e.s. (E).

Part 5: Having defined the (homotopy category) and the derived category of a $\mathbb{K}$-linear abelian category $\mathcal{A}$, we are now concerned with the question of what the appropriate functors are between derived categories of $\mathbb{K}$-linear abelian categories.

The answer is given by the notion of triangulated functors. Let $\mathcal{A}$ and $\mathcal{B}$ $\mathbb{K}$-linear abelian categories with corresponding derived categories $D^*(\mathcal{A})$ and $D^*(\mathcal{B})$, where $* = -, +, b$ nothing with corresponding localization functors $Q_{\mathcal{A}}: K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$ and $Q_{\mathcal{B}}: K^*(\mathcal{B}) \to D^*(\mathcal{B})$.

A $\mathbb{K}$-linear functor $F: K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ is called
or $F: D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$
triangulated if there is a natural isomorphism of functors $\partial: F \circ [1] \xrightarrow{\sim} [1] \circ F$ such that for any triangle $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A^\bullet[1]$ in $K^*(\mathcal{A})$ or $D^*(\mathcal{A})$ the diagram $F(A^\bullet) \xrightarrow{F(f^\bullet)} F(B^\bullet) \xrightarrow{F(g^\bullet)} F(C^\bullet) \xrightarrow{\partial_{A^\bullet} \circ F(h^\bullet)} F(A^\bullet)[1]$ is a triangle in $K^*(\mathcal{B})$ or $D^*(\mathcal{B})$ respectively, where

$$\begin{tikzcd} & F(A^\bullet[1]) \arrow[dr, "\partial_{A^\bullet}", "\sim"'] & \\ F(C^\bullet) \arrow[ur, "F(h^\bullet)"] \arrow[rr, "\partial_{A^\bullet} \circ F(h^\bullet)"'] & & F(A^\bullet)[1] \end{tikzcd}$$

In short the functors F commutes with the shift functors on $K^*(\mathcal{A})$ and $K^*(\mathcal{B})$, or on $D^*(\mathcal{A})$ and $D^*(\mathcal{B})$, and sends triangles in $K^*(\mathcal{A})$, respectively in $D^*(\mathcal{A})$ to triangles in $K^*(\mathcal{B})$ respectively in $D^*(\mathcal{B})$.

A natural source of examples of triangulated functors on $K^*(\mathcal{A})$ is given by $\mathbb{K}$-linear functors $F:\mathcal{A}\to\mathcal{B}$ between $\mathbb{K}$-linear abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then defining (use the same symbol to denote F):

$F: Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{B})$ by

$$\cdots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \longrightarrow \cdots \quad (A^\cdot, d_A^\cdot)$$

$$\cdots \longrightarrow F(A^{n-1}) \xrightarrow{F(d_A^{n-1})} F(A^n) \xrightarrow{F(d_A^n)} F(A^{n+1}) \longrightarrow \cdots \quad (F(A^\cdot), d^\cdot_{F(A)})$$

and if $f^\cdot:(A^\cdot, d_A^\cdot) \longrightarrow (B^\cdot, d_B^\cdot)$ is a chain map, then

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

is sending to

$$\begin{tikzcd}
\cdots \arrow[r] & F(A^{n-1}) \arrow[r, "F(d_A^{n-1})"] \arrow[d, "F(f^{n-1})"'] & F(A^n) \arrow[r, "F(d_A^n)"] \arrow[d, "F(f^n)"'] & F(A^{n+1}) \arrow[r] \arrow[d, "F(f^{n+1})"'] & \cdots \\
\cdots \arrow[r] & F(B^{n-1}) \arrow[r, "F(d_B^{n-1})"] & F(B^n) \arrow[r, "F(d_B^n)"] & F(B^{n+1}) \arrow[r] & \cdots
\end{tikzcd}$$

It is easy to see that this defines a functor $F: Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{B})$ which commutes with the shift functors, and preserves the homotopy relation, so it defines a functor $F: K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ which is triangulated.

If F is exact, then $F: Ch(\mathcal{A}) \to Ch(\mathcal{B})$ commutes with cohomology: $H^n(F(A^\cdot)) \simeq F(H^n(A^\cdot))$, $\forall n \in \mathbb{Z}$ hence F sends quasi-isomorphisms in $Ch(\mathcal{A})$ or $K^*(\mathcal{A})$ to quasi-isomorphisms in $Ch(\mathcal{B})$ or $K^*(\mathcal{B})$.

