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February 19, 2021

Marginalia about roots

These notes are an attempt to maintain a overview collection of facts about and relationships between some situations in which root systems and root data appear. They also serve to track some common identifications and choices. The references include some helpful lecture notes with more examples.

The author of these notes learned this material from courses taught by Zinovy Reichstein, Joel Kamnitzer, James Arthur, and Florian Herzig, as well as many student talks, and lecture notes by Ivan Loseu. These notes are simply collected marginalia for those references. Any errors introduced, especially of viewpoint, are the author's own. The author of these notes would be grateful for their communication to stefand@math.utoronto.ca.

Contents

1 Root systems

The following examples are taken mostly from [8] where they are stated without most of the calculations. Similar stated examples are in [7].

1.1 Root space decomposition

This introductory section is as brief as possible; for a complete introduction see [5] or [7].

Definition 1. If $\mathfrak{g}$ is a complex Lie algebra, then we obtain a bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, the *Killing form* by $B(X, Y) = \mathrm{tr}(\mathrm{ad}\,(X)\mathrm{ad}\,(Y))$.

Theorem 1 (Cartan's criterion for solvability). *A complex Lie algebra is solvable if and only if* $B(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

A more complete statement is that the trace form $B_0(X, Y) = \mathrm{tr}(XY) = 0$ for all $X \in [\mathfrak{g}, \mathfrak{g}]$ when $\mathfrak{g}$ is linear. One then obtains the theorem by using that ad is a faithful representation of $\mathfrak{g}/Z(\mathfrak{g})$, and $Z(\mathfrak{g})$ is already solvable.

Definition 2. A Lie algebra is *semisimple* if it has no solvable ideals. That is, $\mathrm{Rad}(\mathfrak{g}) = 0$. A Lie algebra is *simple* when it has no nontrivial proper ideals and is not abelian.

Theorem 2 (Cartan's criterion for semisimplicity). *Let $\mathfrak{g}$ be a complex Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if B is non-degenerate.*

Theorem 3 (Weyl's theorem). *Complex representations of a semisimple complex Lie algebra are completely reducible.*

Corollary 1. *A complex Lie algebra is semisimple if and only if it is a direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$$

of simple ideals $\mathfrak{g}_i$.

There are two ways to obtain the root space decomposition from here. One can define the *Jordan-Chevally* decomposition and show it is preserved by representations, so that a maximal toral or nilpotent subalgebra will act by semisimple elements. This approach is in [5]. An alternative is as follows. The Killing form is *associative*, meaning $B(\mathrm{ad}\,(X)(Y), Z) = B(Y, \mathrm{ad}\,(X)(Z))$ (this is a calculation with the Jacobi identity). Therefore if we can find a subspace on which B is real and positive-definite, we would have that $\mathrm{ad}\,(X)$ is diagonalizable, and we could find a decomposition into eigenspaces.

Definition 3. For a nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$, define for each $\lambda \in \mathfrak{h}^*$ the space

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \,|\, [H, X] = \lambda(H)(X)\ \forall H \in \mathfrak{h}\}\,.$$

If $\mathfrak{g}_\lambda \neq 0$, we call λ a *root* and $\mathfrak{g}_\lambda$ a *root space*. We say $\mathfrak{h}$ is a *Cartan subalgebra* if $\mathfrak{h} = \mathfrak{g}_0$. We write $\Delta \subset \mathfrak{h}^*$ for the set of roots.

With work, one can show that if $\mathfrak{h} = \mathfrak{g}_0$, then $\mathfrak{h}$ is maximal toral, so *e.g.* [5] begins by defining roots as lying in the dual space of a fixed maximal toral subalgebra $\mathfrak{h}$. There are also other characterizations:

Proposition 1 ([8] Lec. 5 Proposition 1.3). *The following hold:*

1. *A Zariski generic element of $\mathfrak{g}$ is semisimple;*
2. *The centralizer $Z_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \,|\, [X, Y] = 0\}$ is abelian, and a Cartan subalgebra;*
3. *Any two Cartan subalgebras are conjugate by an element of G, where G is the associated Lie group. (This requires algebraic closure.)*

Remark 1. Over $\mathbb{R}$, the conjugacy of Cartan subalgebras is more complicated, but there are only finitely-many conjugacy classes (at least for nice enough real Lie groups).

Example 1. We pick forms such that $\mathfrak{so}(n,\mathbb{C})$ is the set of skew-symmetric matrices with respect to the main *anti*diagonal. More precisely, it is the Lie algebra of the indefinite special orthogonal group of matrices, the identity component of the group determinant one real matrices preserving the quadratic form corresponding to the antidiagonal matrix of 1s.

Thus if $n = 2m+1$ is odd, we take the real form $\mathfrak{so}(m+1, m)$ so that *e.g* $\mathfrak{so}(5,\mathbb{C})$ is viewed as having real form $\mathfrak{so}(3,2)$. This will make computations much more illustrative.

- For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, for a semisimple element we can take X with all eigenvalues distinct. Then any Cartan subalgebra is the subalgebra of elements diagonal in some basis, by the above proposition.

- For $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$, we can pick a semisimple element as in the last example. We can take $\mathfrak{h}$ to the all diagonal matrices in $\mathfrak{so}(n,\mathbb{C})$, so $H \in \mathfrak{h}$ is of the form $H = \mathrm{diag}(x_1, \ldots, x_m, -x_m, \ldots, -x_1)$ if $n = 2m$ is even, and $H = \mathrm{diag}(x_1, \ldots, x_m, 0, -x_m, \ldots, -x_1)$ if $n = 2m+1$.

Theorem 4. *Let $\mathfrak{g}$ be semisimple over $\mathbb{C}$ (or a general algebraically closed field of characteristic zero). We have*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

as we have $\mathfrak{h} = \mathfrak{g}_0$. Moreover, $\dim \mathfrak{g}_\lambda = 1$ *for all λ, and*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \begin{cases} 0 & \text{if } \alpha+\beta \notin \Delta \\ \mathfrak{h} & \text{if } \alpha+\beta = 0 \\ \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta \end{cases}.$$

Proof. The fact $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ follows from the Jacobi identity, and the other facts about containment follow from this one. □

The above can also be obtained by asking that $\mathfrak{h}$ is just maximal toral, after which one must show that $Z_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$. And now two final definitions to be used nowhere below.

Definition 4. Every connected parabolic subgroup P of algebraic group G is the semidirect product of its unipotent radical and a reductive group. This reductive group is the *Levi subgroup* of P. In the Langlands decomposition $P = MAN$, the Levi component is MA.

Theorem 5 (Levi)**.** *Any* real *finite-dimensional Lie algebra is the semidirect product of its radical and a semisimple algebra called the* Levi subalgebra*.*

1.2 Roots, coroots, and reflections

Definition 5. A *base* or *simple system* Π of a root system Δ is a basis of Δ such that every root can be written as an integer linear combination of elements of Δ such that all coefficients are nonnegative or nonpositive. Roots in a base are called *simple roots*.

Definition 6. A root is *positive* if and only if when written in terms of a base, its coefficients are all nonnegative. Note that positivity depends on a choice of base. If a root $\alpha \in \mathfrak{h}^*$ is positive, then fixing the basis $\{H_1, \ldots, H_m\}$ of $\mathfrak{h}$ dual to Π, we have $\alpha(H_1) = \cdots = \alpha(H_k) = 0$ and $\alpha(H_{k+1}) > 0$ for some k. For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, the converse is true [7]. We write $\Delta^+ \subset \Delta$ for the positive roots.

The trace form B_0 is a real multiple of the Killing form.

Proposition 2. *If $\mathfrak{h}$ is a Cartan subalgebra, $B \restriction_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate.*

Proof. A calculation using the associativity of B shows that $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ with respect to B for all α, β such that $\alpha + \beta \neq 0$. Therefore an element orthogonal to $\mathfrak{g}_0 = \mathfrak{h}$ is also orthogonal to every $\mathfrak{g}_\alpha$, so is zero because B is nondegenerate. □

This gives us a way to obtain an inner product on $\mathfrak{h}^*$, via

$$(\alpha, \beta) := B(H_\alpha, H_\beta),$$

where H_β is such that $\beta(H) = B(H, H_\beta)$. Therefore we can define the reflection

$$s_\alpha \colon \beta \mapsto \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\beta. \tag{1}$$

Note that $s_\alpha(\alpha) = -\alpha$. Indeed, s_α reflects about a hyperplane P_α perpendicular to α.

Example 2.