Hence by the universal property of the localizations

$Q_A: K^*(A) \longrightarrow D(\mathcal{A})$

$Q_B: K^*(B) \longrightarrow D(\mathcal{B})$

$$
\begin{tikzcd}
K^*(\mathcal{A}) \arrow[r, "Q_A"] \arrow[d, "F"'] & D(\mathcal{A}) \arrow[d, "\exists! \, \bar{F}"] \\
K^*(\mathcal{B}) \arrow[r, "Q_B"] & D(\mathcal{B})
\end{tikzcd}
$$

there exists a unique, up to isomorphism, functor $\bar{F}: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ such that: $\bar{F} \circ Q_A = Q_B \circ F$.

Hence, any exact functor $F: \mathcal{A} \to \mathcal{B}$, induces a functor $\bar{F}: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ which is, as can easily be seen, triangulated.

However, if F is not exact, the above analysis fails and thus emerges the following problem:

How can we approximate a (non-exact) $\mathbb{K}$-linear functor $F: \mathcal{A} \to \mathcal{B}$, or $F: Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{B})$, or $F: K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ by a triangulated functor $\tilde{F}: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$?

In order to make this precise we need the following definition:

We begin with a triangulated functor $F: K(\mathcal{A}) \to K(\mathcal{B})$, for instance F can be induced by any $\mathbb{K}$-linear functor $F: \mathcal{A} \to \mathcal{B}$ as described before.

Let $F: K(\mathcal{A}) \to K(\mathcal{B})$ be a triangulated functor. A right derived functor of F is a triangulated functor $RF: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ together with a natural morphism of functors $\varepsilon: Q_B \circ F \to RF \circ Q_A$ which is universal:

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "Q_A"] \arrow[d, "F"'] & D(\mathcal{A}) \arrow[d, "RF"] \arrow[d, shift right=6, "G"'] \\ K(\mathcal{B}) \arrow[r, "Q_B"'] & D(\mathcal{B}) \end{tikzcd} \qquad \xleftarrow{\exists! \tilde{\varphi}}$$

for any triangulated functor $G: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ equipped with a natural morphism $\varphi: Q_B \circ F \longrightarrow G \circ Q_A$, there exists a unique natural morphism $\tilde{\varphi}: RF \longrightarrow G$ such that $\tilde{\varphi} \circ Q_A \circ \varepsilon = \varphi$.

$$\begin{tikzcd}
Q_B \circ F \arrow[r, "\varepsilon"] \arrow[dr, "\varphi"'] & RF \circ Q_A \arrow[d, "\widetilde{\varphi} \circ Q_A"] \\
 & C \circ Q_A
\end{tikzcd}$$

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r, "Q_A"] \arrow[d, "F"'] & D(\mathcal{A}) \arrow[d, "G"', bend right] \arrow[d, "LF", bend left] \arrow[d, phantom, "\exists ! \widetilde{\varphi}"] \\
K(\mathcal{B}) \arrow[r, "Q_B"] & D(\mathcal{B})
\end{tikzcd}$$

$$\begin{tikzcd}
 & G \circ Q_A \arrow[dl, "\widetilde{\varphi}\, Q_A"'] \arrow[d, "\psi"] \\
LF \circ Q_A \arrow[r, "\eta"] & Q_B \circ F
\end{tikzcd}$$

Dually: a left derived functor of F is a triangulated functor $LF: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ together with a natural morphism $\eta: LF \circ Q_A \longrightarrow Q_B \circ F$ which is universal: for any triangulated functor $G: D(\mathcal{A}) \to D(\mathcal{B})$ equipped with a natural morphism $\psi: G \circ Q_A \longrightarrow Q_B \circ F$ there exists a unique natural morphism $\widetilde{\varphi}: G \to LF$ such that $\psi = \eta \circ \widetilde{\varphi} \circ Q_A$.

In other words:

- RF is the best triangulated approximation functor $D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ of $F: K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ from the right.
- LF is the best triangulated approximation functor $D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ of $F: K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ from the left.