- **Type A_{n-1} root systems.** As in the above example, let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and let $\epsilon_i \in \mathfrak{h}^*$ take $E_{ii} \mapsto 1$ and take all other matrices to zero. The formula

  $$\mathrm{ad}\,(H)(E_{ij}) = (\epsilon_i(H) - \epsilon_j(H))E_{ij}$$

  shows that $\epsilon_i - \epsilon_j$ are the roots ($i \neq j$), with root space spanned by E_{ij}. Note that for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ we will have $\epsilon_1 + \cdots + \epsilon_n = 0$.

- **Type B_n root systems.** Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ (recall the conventions chosen in example 1). The roots are now $\pm\epsilon_i \pm \epsilon_j$ for $i \neq j$, and $\pm\epsilon_i$. This follows for the same reason as type D_n below, except now we have the possibility *e.g.*

  $$\begin{pmatrix} x_1 & & \\ & 0 & \\ & & -x_1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & & \\ & 0 & \\ & & -x_1 \end{pmatrix} = \begin{pmatrix} 0 & x_1 & 0 \\ 0 & 0 & -x_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

  The point here is the the -1 entry can "hide" thanks to the diagonal zero in H.

- **Type C_n root systems.** If $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, the roots are $\pm\epsilon_i \pm \epsilon_j$ for $i \neq j$, and $\pm 2\epsilon_i$.

- **Type D_n root systems.** If $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ (recall the conventions chosen in example 1), the roots are $\pm\epsilon_i \pm \epsilon_j$ for $i \neq j$ and $i, j \in \{1, \ldots, n\}$. This again follows from the above formula. A good example calculation is

  $$\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & -x_2 & \\ & & & -x_1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & -1 & \end{pmatrix} - \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & -1 & \end{pmatrix} \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & -x_2 & \\ & & & -x_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & & & \\ & & & \\ & & & \\ & & -(x_2 - x_1) & \end{pmatrix}.$$

  Therefore $E_{12} - E_{43}$ lies the root space for $-\epsilon_1 + \epsilon_2$. Note matrices like $E_{12} - E_{43}$ together with the antidiagonal monomial matrices form a basis of $\mathfrak{g}$.

Given this description of the roots, it is easy to now write down choices of base, and hence choices of simple roots.

Example 3.

- **Type A_n root systems.** We can pick n simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Then *e.g.* if $j > i$ we have

  $$\epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \cdots - (\epsilon_{j-1} - \epsilon_j).$$

- **Type B_n root systems.** We can pick n simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$ together with $\alpha_n = \epsilon_n$. Then if $n = 4$ *e.g.*

  $$-\epsilon_1 - \epsilon_3 = -\alpha_i - \alpha_2 - 2\alpha_3 - 2\alpha_4.$$

- **Type C_n root systems.** We can pick n simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$ together with $\alpha_n = 2\epsilon_n$.

- **Type D_n root systems.** We can pick n simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$ together with $\alpha_n = \epsilon_{n-1} + \epsilon_n$. Then *e.g.* $n = 3$ so $2n = 6$,

$$\epsilon_1 + \epsilon_3 = \epsilon_1 - \epsilon_2 + \epsilon_2 + \epsilon_3.$$

Example 4. For $\mathfrak{sl}(3, \mathbb{C})$, we have six possible bases. Note that $\#\mathfrak{S}_3 = 3! = 6$, which is no accident (see theorem 8). From examples 2 and 3, we have roots $\pm\alpha = \pm(\epsilon_2 - \epsilon_3) = \pm(\epsilon_1 + 2\epsilon_2)$, $\pm\beta = \pm(\epsilon_1 - \epsilon_3) = \pm(2\epsilon_1 + \epsilon_2)$, and $\pm\gamma = \pm(\epsilon_1 - \epsilon_2)$. Note that $\beta = \alpha + \gamma$ and α and γ are simple. Note the positive roots are $\Delta^+ = \{\alpha, \beta, \gamma\}$. The bases are:

$$\Pi_1 = \{\alpha, \gamma\}, \quad \Pi_2 = \{\alpha, -\beta\}, \quad \Pi_3 = \{\beta, -\gamma\}, \quad \Pi_4 = \{-\alpha, -\gamma\}, \quad \Pi_5 = \{-\alpha, \beta\}, \quad \Pi_6 = \{-\beta, -\gamma\}.$$

Note that the base Π_1 in the figure above arises as the set of all indecomposable roots on the positive side of P_β, *i.e.* all indecomposable roots μ such that $(\mu, \beta) > 0$. We say such a root is *indecomposable* when it cannot be written as the sum of two roots also on the positive side of P_β. It is a fact [5] that all bases arise this way. Therefore in situations where the root system can be drawn, this affords a way to compute bases geometrically.

If $\alpha \in \Delta$, then $-\alpha \in \Delta$, either by definition or by the observation that otherwise, $\mathfrak{g}_\alpha$ is orthogonal to all of $\mathfrak{g}$ with respect to the Killing form, which is nondegenerate. Therefore our decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_{-\alpha}.$$

Set $\mathfrak{n}^+ := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, the sum of the positive root spaces relative to some base Π. Note that under the bases of the above example, $\mathfrak{n}^+$ is strictly upper-triangular. For in $\mathfrak{sl}(n, \mathbb{C})$ positivity of $\lambda = \epsilon_i - \epsilon_j$ is equivalent to $i < j$, so $\mathfrak{g}_\lambda$ is spanned by E_{ij}, which lies above the main diagonal. Note that as $\mathfrak{n}^+$ has only positive weights, it is nilpotent; either because weight strings have finite length, or more concretely because $\mathfrak{n}$ is strictly upper triangular in the given basis. Note that we have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

Definition 7. The *Borel subalgebra* associated to $\mathfrak{h}$ and Π is

$$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+.$$

It is solvable but not nilpotent, as $\mathfrak{h}$ acts of course by scalars under ad.

Therefore choosing a base, which includes choosing a set of positive roots, chooses a Borel subalgebra. Exponentiating (for Lie groups) it also chooses a Borel subgroup. We recall that for algebraic groups when the field is algebraically closed, all Borel subgroups are conjugate.

Definition 8. Any subalgebra $\mathfrak{q}$ fitting into a filtration $\mathfrak{b} \subset \mathfrak{q} \subset \mathfrak{g}$ is called *parabolic*. Recall this condition on the group level is one of the equivalent definitions of a parabolic subgroup.

In the same way that Borel subalgebras are (up to conjugacy) upper triangular (thanks either to Lie's theorem or to the above concrete discussion), parabolic subalgebras are block-upper-triangular. For $\mathfrak{sl}(3, \mathbb{C})$, this leaves few options for each parabolic subalgebra; there is room to add only a single 2×2 block. For $\mathfrak{sl}(2, \mathbb{C})$ there is no room at all for such a filtration.

Theorem 6. *The parabolic subalgebras $\mathfrak{q}$ contained in $\mathfrak{b}$ are parametrized by the set of subsets of Π. The correspondence sends*

$$\Pi' \subset \Pi \mapsto \mathfrak{q}_{\Pi'} := \mathfrak{b} \oplus \bigoplus_{\alpha \in \operatorname{span} \Pi'} \mathfrak{g}_{-\alpha},$$

where the right-hand side means by definition the subalgebra generated by all $\mathfrak{g}_\alpha$, $\alpha \in \Pi$, and $\mathfrak{g}_{-\beta}$, $\beta \in \Pi'$; the span *notation is abused here.*

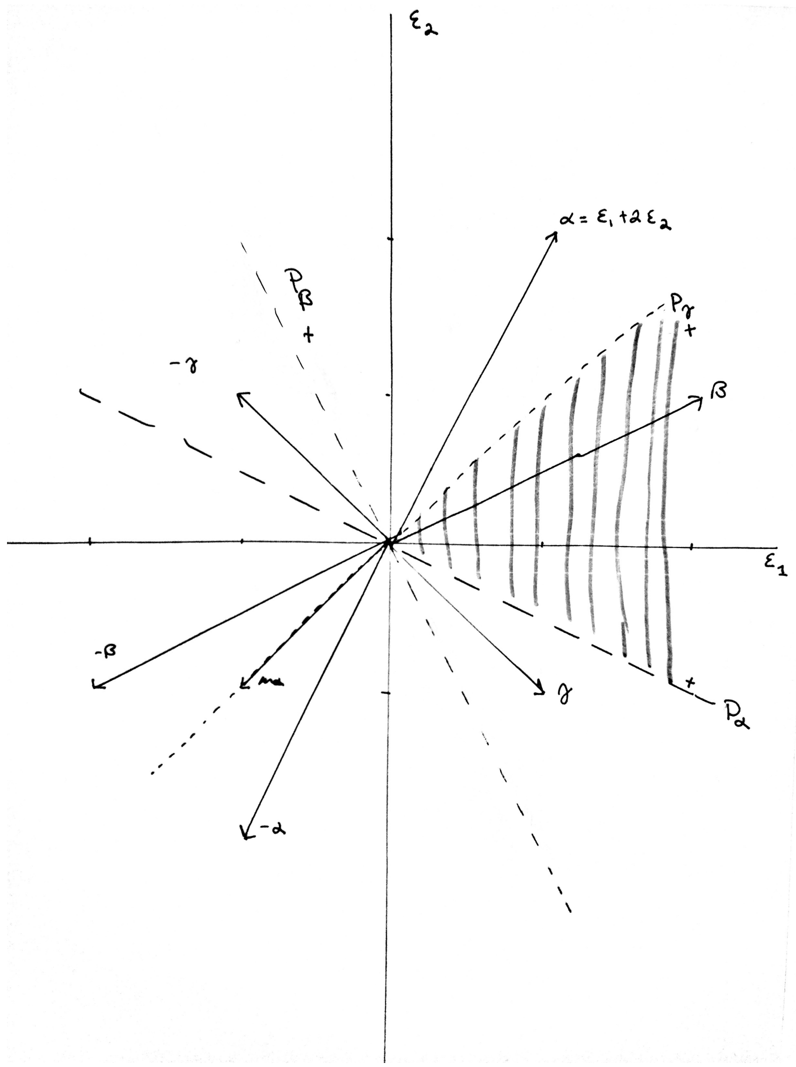


Figure 1: The root system A_2. The shaded region is the *fundamental* Weyl chamber relative to the base $\Pi_1 = \{\alpha, \gamma\}$ (*i.e.* in theorem 8 we identify W with the orbit $W \cdot \Pi_1$, so the shaded region corresponds to the identity). The + symbols indicate which sides of the hyperplanes P_μ are "positive" with respect to the inner product coming from the trace form. Note this chamber contains neither α nor γ! Indeed, it is contains (see the remark following theorem 7) precisely roots pairing strictly positively with α and γ; equivalently, lying on the positive sides of the P_γ and P_α.

Example 5. For $\mathfrak{sl}(3, \mathbb{C})$, chose the base $\Pi = \Pi_1 = \{\alpha, \gamma\}$. This determines positive roots $\{\alpha, \beta = \alpha + \gamma, \gamma\}$, and hence the Borel subalgebra of upper triangular matrices, in accordance with example 2. The parabolic subalgebras are therefore

$$\begin{array}{ccc} \emptyset & \leftrightarrow & \mathfrak{b} \\ \{\alpha\} & \leftrightarrow & \mathfrak{b} \oplus \mathfrak{g}_{-\alpha} \\ \{\gamma\} & \leftrightarrow & \mathfrak{b} \oplus \mathfrak{g}_{-\gamma} \\ \Pi_1 & \leftrightarrow & \mathfrak{g} \end{array}$$

1.2.1 Abstract root systems

Definition 9. An *abstract (reduced) root system* is a Euclidean space E with a choice of symmetric bilinear form, and a finite subset $\Delta \subset E \setminus \{0\}$. For $\alpha \in \Delta$, write $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ (c.f. proposition 4). A root system must satisfy the axioms

1. The only scalar multiples of $\alpha \in \Delta$ also in Δ are $\pm\alpha$.
2. The map $s_\beta \colon v \mapsto v - (\beta^\vee, v)\beta$ is an automorphism and maps $\Delta \to \Delta$.
3. Δ spans E.
4. We have $(\alpha^\vee, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$.

Remark 2. There is some redundancy in these axioms [5].

If we take $(\cdot, \cdot)$ to be the trace or Killing forms, we see our root systems above are abstract root systems [5], [8].

1.2.2 Coroots, fundamental weights and Cartan matrices

From a root system we obtain an *Cartan matrix* with entries $n_{ij} = \alpha_i^\vee(\alpha_j)$. In particular the diagonal entries are

$$\alpha_i^\vee(\alpha_i) = (\alpha_i^\vee, \alpha_i) = 2\frac{(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} = 2.$$

Because the Weyl group (see below) acts on Δ, the Cartan matrix is defined up to conjugation by a monomial matrix (frequently just a permutation matrix), corresponding to different choices of base.

Note that we do not have $\alpha_i^\vee(\alpha_j) = \alpha_j^\vee(\alpha_i)$ unless $\|\alpha_i\|^2 = \|\alpha_j\|^2$.

Definition 10. An *abstract Cartan matrix* is a square matrix A such that

1. $a_{ii} = 2$;
2. $a_{ij} \leq 0$ if $i \neq j$;
3. $a_{ij} = 0 \iff a_{ji} = 0$.

The a_{ij} are the *Cartan integers*.

Note this means the Cartan matrix is the matrix corresponding to the form $(\cdot, \cdot)$ with respect to the standard basis. In other words

$$(\alpha, \beta) = \alpha^T A \beta.$$

The negative off-diagonal entries correspond to the fact that the difference $\alpha_i - \alpha_j$ for $\alpha_i \neq \alpha_j$ in Π is not a root; one coefficient is positive and the other is negative. The Cartan matrix determines Δ up to isomorphism of root spaces, and there is an algorithm for recovering Δ from the matrix [5]. One can also define Lie algebras directly from the matrix [8].

From a Cartan matrix A, draw vertices for every simple root *i.e.* n vertices if A is $n \times n$. Then draw $a_{ij}a_{ji}$ edges between the vertices for α_i and α_j. The edges become oriented if we draw an arrow from the longer root to the shorter root. We recall that the only possible values of $\|\beta\|^2/\|\alpha\|^2$ are 1, 2, and 3 [5].

Example 6 (Cartan matrix for $\mathfrak{sl}(3, \mathbb{C})$ with respect to the trace form)**.** We take the first base from example 4.

By example 3, the matrix will be 2×2, and as our form is symmetric, we will need to calculate only one inner product to fill out the matrix completely, the one corresponding to the off-diagonal entries. Picking the base $\{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3\}$ we have, using (2),

$$(\alpha_1, \alpha_2) = B_0(H_{\alpha_1}, H_{\alpha_2}) = \operatorname{tr}\left(\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} 0 & & \\ & -1 & \\ & & 0 \end{pmatrix} = -1 \leq 0.$$

Therefore the Cartan matrix is (assuming for now $\|\alpha_1\| = \|\alpha_2\|$)

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

so the Dynkin diagram starts out as (the edge should actually connect the two vertices if typeset properly)

$$\bullet \text{ —— } \bullet$$

so to finish we need just calculate the lengths. We have

$$\|\alpha_1\|^2 = B_0(\alpha_1, \alpha_1) = \operatorname{tr}\left(\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} = 2$$

and likewise for $\|\alpha_2\|$. Therefore the roots have the same length and we draw no arrows. The A_2 diagram is then

$$\bullet \text{ —— } \bullet$$

Example 7 (Dynkin diagram and Cartan matrix for type B_2)**.** See the above examples and example 3 for set-up. If we take $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$ (recall the conventions chosen in example 1) and $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2$, we have

$$(\alpha_1, \alpha_2) = \operatorname{tr}\left(\begin{pmatrix} 1/2 & & & & \\ & -1/2 & & & \\ & & 0 & & \\ & & & 1/2 & \\ & & & & -1/2 \end{pmatrix}\begin{pmatrix} 0 & & & & \\ & 1/2 & & & \\ & & 0 & & \\ & & & -1/2 & \\ & & & & 0 \end{pmatrix}\right) = -1/2.$$

Now the lengths are

$$\|\alpha_1\|^2 = (\alpha_1, \alpha_1) = \operatorname{tr}\left(\begin{pmatrix} 1/2 & & & & \\ & -1/2 & & & \\ & & 0 & & \\ & & & 1/2 & \\ & & & & -1/2 \end{pmatrix}\begin{pmatrix} 1/2 & & & & \\ & -1/2 & & & \\ & & 0 & & \\ & & & 1/2 & \\ & & & & -1/2 \end{pmatrix}\right) = 1$$

and

$$\|\alpha_2\|^2 = (\alpha_2, \alpha_2) = \operatorname{tr}\left(\begin{pmatrix} 0 & & & & \\ & 1/2 & & & \\ & & 0 & & \\ & & & -1/2 & \\ & & & & 0 \end{pmatrix}\begin{pmatrix} 0 & & & & \\ & 1/2 & & & \\ & & 0 & & \\ & & & -1/2 & \\ & & & & 0 \end{pmatrix}\right) = 1/2.$$

Therefore we have

$$\alpha_2^\vee(\alpha_1) = \frac{2(\alpha_2, \alpha_1)}{(\alpha_2, \alpha_2)} = 2 \cdot 2 \cdot \left(-\frac{1}{2}\right) = -2$$

while

$$\alpha_1^\vee(\alpha_2) = 2 \cdot 1 \cdot \left(-\frac{1}{2}\right) = -1.$$

Therefore the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ and the Dynkin diagram is

$$\bullet = \rangle = \bullet$$

For more, see [5] or [7].