The definitions of $RF, LF: D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ make sense if we restrict to $D^-(\mathcal{A})$, $D^+(\mathcal{A})$, $D^b(\mathcal{A})$ giving triangulated functors $R^{\pm}_{b}F: D^{\pm}_{b}(\mathcal{A}) \longrightarrow D(\mathcal{B})$, $L^{\pm}_{b}F: D^{\pm}_{b}(\mathcal{A}) \longrightarrow D(\mathcal{B})$

Main Problem: Do the left or right derived functors exist? If so, how to compute them?

We will show that, in certain cases, left or right derived functors exist and can be computed via certain "resolutions" of complexes.

To this end we need the notion of left or right adapted classes of objects to $\mathbb{K}$-linear functors defined on $\mathbb{K}$-linear abelian categories

Adapted Classes and Resolutions

We fix a $\mathbb{K}$-linear functor $F: \mathcal{A} \to \mathcal{B}$ between $\mathbb{K}$-linear abelian categories $\mathcal{A}$ and $\mathcal{B}$.

① If F is left exact, then a right adapted class for F is a full subcategory $\mathcal{L} \subseteq \mathcal{A}$ satisfying the following conditions:

(α) For any object $A \in \mathrm{Ob}(\mathcal{A})$, there is a monomorphism $A \rightarrowtail X$, where $X \in \mathcal{L}$.

(β) If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence in $\mathcal{A}$ and $X', X \in \mathcal{L}$, then $X'' \in \mathcal{L}$.
($\mathcal{L}$ is closed under cokernels of monomorphisms)

(γ) For any short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, where $X' \in \mathcal{L}$, the sequence

$$0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0 \quad \text{is exact.}$$

(F is exact on s.e.s. with left first term in $\mathcal{L}$).

② If F is right exact, then a left adapted class for F is a full subcategory $\mathcal{L} \subseteq \mathcal{A}$ satisfying the following conditions:

(a) For any object $A \in Ob(\mathcal{A})$, there is an epimorphism

$$X \twoheadrightarrow A, \text{ where } X \in \mathcal{L}$$

(β) If $0 \to X' \longrightarrow X \longrightarrow X'' \to 0$ is a short exact sequence in $\mathcal{A}$ and $X, X'' \in \mathcal{L}$, then $X' \in \mathcal{L}$.

($\mathcal{L}$ is closed under kernels of epimorphisms)

(γ) For any short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ where $X'' \in \mathcal{L}$, the sequence

$$0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0 \quad \text{is exact.}$$

(F is exact on s.e.s. with right first term in $\mathcal{L}$).

Remark: ① If $\mathcal{A}$ has enough injectives: for any object $A \in \mathcal{A}$ there is a monomorphism $A \rightarrowtail I$, where I is the injective, then the full subcategory $\mathrm{Inj}(\mathcal{A})$ is a right adapted class for any left exact functor $F: \mathcal{A} \to \mathcal{B}$.

② If $\mathcal{A}$ has enough projectives: for any object $A \in \mathcal{A}$, there is an epimorphism $P \twoheadrightarrow A$, where P is projective, then the full subcategory $\mathrm{Proj}(\mathcal{A})$ is a left adapted class for any right exact functor $F: \mathcal{A} \to \mathcal{B}$. ■

In order to state the first main result on the existence of left or right derived functors, we need two lemmas:

Lemma ①: Let $\mathcal{A}$ be a ($\mathbb{K}$-linear) abelian category.

(a) If $\mathcal{L} \subseteq \mathcal{A}$ is a right adapted class, then for any complex $X^\bullet \in Ch^+(\mathcal{A})$, there is a quasi-isomorphism

$$q: X^\bullet \longrightarrow Q^\bullet, \text{ where } Q^\bullet \in Ch^+(\mathcal{L})$$

(b) If $\mathcal{L} \subseteq \mathcal{A}$ is a left adapted class, then for any complex $X^\bullet \in Ch^-(\mathcal{A})$, there is a quasi-isomorphism

$$q: Q^\bullet \longrightarrow X^\bullet, \text{ where } Q^\bullet \in Ch^-(\mathcal{L}).$$ ■

Definition: ⊙ A quasi-isomorphism $q^\cdot: X^\cdot \to Q^\cdot$, where $X^\cdot \in Ch^+(\mathcal{A})$, $Q^\cdot \in Ch^+(\mathcal{L})$ is called a right $\mathcal{L}$-resolution of $X^\cdot$.