This next terminology is from [1] which used different notation than the other references for this section. We have tried to translate it.

Remark 3. A complete table [4] of Dynkin diagrams shows that for small n, some types coincide. This is an explanation of the *exceptional isomorphisms*, *e.g.*

- Diagrams A_1, B_1 and C_1 coincide, so $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sp}(2,\mathbb{C}) \subseteq \mathfrak{sl}(2,\mathbb{C})$;
- Diagrams B_2 and C_2 coincide, so $\mathfrak{so}(5,\mathbb{C}) \simeq \mathfrak{sp}(4,\mathbb{C}) \subset \mathfrak{sl}(4,\mathbb{C})$;
- Correctly, understood, the D_2 diagram is two vertices with no edge between them, and so $D_2 \simeq A_1 \times A_1$ and $\mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$;
- Diagrams $A_3 = D_3$ so $\mathfrak{sl}(4,\mathbb{C}) \simeq \mathfrak{so}(6,\mathbb{C})$.

Definition 11. Let Π be a base of Δ.

- A *coroot* is an element $\alpha^\vee$ for $\alpha \in \Delta$ a root. Then $\Delta^\vee$ is a root system [5], and $\Pi^\vee$ gives a basis.
- Define $\omega_i \in \mathfrak{h}^*$ by $\omega_i(\alpha_j^\vee) = \delta_{ij}$, so the ω_i give a dual basis to the $\alpha_j^\vee$. We call the ω_i the *fundamental weights.* We write $\widehat{\Pi}$ for this basis for the corresponding basis.
- The *coweights* are dual to the fundamental weights. We write $\left(\widehat{\Pi}\right)^*$.

The fundamental weights are *not* the basis of roots, despite being dual to the coroots. Indeed, a root pairs with with coroots by evaluation as $(\alpha_j^\vee, \alpha_i) = n_{ij}$ to form entries of the Cartan matrix. In particular $(\alpha_i^\vee, \alpha_i) = 2$, not 1.

Example 8. We give an idea of what fundamental weights look like. See [8], and [3] for more detailed calculations.

- **Fundamental weights for $\mathfrak{g} = \mathfrak{sl}_{n+1}$.** In this case we have $\omega_i = \sum_{j=2}^{i} \epsilon_j$. We have
$$\omega_i(\alpha_j^\vee) = \sum_{k=1}^{i} \epsilon_k(E_j - E_{j+1}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i > j \text{ } i.e. \text{ } i \geq j+1 \\ 0 & \text{if} i < j \end{cases} .$$
- **Fundamental weights for $\mathfrak{g} = \mathfrak{so}_{2n+1}$.** In this case we have $\omega_i = \sum_{j=1}^{i} \epsilon_j$ if $i < n$ and $\omega_n = \frac{1}{2}\sum_{j=1}^{n} \epsilon_j$.

1.2.3 Roots vs weights

The preservation of the Jordan-Chevally decomposition shows that $\mathfrak{h}$ acts semisimply even when the representation of $\mathfrak{g}$ is not ad. In this case we still get a decomposition into spaces corresponding to elements $\lambda \in \mathfrak{h}^*$, but now we call the λ *weights.* The roots are the weights of the adjoint representation. In general the weight spaces need not be one-dimensional, but the highest weight space will be [5].

1.2.4 Roots at the group level

Sometimes roots are defined as multiplicative characters on a maximal torus of the group G. Then *e.g.* corresponding to the root α of $\mathfrak{sl}(3,\mathbb{C})$ sending

$$\alpha \colon \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & -h_1 - h_2 \end{pmatrix} \mapsto h_1 - h_2$$

we have the "root"

$$\begin{pmatrix} e^{h_1} & & \\ & e^{h_2} & \\ & & e^{-h_1-h_2} \end{pmatrix} \mapsto e^{h_1}/e^{h_2} = e^{h_1 - h_2},$$

so the multiplicative "root" on G is obtained by exponentiation of the actual root α.

This example is from a course given by Kamnitzer. The below material can be found in [6] as notes for a different course.

Example 9. Consider the algebraic group $G = \mathrm{GL}_n(\mathbb{C})$. Recalling the only morphism of algebraic groups $\mathbb{C}^\times \to \mathbb{C}^\times$ is $z \mapsto z^n$, we have that any multiplicative character of a maximal torus $T \simeq (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ is of the form

$$\lambda\colon \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n}$$

for $\lambda_i \in \mathbb{Z}^n$. We can then call the tuple $(\lambda_i) \in \mathbb{Z}^n$ a weight. Note we still have a natural action by permutation of $\mathfrak{S}_n$, which is good as $\mathrm{GL}_n(\mathbb{C})$ is type A_{n-1} as an algebraic group.

The *positive* roots corresponding to "exponentiations" of the positive roots (with respect to the upper-triangular Borel subgroup) $\epsilon_i - \epsilon_j$ for $j > i$ are tuples $\alpha = (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0 \ldots, 0)$. They send $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i/t_j$.

The condition of λ being *dominant* then translates to $\lambda \in \mathbb{Z}_+^n := \{(\lambda_1, \ldots, \lambda_n \,|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$. Indeed, the dot product gives us a pairing and if α is as above

$$(\mu, \alpha) = \mu_i - \mu_j \geq 0.$$

Therefore the increasing condition on the μ_i are equivalent to lying inside the Weyl chamber defined by the simple roots $\beta_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$, which is what it should mean to be dominant.

In this context it is helpful to use lemma A of [5] §10.3 to understand length: The length of some $w \in \mathfrak{S}_n$ is $\#\{(i,j) \,|\, i > j \text{ and } wi < wj\}$, *i.e.* the number of positive roots α such that $w\alpha < 0$.

1.3 The Weyl group

Definition 12. Given a root system $\Delta \subset E$, the subgroup $W \subset \mathrm{O}(E)$ generated by the simple reflections (1) is the *Weyl group* of Δ.

If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, taking the trace form B_0 as the inner product on $E \supset \Delta$ amounts to taking the dot product. For $\lambda = \epsilon_i - \epsilon_j$, we take $H_\lambda = E_{ii} - E_{jj}$. Indeed, if $n = 4$ and $\lambda = \epsilon_1 - \epsilon_2$, we have *e.g.*

$$B_0(H, H_\lambda) = \mathrm{tr}\left(\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} a & & & \\ & -b & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = a - b. \tag{2}$$

A corollary of the second root space axiom is that the Weyl group permutes the roots in Δ. In particular, it is finite and naturally a subgroup of $\mathfrak{S}_{\#\Delta}$.

Example 10. Throughout, we do the calculations below writing roots in terms of the matrices E_{ij} as above. We identify α with H_α via B_0.

- **Weyl groups for type A_{n-1}.** If we take the dot product as the inner product on E as above, then $(\beta, \beta) = 2$ for any $\beta \in \Delta$, so $\beta^\vee = \beta$. Then s_β sends

$$s_\beta\colon v \mapsto v - B_0(H_{\beta^\vee}, v)H_\beta = \begin{pmatrix} v_1 \\ \vdots \\ v_i \\ v_{i+1} \\ \vdots \\ v_n \end{pmatrix} - (v_i - v_{i+1})\begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_i - v_i + v_{i+1} \\ v_{i+1} - v_{i+1} + v_i \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_{i+1} \\ v_i \\ \vdots \\ v_n \end{pmatrix}, \tag{3}$$

  We see that the simple roots $\epsilon_i - \epsilon_{i+1}$ correspond to simple transpositions, and that $W = \mathfrak{S}_n$. Note the simple transpositions generate $\mathfrak{S}_n$. Note the root β is sent by s_β to $-\beta$.