⊙ A quasi-isomorphism $q^\cdot: Q^\cdot \to X^\cdot$ where $X^\cdot \in Ch^-(\mathcal{A})$, $Q^\cdot \in Ch^-(\mathcal{L})$ is called a left $\mathcal{L}$-resolution of $X^\cdot$.

Lemma ②: Let $F: \mathcal{A} \to \mathcal{B}$ be a ($\mathbb{K}$-linear) functor between ($\mathbb{K}$-linear) abelian categories.

(a) Let F be left exact, and $\mathcal{L}$: right adapted class for F.

⊙ $X^\cdot \in Ch^+(\mathcal{L})$ is acyclic $\Rightarrow F(X^\cdot)$ is acyclic

⊙ $f^\cdot: X^\cdot \to Y^\cdot$ is quasi-isomorphism in $Ch^+(\mathcal{L})$, then $F(f^\cdot): F(X^\cdot) \to F(Y^\cdot)$ is quasi-isomorphism

(any left exact functor preserves acyclicity and quasi-isomorphisms between bounded below complexes of $\mathcal{L}$.)

(b) Let F be right exact, and $\mathcal{L}$: left adapted class for F.

⊙ $X^\cdot \in Ch^-(\mathcal{L})$ is acyclic $\Rightarrow F(X^\cdot)$ is acyclic.

⊙ $f^\cdot: X^\cdot \to Y^\cdot$ is quasi-isomorphism in $Ch^-(\mathcal{L})$, then $F(f^\cdot): F(X^\cdot) \to F(Y^\cdot)$ is quasi-isomorphism.

Proof: We only prove part (a) since the proof of (b) is dual.

Let $X^\bullet$ be acyclic. Then $X^\bullet$ is of the form:

$$0 \to X^{-k} \xrightarrow{d_X^{-k}} X^{-k+1} \xrightarrow{d_X^{-k+1}} X^{-k+2} \to \cdots \to X^{-1} \xrightarrow{d_X^{-1}} X^{0} \xrightarrow{d_X^{0}} X^{1} \to \cdots \to X^{n} \xrightarrow{d_X^{n}} X^{n+1} \to \cdots$$

Since $X^\bullet$ is acyclic, we have in any degree n:

$\mathrm{Im}(d_X^n) = \mathrm{Ker}(d_X^{n+1})$, $\forall n \geq -k$, and exactness in degree $-k$ means that d_X^{-k} is a monomorphism. It follows that we have s.e.s.:

$$0 \to X^{-k} \to X^{-k+1} \to \tilde{X}^{-k+2} \to 0, \text{ where } \tilde{X}^{-k+2} = \mathrm{Ker}(d_X^{-k+2})$$

$$0 \to \tilde{X}^{-k+2} \to X^{-k+2} \to \tilde{X}^{-k+3} \to 0, \text{ where } \tilde{X}^{-k+3} = \mathrm{Ker}(d_X^{-k+3})$$

$\vdots$

Since all components $X^n \in \mathcal{L}$ and $\mathcal{L}$ is right adapted, we have $\tilde{X}^{-k+2} \in \mathcal{L}$. Using this we have $\tilde{X}^{-k+3} \in \mathcal{L}$, and so on.

Continuing in this way the acyclic complex $X^\bullet$ is of the form

$$\begin{tikzcd}
\cdots 0 \arrow[r] & X^{-k} \arrow[r] & X^{-k+1} \arrow[rr] \arrow[dr, two heads] & & X^{-k+2} \arrow[rr] \arrow[dr, two heads] & & X^{-k+3} \arrow[rr] \arrow[dr, two heads] & & X^{-k+4} \arrow[r] & \cdots \\
& & & \tilde{X}^{-k+2} \arrow[ur, hook] & & \tilde{X}^{-k+3} \arrow[ur, hook] & & \tilde{X}^{-k+4} \arrow[ur, hook] & &
\end{tikzcd}$$

and the short exact sequences

$0 \to X^{-k} \to X^{-k+1} \to \tilde{X}^{-k+2} \to 0$, $0 \to \tilde{X}^{-k+2} \to X^{-k+2} \to \tilde{X}^{-k+3} \to 0$,

$0 \to \tilde{X}^{-k+3} \to X^{-k+3} \to \tilde{X}^{-k+4} \to 0$ ... have all their terms in $\mathcal{R}$.

Applying the left exact functor F to these s.e.s. and using that $\mathcal{R}$ is right adapted we have s.e.s.

$$0 \to F(X^{-k}) \to F(X^{-k+1}) \to F(\tilde{X}^{-k+2}) \to 0,$$

$$0 \to F(\tilde{X}^{-k+2}) \to F(X^{-k+2}) \to F(\tilde{X}^{-k+3}) \to 0,$$

$$0 \to F(\tilde{X}^{-k+3}) \to F(X^{-k+3}) \to F(\tilde{X}^{-k+4}) \to 0, \dots$$

and therefore an acyclic complex:

$$\dots \to F(X^{-k}) \to F(X^{-k+1}) \to F(X^{-k+2}) \to \dots \to F(X^{\cdot}) \to F(X^{\cdot}) \to F(X^{\cdot}) \to \dots$$

$\Rightarrow F(X^{\cdot})$: acyclic.

⊙ Let $f^{\cdot}: X^{\cdot} \to Y^{\cdot}$ be a quasi-isomorphism, where $X^{\cdot}, Y^{\cdot} \in Ch^{+}(\mathcal{R})$.
From the diagram (**) in Cone Construction, we have a s.e.s. of complexes:

$$0 \to Y^{\cdot} \xrightarrow{g^{\cdot}} \mathrm{Cone}(f^{\cdot}) \xrightarrow{h^{\cdot}} X[1] \to 0 \quad (\ast\ast\ast)$$

Note that the chain maps $g^{\cdot}, h^{\cdot}$ induce degree wise s.e.s. in $\mathcal{A}$

$$0 \to Y^{n} \xrightarrow{\binom{0}{1_{Y^n}}} X^{n+1} \oplus Y^{n} \xrightarrow{\left(1_{X^{n+1}}\ 0\right)} X^{n+1} \to 0 \quad \forall n \in \mathbb{Z}.$$

hence, applying F to $(***)$, we have a s.e.s. of chain complexes in $Ch^+(\mathcal{B})$:

$$0 \longrightarrow F(Y^\cdot) \xrightarrow{F(g^\cdot)} F\,Cone(f^\cdot) \xrightarrow{F(h^\cdot)} F(X^\cdot[1]) \longrightarrow 0 \quad (+)$$

The s.e.s. $(***)$ induces a long exact sequence:

$$\begin{tikzcd}
H^{n-2}(X^\cdot[1]) \arrow[r, "H^{n-1}(f^\cdot)"] \arrow[d, phantom, "\cong" rotate=90] & H^{n-1}(Y^\cdot) \arrow[r] & H^{n-1}(Cone(f^\cdot)) \arrow[r] & H^{n-1}(X^\cdot[1]) \arrow[r, "H^n(f^\cdot)"] \arrow[d, phantom, "\cong" rotate=90] & H^n(Y^\cdot) \arrow[r] & H^n(Cone(f^\cdot)) \arrow[r] & \cdots \\
H^{n-1}(X^\cdot) & & & H^n(X^\cdot) & & &
\end{tikzcd}$$

Since $f^\cdot$: quasi-isomorphism, we have $H^n(f^\cdot)$: isomorphism, hence $H^n(Cone(f^\cdot)) = 0$, $\forall n \in \mathbb{Z} \Rightarrow Cone(f^\cdot)$: acyclic.

Since the components of $Cone(f^\cdot)$ are objects in $\mathcal{L}$, and clearly $Cone(f^\cdot) \in Ch^+(\mathcal{L})$, by the first part, the complex $F(Cone(f^\cdot))$ is acyclic.

Consider the long exact sequence arising from the s.e.s. $(+)$

$$\begin{tikzcd}
\cdots H^{n-2}(F(X^\cdot[1])) \arrow[r] \arrow[d, "\wr"] & H^{n-1}(F(Y^\cdot)) \arrow[r] & H^{n-1}(F(Cone(f^\cdot))) = 0 \arrow[r] & H^{n-1}(F(X^\cdot[1])) \arrow[d] \arrow[dl, "\sim"'] \\
H^{n-2}(F(X^\cdot)[1]) \arrow[d, "\wr"] & & H^n(F(X^\cdot)) \arrow[r, "H^n(F(f^\cdot))"] & H^n(F(Y^\cdot)) \arrow[d] \\
H^{n-1}(F(X^\cdot)) \arrow[uur, dashed, "H^{n-1}(F(f^\cdot))"'] & & & H^n(F(Cone(f^\cdot))) = 0 \arrow[d] \\
& & & \vdots
\end{tikzcd}$$