- **Weyl groups for type B_n and C_n.** In this case the Weyl group is $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{\times n}$. The factors $\mathbb{Z}/2\mathbb{Z}$ switch the signs of the basis vectors ϵ_i. Indeed, if $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$ (type B_3) (recall the conventions chosen in example 1) and $\alpha = \epsilon_3$, then

$$H_{\epsilon_3} = \mathrm{diag}(0, 0, 1/2, 0, -1/2, 0, 0)$$

and $\epsilon_n^\vee = 4\epsilon_n$. Then we have

$$s_\alpha \colon v \mapsto v - B_0(H_{\alpha^\vee}, v)H_\alpha = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \\ -v_3 \\ -v_2 \\ -v_1 \end{pmatrix} - \mathrm{tr}\left(\begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \\ -v_3 \\ -v_2 \\ -v_1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ -1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ -v_3 \\ 0 \\ v_3 \\ -v_2 \\ -v_1 \end{pmatrix}, \tag{4}$$

where we again write column vectors for diagonal matrices. Thus s_{ϵ_n} and its conjugates act by elements of $\mathfrak{S}_n$, permuting pairs of entries $(v_i, -v_i) \mapsto (v_j, -v_j)$ by swapping entries across the middle diagonal 0 entry.

- **Weyl groups for type D_n.** In this case we obtain the index two normal subgroup of $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{\times n}$ consisting of matched (*i.e.* mirrored on the second half of the main diagonal) permutations, and elements which change an even number of signs. Indeed, if $\mathfrak{g} = \mathfrak{so}(6, \mathbb{C})$ (type D_3) (recall the conventions chosen in example 1), we have $\alpha = \alpha_3 = \epsilon_2 + \epsilon_3$, and we see

$$H_\alpha = \begin{pmatrix} 0 & & & & & \\ & 1/2 & & & & \\ & & 1/2 & & & \\ & & & -1/2 & & \\ & & & & -1/2 & \\ & & & & & 0 \end{pmatrix},$$

so that $\alpha^\vee = 2\alpha$.

$$\begin{aligned} s_\alpha \colon v \mapsto v - B_0(H_{\alpha^\vee}, v)H_\alpha &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ -v_3 \\ -v_2 \\ -v_1 \end{pmatrix} - \mathrm{tr}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ -v_3 \\ -v_2 \\ -v_1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ -v_3 \\ -v_2 \\ -v_1 \end{pmatrix} - (2v_2 + 2v_3)\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ -v_3 \\ v_2 \\ -v_2 \\ v_3 \\ v_1 \end{pmatrix}, \end{aligned}$$

where we again write column vectors for diagonal matrices. Then we see that W is as claimed, as we can now compose this s_α with permutations.

1.3.1 Weyl Chambers

The hyperplanes $\alpha^\perp$ across which s_α reflects divide E into *Weyl chambers*. The interiors of these chambers, *i.e.* without the walls, are the *open Weyl chambers*.

Definition 13. An element of E is

- *regular* if it lies in the complement of the walls;
- *dominant* if it lies in the closure of a Weyl chamber corresponding to a fixed base;
- *strongly dominant* if it lies inside the the interior of a Weyl chamber.

The term dominant is motivated by the fact that $\sigma\lambda \leq \lambda$ for all σ in the Weyl group (see below).

Every root is conjugate to a unique dominant root. One uses (3) from the below theorem and the fact that every point in E is conjugate to a point in the closure of the Weyl chamber for a fixed base, then following [5], one extends $\leq$ to all of E by allowing coefficients in $\mathbb{R}$, take σ such that $\sigma\mu$ is maximal; it will land in the closure.

Every regular root is therefore in exactly one Weyl chamber, and two regular roots define the same Weyl chamber iff they on the same side of every Wall. One can show (recall example 4) that the set of simple roots which pair positively with some fixed element $\gamma \in E$ is a base, so

Theorem 7. *Weyl chambers are in bijection with bases.*

If a chamber C is defined by a base Π, then $C = \{\mu \in \Phi \,|\, (\alpha, \mu) > 0 \; \forall \alpha \in \Pi\}$.

Viewing W as reflecting across the hyperplane $P_\alpha = \alpha^\perp$, we have

Lemma 1. *The reflection s_α sends $\alpha \mapsto -\alpha$ and permutes the other positive roots when α is simple.*

This lemma as proved in [5] uses the property that all coefficients relative to a base are nonnegative or nonpositive.

Theorem 8. 1. *W permutes the Weyl chambers simply transitively;*

2. *If $\dim E = n$, i.e. the number of simple roots is always n, then every chamber has n walls. Therefore there are roots $\alpha_1, \ldots, \alpha_n$ such that the chamber is $\{v \in E \,|\, (\alpha_i, v) \geq 0 \; \forall i = 1, \ldots, n\}$;*

3. *Each Weyl chamber is a fundamental domain for W. In fact if $\sigma\mu = \lambda$ for μ, λ in the closure of the same chamber, σ is a product of reflections fixing λ;*

4. *The* simple reflections s_{α_i} *generate W.*

A proof is in [5]. Some notes are:

Proof. Transitivity is visually clear, and no chamber can be stabilized; W reflects across the walls. The last item is proved using the third. □

Definition 14. The *Bruhat order* on W is the relation $\prec$, where we write $u \prec w$ if $w = s_{\beta_k} \cdots s_{\beta_1} u$ with $\ell(w) > \ell(s_{\beta_{k-1}} \cdots s_{\beta_1} u) > \cdots > \ell(u)$ for roots β_i. [8] specifies the roots need not be simple. The longest element is written w_0. The shortest is the identity.

See example 18 for an example longest element.

Lemma 2. 1. *If $u \prec w$ then $u \cdot 0 > w \cdot 0$, where the dot action is $u \cdot \lambda = u(\lambda + \rho) - \rho$;*

2. *If u is obtained by deleting reflections in a reduced expression for w, then $u \prec w$.*

3. *$u \prec w$ iff $w_0 w \prec w_0 u$, so multiplying by w_0 is an antitone automorphism of W as a set.*

We sketch the proof of 1 and 3.

Let $\alpha \in \Pi$ be simple in some fixed base. We claim if $w = s_\alpha u$ with $\ell(s_\alpha u) = \ell(u)+1$, then $u \cdot 0 - w \cdot 0 > 0$. We have

$$u \cdot \rho - \rho - (w \cdot \rho - \rho) = \rho - \sum_{i=1}^{\ell(u)} \gamma_i - \rho - (s_\alpha u)\rho + \rho = -\sum_{i=1}^{\ell(u)} \gamma_i + \alpha + s_\alpha \cdot \sum_{i=1}^{\ell(u)} \gamma_i, \tag{5}$$

where if $u = s_{\nu_1} \cdots s_{\nu_n}$, the γ_i are ν_1, $s_{\nu_1}(\nu_2)$, $s_{\nu_1}s_{\nu_2}(\nu_3)$, and so on. Note $\gamma_i > 0$. If $\alpha \neq \gamma_i$ for all i, then the coefficient of α in (5) is nonegative, so because Π is a base, the root (5) is positive. We cannot have $\alpha = \nu_1$ by the length hypothesis, and we cannot have $\alpha = s_{\nu_1}(\nu_2)$ etc., as s_{ν_1} sends Π into a different Weyl chamber. We likewise cannot have $\alpha = \nu_2$, and so on. This shows 1.

We claim $\ell(w_0 u) = \ell(w_0) - \ell(u)$. If $u(\beta) < 0$ for $\beta > 0$, then $(w_0 u)(\beta) = w_0(u\beta) > 0$. Therefore $\ell(w_0 u) \leq \ell(w_0) - \ell(u) = \#\Delta^+ - \ell(u)$. But w_0 still flips every positive root λ such that $v\lambda > 0$, so we have equality. Now 3 follows from the definition of the Bruhat order.

Example 11. When $W = \mathfrak{S}_n$, we have $u \prec w$ iff $m(u)_{ij} \geq m(w)_{ij}$ for all i, j, where

$$m(w)_{ij} = \# \{ r \leq j \,|\, w(f) \leq i \} .$$

The Bruhat order is used in many places when studying Hecke algebras. The proof of existence of Kazhdan-Lusztig bases given in [8] uses induction on the Bruhat order.

1.3.2 The Weyl group as a subquotient for compact Lie groups

In the case of algebraic groups or compact Lie groups (for noncompact groups see below) we can define the "analytic" Weyl group as $N_G(T)/Z_G(T)$, the normalizer modulo the centralizer of a chosen maximal torus. Then if $\bar{w}$ is an equivalence class, it acts on roots λ by

$$(\bar{w}\lambda)(H) = \lambda(\mathrm{Ad}(w)^{-1}(H)) = \lambda(w^{-1}Hw).$$

Theorem 9. *The analytic and algebraic Weyl groups coincide for compact Lie groups and algebraic groups.*

For more examples like the below and a proof of the theorem, see [7].

Example 12. Let G be the real Lie group $\mathrm{SU}(n+1)$. We show that $N_G(T)/T$ is isomorphic to $\mathfrak{S}_n$, the Weyl group of type A_n. Indeed, we can take T to be the diagonal matrices in G. Belonging to G is equivalent to the entries lying each lying on the complex unit circle. Then G acts on $\mathbb{C}^{n+1}$ unitarily, so sends orthonormal bases to orthonormal bases. As a representation of the abelian group T, $\mathbb{C}^{n+1}$ decomposes into one-dimensional subrepresentations, which clearly must be the coordinate axes. Therefore if $gtg^{-1} = t'$ for some $t' \in T$ then $gt = t'g$ so gv is an eigenvector of t' when v is an eigenvector of t. Therefore g permutes the coordinate axes and there is a monomial matrix w such that $gw = 1$. That is, if

$$g \begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & \ddots & & \\ & & & t_n & \\ & & & & t_1^{-1} \cdots t_n^{-1} \end{pmatrix} g^{-1} = \begin{pmatrix} t_1' & & & & \\ & t_2' & & & \\ & & \ddots & & \\ & & & t_n & \\ & & & & (t_1')^{-1} \cdots (t_n')^{-1} \end{pmatrix},$$

then we see g differs from a *permutation* matrix by an element of T, if at all. Therefore we can take $W \in \mathfrak{S}_{n+1} \hookrightarrow \mathrm{SU}(n+1)$ and the claim is proved.

1.3.3 The Weyl group as a subquotient for noncompact Lie groups

For non-compact groups, one defines the "analytic" Weyl group, and the algebraic Weyl group, in a more complicated way. Let θ and Θ be a chosen Cartan involution on $\mathfrak{g}$ and G, respectively. Then we obtain the *Cartan decomposition* $\mathfrak{g} = \mathrm{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ into eigenspaces corresponding to 1 and -1, respectively. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let A be the closed subgroup with Lie algebra $\mathfrak{a}$. There is an inner product on $\mathfrak{g}$ as a real vector space for which every $\operatorname{ad} X$ for $X \in \mathfrak{p}$ is symmetric, and so we obtain a decomposition of the real Lie algebra $\mathfrak{g}$ into root spaces for $\mathfrak{a}$. This turns out to give an abstract root system in the sense of these notes. Let $W(\mathfrak{a})_{\mathrm{alg,nc}}$ be its abstract Weyl group. Using a fixed maximal compact subgroup K, we can set $W(A : G) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

Theorem 10. *The groups $W(\mathfrak{a})_{alg,nc}$ and $W(A : G)$ are isomorphic for a non-compact real linear connected reductive Lie group.*

It is this group $W = W(A : G)$ that appears in the Bruhat decomposition in 3.

One can define the still more subtle notion of a Weyl group with respect to a Θ-stable Cartan subgroup, which is by definition the centralizer of a θ-stable Cartan subalgebra, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ maximal among abelian θ-stable subalgebras of $\mathfrak{g}$. These subgroups need not be conjugate. If $G = \mathrm{SL}(2, \mathbb{R})$, then the group H_1 of 2×2 real rotation matrices and the subgroup H of matrices $\pm\mathrm{diag}(e^t, e^{-t})$ form two non-conjugate θ-stable subalgebras of $\mathfrak{sl}(2, \mathbb{R})$. Using the notion of compact forms, one can show that after complexifying $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ gives a root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ and we have an abstract Weyl group $W(\mathfrak{h}^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}})$, and an analytic Weyl group $W(H : G) = N_K(H)/Z_K(H)$. Some work shows we have an inclusion $W(H : G) \subset W(\mathfrak{h}^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}})$ but in general this inclusion is strict: for different H the group $W(H : G)$ can change but $W(\mathfrak{h}^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}})$ is independent of $\mathfrak{h}$.

Example 13. Let $G = \mathrm{SL}(2, \mathbb{R})$, and H and H_1 be as above. Then $W(H_1 : G) = 1$ and $W(H : G) \simeq \mathbb{Z}/2\mathbb{Z}$. Indeed, $H_1 = K$, and a direct calculation shows that

$$N_K(H) = \left\{ \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, \begin{pmatrix} & \pm 1 \\ \mp 1 & \end{pmatrix} \right\} \simeq \mathbb{Z}/4\mathbb{Z},$$

and that $Z_K(H) = \{\pm 1\} \simeq \mathbb{Z}/2\mathbb{Z}$ so the quotient is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ also.

We can also define the Weyl group as a subquotient of finite groups of Lie type.

Example 14. Let $G = \mathrm{GU}_n(\mathbb{F}_q) = \left\{ g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \,\middle|\, \bar{g}^T J g = J \right\}$ be the finite unitary group. Here J is the antidiagonal matrix of 1s and $\bar{g} = \mathsf{Fr}_q(g)$ is the image of the Frobenius morphism. Assume that q is sufficiently large; say, so that $q \nmid n$. Let $T \subset G$ be the subgroup of diagonal matrices. Note that if $t = \mathrm{diag}(a_1, \ldots, a_n) \in T$, then

$$a_i = (a_{n+1-i}^q)^{-1}. \tag{6}$$

We show that

$$N_G(T)/T \simeq \mathfrak{S}_{\lfloor \frac{n}{2} \rfloor} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\times \lfloor \frac{n}{2} \rfloor}$$

is the Weyl group of type $B_{\lfloor \frac{n}{2} \rfloor}$. Now, G preserves the Hermitian form for J and Fr_q, *i.e.* acts unitarily on $(\mathbb{F}_{q^2})^n$, and T is abelian, and it follows from follows from q being large enough that the only one-dimensional subspaces stable under T are the coordinate axes, as there is an element in T with pairwise distinct entries. Therefore we can calculate that if $gtg^{-1} = t'$, then g can permute the first $\lfloor \frac{n}{2} \rfloor$ coordinate axes and the corresponding axes in the last $\lfloor \frac{n}{2} \rfloor$, or swap symmetric pairs (*e.g.* first and n-th coordinate axes, second and $(n-1)$-st). Let $s \in \mathfrak{S}_{\lfloor \frac{n}{2} \rfloor}$ and $f \in (\mathbb{Z}/2\mathbb{Z})^{\times \lfloor \frac{n}{2} \rfloor}$ (viewed as block matrices of transpositions) be the corresponding matrices, which lie also in G. Another calculation using (6) shows that if $g \in N_G(T)$ scales any coordinate axis, then g differs from sf by an element of T.

Proposition 3 (Bruhat decomposition)**.** *Let G be a connected affine reductive algebraic group (hypothesis of* [2]*), a semisimple complex Lie group (hypothesis of* [4]*), a linear connected reductive real Lie group (hypothesis of* [7]*) or a finite group of Lie type (hypothesis of* [8]*). Then*

$$G = \coprod_{[w] \in W} BwB$$

The idea, especially for groups like $\mathrm{SL}(n, \mathbb{R})$ where W is a symmetric group, is that transpositions (simple reflections) s_i flip upper triangular entries to lower triangular entries, and the longest element flips the most (see example 18). In this way we get the whole group.

Remark 4. The Bruhat decomposition is linked to the flag variety G/B via the *Schubert cells*. See 2.3.2.

Definition 15. The double costs $BwB \subset G$ are the *Bruhat cells* of G. Some authors such as [4] appear to call the Schubert cells Bruhat cells.