Since $F(\mathrm{Cone}(f^\cdot))$ is acyclic $\Rightarrow H^n(F(\mathrm{Cone}(f^\cdot)))=0$, $\forall n\in\mathbb{Z}$, and therefore the above long exact sequence shows that $H^n(F(f^\cdot))$: isomorphism $\forall n\in\mathbb{Z} \Rightarrow F(f^\cdot)$: quasi-isomorphism. ■

Theorem: Let $F:\mathcal{A}\to\mathcal{B}$ be a $\mathbb{K}$-linear functor between $\mathbb{K}$-linear abelian categories

① Let F be left exact and assume that there exists a right adapted class for F in $\mathcal{A}$. Then there exists a right derived functor $R^+F: D^+(\mathcal{A})\longrightarrow D^+(\mathcal{B})$

② Let F be right exact and assume that there exists a left adapted class for F in $\mathcal{A}$. Then there exists a left derived functor $L^-F: D^-(\mathcal{A})\to D^-(\mathcal{B})$.

Proof: (Sketch) for ①, the sketch of the Proof of ② is dual.

- $\forall X^\cdot\in Ch^+(\mathcal{A})$, equivalently $\forall X^\cdot\in D^+(\mathcal{A})$ choose a quasi-isomorphism $q_{X^\cdot}: X^\cdot\to Q_X^\cdot$ where $Q_X^\cdot\in Ch^+(\mathcal{L})$ where $\mathcal{L}$ is a right adapted class for F, which exists by Lemma ①.
Define: $R^+F(X^\cdot)=F(Q_X^\cdot)\in Ob(Ch^+(\mathcal{B}))=ob(D^+(\mathcal{B}))$

• Let $f^\cdot : X^\cdot \to Y^\cdot$ be a morphism in $D^+(\mathcal{A})$, represented by a roof

$$\begin{tikzcd} & A^\cdot \arrow[dl, "s^\cdot"'] \arrow[dr, "f^\cdot"] & \\ X^\cdot & & Y^\cdot \end{tikzcd}$$

where $s^\cdot$: quasi-isomorphism.

Then one shows that this morphism is equal (in $D^+(\mathcal{A})$) to the equivalence class of a roof of the form

$$\begin{tikzcd} & \tilde{A}^\cdot \arrow[dl, "t^\cdot"'] \arrow[dr, "g^\cdot"] & \\ Q_X^\cdot & & Q_Y^\cdot \end{tikzcd}$$

where $t^\cdot$: quasiisomorphism and $\tilde{A}^\cdot \in Ch^+(\mathcal{L})$.

Define $(R^+F)(f^\cdot) =$

$$\begin{tikzcd} & F(\tilde{A}^\cdot) \arrow[dl, "F(t^\cdot)"'] \arrow[dr, "F(g^\cdot)"] & \\ F(Q_X^\cdot) & & F(Q_Y^\cdot) \end{tikzcd}$$

By Lemma ② this makes sense, and a lot of work is needed to check that this gives a well defined (triangulated) functor $R^+F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r, "Q_A"] \arrow[d, "F"'] \arrow[dr, bend left, "R^+F \circ Q_A"] \arrow[dr, bend right, "Q_{\mathcal{B}} \circ F"'] \arrow[dr, phantom, "\Uparrow \varepsilon"] & D^+(\mathcal{A}) \arrow[d, "R^+F"] \\ K^+(\mathcal{B}) \arrow[r, "Q_{\mathcal{B}}"'] & D^+(\mathcal{B}) \end{tikzcd}$$

• To define $\varepsilon : Q_{\mathcal{B}} \circ F \to R^+F \circ Q_A$ the natural transformation let $X^\cdot \in K^+(\mathcal{A})$ and define

$$\varepsilon_X : (Q_{\mathcal{B}} \circ F)(X^\cdot) = Q_{\mathcal{B}}(F(X^\cdot)) \longrightarrow Q_{\mathcal{B}}(F(Q_X^\cdot)) = R^+F(X^\cdot) = (R^+F)(Q_A(X^\cdot)) = (R^+F \circ Q_A)(X^\cdot)$$

Then one checks that ε has the required universal property.

Remark: The above Theorem is due to Verdier (1963-1967). After more than 20, Spaltenstein (1988) extended this to the unbounded derived categories of certain abelian categories (Grothendieck abelian categories) and later Böckstedt-Neeman (1993) extended Spaltenstein's result to cocomplete abelian categories with exact coproduct, complete abelian categories with exact products and enough projectives and injectives respectively.

The above results cover the categories of Sheaves over topological spaces, e.g. over a wide class of schemes (quasi-separated, quasi-compact) ■

Finally consider right derived functors of composition of functors:

Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ $\mathbb{K}$-linear left exact functors between $\mathbb{K}$-linear abelian categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and assume that the (suitably bounded versions of) the derived functors $R^+F, R^+G, R^+(G\circ F)$ exist. Then, using the universal property of the derived functors one can construct a natural morphism

$$R(G\circ F) \longrightarrow R(G)\circ R(F)$$

which in some cases is invertible:

Proposition: As before, consider $\mathbb{K}$-linear left exact functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

between $\mathbb{K}$-linear abelian categories, and assume that:

- there is a right adapted class $\mathcal{L} \subseteq \mathcal{A}$ for F
- there is a right adapted class $\mathcal{J} \subseteq \mathcal{B}$ for G
- $F(\mathcal{L}) \subseteq \mathcal{J}$

Then the natural morphism

$$R^+(G \circ F) \xrightarrow{\simeq} R^+(G) \circ R^+(F) : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{C}) \text{ is an isomorphism}$$

And there is a dual version for right exact functors. ∎

Note: The natural morphism $R^+(G \circ F) \to R^+(G) \circ R^+(F)$ is the derived version of Grothendieck's spectral sequence for the composition of left exact functors, the latter follows by considering the cohomology of the natural (iso)morphism

$$R^+(G \circ F) \longrightarrow R^+(G) \circ R^+(F).$$ ∎

Part 6: Finally we discuss an example of derived functors of a functor which is of two variables and is not induced by a functor between the underlying abelian categories, but it plays an important role in analyzing their structure.

As before, let $\mathcal{A}$ be a $\mathbb{K}$-linear abelian category. It is important to note that the category of complexes $Ch(\mathcal{A})$ is equipped with an "internal" Hom functor of 2 variables, extending the usual Hom-functor $\mathrm{Hom}: \mathcal{A}^{op}\times\mathcal{A}\to\mathbb{K}\text{-Mod}$. More precisely:

$$\mathrm{Hom}^{\bullet}: Ch^{-}(\mathcal{A})^{op}\times Ch^{+}(\mathcal{A})\longrightarrow Ch^{+}(\mathcal{A}) \text{ where}$$

$\forall A^{\bullet}\in Ch^{-}(\mathcal{A})$: $\mathrm{Hom}^{\bullet}(A^{\bullet},B^{\bullet})$ is the chain complex, where
$\forall B^{\bullet}\in Ch^{+}(\mathcal{A})$:
$$\mathrm{Hom}^{n}(A^{\bullet},B^{\bullet})=\bigoplus_{j-i=n}\mathrm{Hom}_{\mathcal{A}}(A^{i},B^{j})$$

and the differential is given by the formula: if $f\in\mathrm{Hom}(A^i,B^j)$ then $d_{\mathrm{Hom}(A,B)}$ is given on the components by

$$d(f)=d_{B^{\bullet}}\circ f+(-1)^{j-i+1}f\circ d_{A^{\bullet}} \quad \text{that is:}$$

$$\begin{tikzcd}
\mathrm{Hom}^{n}(A^{\bullet},B^{\bullet}) \arrow[r, "d^{n}_{\mathrm{Hom}^{\bullet}(A^{\bullet},B^{\bullet})}"] \arrow[d, equal] & \mathrm{Hom}^{n+1}(A^{\bullet},B^{\bullet}) \arrow[d, equal] \\
\bigoplus_{j-i=n}\mathrm{Hom}_{\mathcal{A}}(A^{i},B^{j}) & \bigoplus_{j-i=n+1}\mathrm{Hom}_{\mathcal{A}}(A^{i},B^{j})
\end{tikzcd}$$