For $G = \mathrm{GL}(n, \mathbb{C})$ (and GL_n as an algebraic group over $\mathbb{Q}$, see below) we get therefore a bijection between parabolic subgroups and partitions of n; one just picks how big each block on the diagonal should be. The second point in the example below shows that for other groups, not every block-upper-triangular matrix corresponds to a parabolic subgroup, so only certain partitions are possible. Note that B corresponds to the partition $(1, \dots, 1)$ and parabolic subgroups for other partitions stabilize *partial* flags. The bijection is as follows [1]: Note that GL_n is an algebraic group is type A_n so we can take simple roots $\alpha_i \in \Pi$ as in 3. Call these Π_0 in an amalgamtion of the notation of [7] and [1]. Then we have

$$\begin{aligned}\{\text{partitions } (n_1, \dots, n_p) \text{ of } n\} &\longleftrightarrow \{\text{standard parabolics } P \supset P_0\} \\ &\longleftrightarrow \{\text{subsets } \Pi_0^P = \{\alpha \in \Delta_0 \,|\, i \neq n_1 + n_2 + \cdots + n_k \ \forall 1 \leq k < p\}\}.\end{aligned}$$

Example 15. Let $G = \mathrm{GL}_4$. Then the parabolic subgroup of matrices

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$$

corresponds to the partition $(2, 1, 1)$ of $n = 4$. Recall from 6 that this parabolic subgroup (viewed as corresponding its Lie algebra) is obtained by adding $-\alpha_1 = \epsilon_2 - \epsilon_1$ corresponding to E_{21}. Indeed, we have that $i \neq n_1 + n_2 \implies i \neq 3$ and $i \neq n_1 \implies i \neq 2$, so $i = 1$ and $\Pi_0^P = \{\alpha_1\}$ as required.

Recall parabolic subalgebras were obtained by adding in some negative roots, and are block-upper-triangular. This corresponds exactly to the flipping:

Example 16.

- Let $G = \mathrm{SL}(3, \mathbb{C})$. Then the parabolic subalgebras are $\mathfrak{q} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha}$ and $\mathfrak{q} = \mathfrak{b} \oplus \mathfrak{g}_{-\gamma}$ in the notation from example 5. We have $\gamma = \epsilon_1 - \epsilon_2$, which we called α_1 in example 3. It has eigenspace spanned by E_{12} as explained in example 2. We have

$$Bs_\alpha B = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \right\},$$

  which differs from B precisely by an entry in E_{21}. The corresponding parabolic subgroup is then

$$P = (Bs_\alpha B) \coprod B.$$

  Note the "flipping" language is figurative; while a new nonzero entry appears in the double coset, multiplying on the left or right by a permutation matrix corresponds of course to a swapping of whole row or columns.

- For $G = \mathrm{GU}_n(\mathbb{F}_q)$, we get different looking parabolic subgroups, as we have different simple reflections, they are "mirrored" or "matched" as mentioned above. If we choose simple roots for $B_{\lfloor \frac{n}{2} \rfloor}$ as before, then $\alpha_i = \epsilon_i - \epsilon_{i+1}$ is simple and corresponds to $[w] = (i\ i+1)$ in W by example 14. We then have

$$
\begin{aligned}
&BwB \\
&= \left\{ \begin{pmatrix} * & & \cdots & & * \\ & * & & & \\ & & \ddots & & \\ & & & \ddots & \vdots \\ & & & & \\ & & & & * \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 1 & \\ & & & & & 1 & 0 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} * & & \cdots & & * \\ & * & & & \\ & & \ddots & & \\ & & & \ddots & \vdots \\ & & & & \\ & & & & * \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} * & & & \cdots & & & * \\ & \ddots & & & & & \\ & & * & * & & & \\ & & * & * & & & \\ & & & & \ddots & & \vdots \\ & & & & & * & * \\ & & & & & * & * \\ & & & & & & \ddots \\ & & & & & & & * \end{pmatrix} \right\}.
\end{aligned}
$$

The corresponding parabolic subgroup is then $(BwB) \coprod B$ again.

Definition 16. The subgroup W_X of a Weyl group Wgenerated by a subset $X \subset S$ of the simple reflections is called a *parabolic subgroup* of W.

If $X \subset \Pi$ corresponds to a parabolic subgroup $P = MAN$ up of G, then W_X is the Weyl group of M.

2 Root data

See [2] and [9].

2.1 Root data

Definition 17. A *root datum* is a tuple $(X, \Phi, X^\vee, \Phi^\vee)$ where

- X and $X^\vee$ are finitely-generated abelian groups;
- There is a perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z}$;
- $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$ are finite subsets with a bijection $\Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$. The following two axioms must be satisfied:
  1. $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$;
  2. $s_\alpha(\Phi) = \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$ for all $\alpha \in \Phi$, where
  $$s_\alpha(x) = \beta - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$$
  for all $x \in X$ and $y \in X^\vee$.

Note the resemblance of the s_α and $s_{\alpha^\vee}$ to the reflections in (1), and recall that in definition 9 we *defined* $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$.

Proposition 4. *Let $(X, \Phi, X^\vee, \Phi^\vee)$ be a root datum and define $p\colon X \to X^\vee$ by $p(x) = \sum_{\alpha\in\Phi} \langle x, \alpha^\vee\rangle \alpha^\vee$. Then*

1. *$\langle x, p(x)\rangle \geq 0$ for all $x \in X$ and $\langle \beta, p(\beta)\rangle > 0$ for all $\beta \in \Phi$;*
2. *$\langle s_\beta(x), p(s_\beta(x))\rangle = \langle x, p(x)\rangle$ for all $x \in X$;*
3. *$\langle \beta, p(\beta)\rangle \beta^\vee = 2p(\beta)$, so that*
$$\beta^\vee = \frac{2p(\beta)}{\langle \beta, p(\beta)\rangle};$$
4. *p induces a $\mathbb{Q}$-linear isomorphism $\mathbb{Q}\Phi \to \mathbb{Q}\Phi^\vee$.*

Root data arise from connected reductive algebraic groups:

Theorem 11. *Let G be connected reductive with maximal torus T. Then $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is a root datum. Here the roots $\Phi = \Phi(T, T) \subset X^*(T) \setminus \{0\}$ are all nontrivial characters α such that*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(t)X = \alpha(t)X \ \forall t \in T\} \neq \emptyset.$$

The required pairing is the natural one: $\langle \lambda, \mu\rangle = \lambda \circ \mu\colon \mathbb{G}_m \to T \to \mathbb{G}_m$.

Clearly, a root datum is not a root system; there is no automatic real vector space structure. However, if $V = \mathbb{R}\Phi$ is the subspace generated by Φ in $X \otimes_{\mathbb{Z}} \mathbb{R}$, then V is a root system [2] §14.7. Note we have $(X \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R} = X \otimes_{\mathbb{Z}} (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}) = X \otimes_{\mathbb{Z}} \mathbb{R}$, so by the next proposition $V = X \otimes_{\mathbb{Z}} \mathbb{R}$ iff G is semisimple, in the setting when our root datum arises from an algebraic group as above. When $V = X \otimes_{\mathbb{Z}} \mathbb{R}$, (X, Φ) determines $(X, \Phi, X^\vee, \Phi^\vee)$.

Theorem 12. *Let G be connected reductive with a maximal torus T, roots Φ and coroots $\Phi^\vee$ with respect to T. Then the following are equivalent:*

1. *G is semisimple;*
2. *$\mathbb{Q}\Phi = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$;*
3. *$\mathbb{Q}\Phi^\vee = X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$;*
4. *$G = \langle U_\alpha \mid \alpha \in \Phi\rangle$;*
5. *$G = [G, G]$.*

See [9] §8.1 for a proof. The theorem uses the notation of [9] where, $G_\alpha := \mathcal{Z}_G(T_\alpha)$, $T_\alpha = (\ker \alpha)^0$, and $\mathrm{Lie}\, U_\alpha = \mathfrak{g}_\alpha$. Alternatively, U_α is the unipotent part of the Borel subgroup B_α of G_α containing T. We have $\mathrm{Lie}\, B_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha$. This sets up the group-level linear algebraic group version of reducing many statements to statements for SL_2 via "$\mathfrak{sl}_2$-triples."

Theorem 13. *Two connected reductive algebraic groups are isomorphic iff their root data are isomorphic.*

Theorem 14 (An analogue of Lie's theorems.)**.** *Given a root datum, there exists a (connected) reductive group with that root datum.*

See again [9].

2.2 The Langlands dual group

This theorem allows one to define the *Langlands dual* ${}^LG$ of a connected reductive group G; ${}^LG$ is the group whose root datum is $(X^\vee, \Phi^\vee, X, \Phi)$ is the dual root datum of G. Note that this really is a root datum. This duality corresponds to duality as lattices [1], so when G is "large", ${}^LG$ is "small."

2.3 The flag variety

In this section we write Φ^+ and not Δ^+ for the positive roots. This section is based on courses by Joel Kamnitzer (for which some handwritten notes are available at `http://www.math.toronto.edu/~jkamnitz/courses/flagvarieties/index.html`) and Florian Herzig. Some material is in [4].