$$f\in\mathrm{Hom}_{\mathcal{A}}(A^{i},B^{j}) \longmapsto d^{n}_{\mathrm{Hom}^{\bullet}}(f)\in\mathrm{Hom}_{\mathcal{A}}(A^{i},B^{j+1})$$

$$A^{i}\xrightarrow{f}B^{j} \qquad\qquad A^{i}\xrightarrow{d_B^{j}\circ f+(-1)^{j-i+1}f\circ d_A^{i}}B^{j+1}$$

$$\begin{tikzcd}
A^{i} \arrow[r, "f"] \arrow[d, "d_A^{i}"'] & B^{j} \arrow[d, "d_B^{j}"] \\
A^{i+1} \arrow[r, "f"] & B^{j+1}
\end{tikzcd}$$

Applying the machinery introduced before, suitably modified, it follows that there is induced a right derived functor of 2 variables

$$\mathrm{RHom}^\bullet : D^-(\mathcal{A})^{op} \times D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$

which is triangulated and is equipped with a natural morphism

$$\varepsilon : Q_{\mathcal{A}} \circ \mathrm{Hom}^\bullet \longrightarrow \mathrm{RHom}^\bullet \circ Q$$

satisfying the required universal property and $\mathrm{RHom}^\bullet$ is called the **derived** Hom functor.

$$\begin{tikzcd} K^-(\mathcal{A})^{op} \times K^+(\mathcal{A}) \arrow[r, "Q_{\mathcal{A}}^- \times Q_{\mathcal{A}}^+ = Q"] \arrow[d, "\mathrm{Hom}^\bullet"'] & D^-(\mathcal{A})^{op} \times D^+(\mathcal{A}) \arrow[d, "\mathrm{RHom}^\bullet"] \\ K^+(\mathcal{A}) \arrow[r, "Q_{\mathcal{A}}"] & D^+(\mathcal{A}) \end{tikzcd}$$

The following final result indicates an important connection between the complex $\mathrm{RHom}^\bullet(A^\bullet, B^\bullet)$ and homological properties of $\mathcal{A}$.

Proposition: The derived Hom functor $\mathrm{RHom}^\bullet$ exists, if $\mathcal{A}$ has either enough projectives or enough injectives. Moreover:

① $\forall A^\bullet \in D^-(\mathcal{A})$, $\forall B^\bullet \in D^+(\mathcal{A})$: $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet) \cong H^0\,\mathrm{RHom}^\bullet(A^\bullet, B^\bullet)$

② $\forall A, B \in \mathcal{A}$: $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[n]) \cong \mathrm{Ext}^n_{\mathcal{A}}(A, B) \cong H^n\,\mathrm{RHom}^\bullet(A^\bullet, B^\bullet)$
where we consider $A^\bullet, B^\bullet$ as (stalk) complexes concentrated in degree 0.

Epilogue: The theory developed can be applied to the following important situation.

Let X be a topological space. We consider the poset of open subset $\mathcal{O}$ of X as a category and then for any category $\mathcal{A}$, one forms the functor category

$$\text{Presheaves}(X) = [\mathcal{O}^{op}, \mathcal{A}] \quad (\text{functor category})$$

called the category of presheaves on X with values in $\mathcal{A}$.

We restrict to the case $\mathcal{A} = \text{Mod-}\mathbb{Z}$ or $\mathcal{A} = \text{Mod-}\mathbb{C}$. A sheaf on X with values in $\mathcal{A}$ is a presheaf $F: \mathcal{O}^{op} \to \mathcal{A}$ such that: for any open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of the open subset $U \in \mathcal{O}$, and for any elements $f_i \in F(U_i)$, $i \in I$, such that: $f_i|_{F(U_i \cap U_j)} = f_j|_{F(U_i \cap U_j)}$ there exists a unique $f \in F(U)$ such that $F(U) \to F(U_i)$, $i \in I$
$f \mapsto f_i$.

The full subcategory $Sh(X, \mathcal{A})$ of the presheaf category Presheaves(X), is defined to be consisting of all sheaves & it follows that is an abelian category (but not an abelian subcategory of Presheaves(X))

Grothendieck: $Sh(X, \mathcal{A})$ is a Grothendieck category (a cocomplete abelian category with a generator and exact filtered colimits) and has enough injectives.