2.3.1 Bruhat decomposition revisited

Each double coset BwB in the Bruhat decomposition of a reductive algebraic group G is an orbit under the action map $B \times B \to G$ given by $(b_1, b_2) \cdot g = b_1 g b_2$, so is locally closed. We can describe the closure of a double coset as

$$\overline{BwB} = \bigcup_{w' \prec w} Bw'B,$$

where $\prec$ is the Bruhat partial order. By the Bruhat decomposition we have

$$G/B = \coprod_{w \in W} BwB/B$$

Let U_α be as is in theorem 12. Then the multiplication map

$$\prod_{\substack{\alpha > 0 \\ w^{-1}\alpha < 0}} U_\alpha \to B_u$$

is a closed immersion of varieties. (This is especially clear on the Lie algebra level from the direct sum structure of $\mathfrak{b}$.) The image is a subvariety U^w. We have an isomorphism of varieties

$$U^w \times B \to BwB$$

by

$$(u, b) \mapsto (uwb),$$

identifying the coset w with any of its representatives as we have been doing. Consulting calculation (8), we have

$$BwB = B_u TwB = B_u wTB = B_u wB = U^w \prod_{\substack{\alpha > 0 \\ w^{-1}\alpha > 0}} U_\alpha wB = U^w w \prod_{\substack{\alpha > 0 \\ w^{-1}\alpha > 0}} U_{w^{-1}\alpha},$$

and $\prod_{\substack{\alpha > 0 \\ w^{-1}\alpha > 0}} U_{w^{-1}\alpha} \subset B$. This shows that $BwB/B \simeq U^w$ as varieties, and U^w is affine. Now we have

$$\dim(BwB/B) = \dim(U^w) = \#\left\{\alpha \in \Phi \,\middle|\, \alpha > 0 \text{ and } w^{-1}\alpha < 0\right\} = \ell(w). \tag{7}$$

Definition 18. Let G be a connected algebraic group. As a set, the *flag variety of* G is $\mathrm{Fl} = \mathrm{Fl}(G) := G/B$ for a Borel subgroup B of G.

Remark 5. As all Borel subgroups are conjugate and equal to their own normalizers, we have a bijection $G/B \to \{\text{Borel subgroups } B' \subset G\}$ given by $gB \mapsto gBg^{-1}$.

Example 17. Let $G = \mathrm{GL}_n$ and $B = B_n$. Then the flag variety is the set of all flags

$$E_\bullet = (0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = k^n).$$

2.3.2 Schubert cells

For this section we specialize to $G = \mathrm{GL}_n(\mathbb{C})$, taking $B = B_n$, T to be the diagonal matrices, and identifying $W(G, T)$ with $\mathfrak{S}_n$.

Definition 19. Flags $V_\bullet$ and $W_\bullet$ are in *relative position* $w \in \mathfrak{S}_n$ if $\dim(V_i \cap W_j) = m(w)_{ij} = \#\{r \leq j \,|\, w(f) \leq i\}$. Define the *Schubert cell*

$$X_w = \{V_\bullet \in \mathrm{Fl} \,|\, E_\bullet \text{ and } V_\bullet \text{ are in relative position for } w\},$$

where $E_\bullet$ is the *standard flag* with $E_i = \mathrm{span}\{e_1, \ldots, e_i\}$ for the standard basis of $\mathbb{C}^n$.

Equivalently, $X_w = B \cdot E^w_\bullet$, where $E^w_i = \mathrm{span}\{e_{w(1)}, \ldots, e_{w(i)}\}$. We can also define Schubert cells from Bruhat cells, with $X_w = BwB/B$. Recall that for Weyl group $\mathfrak{S}_n$, we have $w \prec u$ in the Bruhat order iff $m(w)_{ij} \geq m(u)_{ij}$. Let w_0 be the longest element of W, so that $\ell(w_0) = \#\Phi^+ = \#\{\alpha > 0 \,|\, w_0\alpha < 0\}$. Then $w_0 B w_0^{-1}$ is a Borel subgroup of G, corresponding to the roots Φ^-. Indeed, say $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. Then $\mathrm{Lie}(\mathrm{Ad}(w_0)B) = \mathrm{Ad}(w_0)\mathfrak{b}$, and for $X \in \mathrm{Ad}(w_0)\mathfrak{g}_\alpha$ and t in T, let $w_0^{-1} t w_0 = t'$, so that

$$\begin{aligned} \mathrm{Ad}(t)\mathrm{Ad}(w_0)X &= t w_0 X w_0^{-1} t^{-1} \\ &= w_0 t' X (t')^{-1} w_0^{-1} \\ &= w_0 \alpha(t') X w_0^{-1} \\ &= \alpha(t') w_0 X w_0^{-1} \\ &= \alpha(w_0^{-1} t w_0) w_0 X w_0^{-1} \\ &= (w_0\alpha)(t) w_0 X w_0^{-1}. \end{aligned} \tag{8}$$

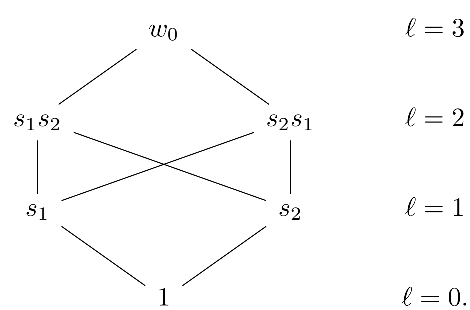
Example 18. With G, B, and T as above, the longest element w_0 is the coset of the antidiagonal matrix of 1s, which conjugates upper-triangular matrices to lower-triangular.

By the last paragraph and (7), we then have

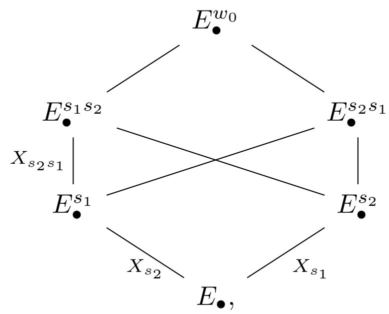
$$\dim(Bw_0B/B) = \#\Phi^+ = \dim(G/B) = \dim(\mathrm{Fl}).$$

Now, $G/B = \mathrm{Fl}$ is irreducible, so Bw_0B/B is open because it is locally closed. We call this dense Schubert cell $Bw_0B/B = X_{w_0}$ the *big cell*.

Example 19. For GL_3, we have simple reflections $\{s_1, s_2\}$ and Weyl group $W = \langle s_1, s_2 \,|\, s_1^2 = s_2^2 = (s_1 s_2)^3 = 1\rangle$. The longest element is then $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ with length $\ell(w_0) = 3$. We draw the picture



People also draw pictures like



which are apparently related to moment maps.

3 Adelic groups

These notes relate to a course given by Arthur, based on [1]. We highlight only a few things. In the case of real Lie groups, the Lie algebra $\mathfrak{a}$ of a maximal abelian subgroup A *i.e.* the thing appearing in the decomposition $G = KAK$ is a place to do analysis. One can do things such as endow it with useful coordinates and study the asymptotics of certain functions, *e.g.* the τ-*spherical* functions [7]. For an adelic group defined over $\mathbb{Q}$, the Lie algebra is also defined over $\mathbb{Q}$, but following the above example would like to work in a real vector space.

It is a fact that we can write $G(\mathbb{A}) = N_P(\mathbb{A})M_P(\mathbb{A})K$ where N_P is the unipotent radical of a standard (*i.e.* containing the usual upper-triangular Borel subgroup) parabolic P, M_P is its Levi component, and K is a maximal compact subgroup. This is a version of the Iwasawa decomposition and is proved by the usual Gramm-Schmidt argument at the archimedean place, and with a p-adic analogue elsewhere. We define $X(M_P)_\mathbb{Q} := \mathrm{Hom}(M_P, \mathbb{G}_m)$, where the morphisms are those of algebraic groups over $\mathbb{Q}$. Then our surrogate Lie algebra is

$$\mathfrak{a}_P := \mathrm{Hom}_\mathbb{Z}(X(M_P)_\mathbb{Q}, \mathbb{R})$$

and

$$\mathfrak{a}_P^* = X(M_P)_\mathbb{Q} \otimes_\mathbb{Z} \mathbb{R}.$$

3.1 Weyl sets

At certain places the notion of *Weyl set* is needed.

Definition 20. For two standard parabolic subgroups P and P' the *Weyl set* $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is the set of restrictions of the Weyl group (as defined using the upper-triangular Borel) which give linear isomorphisms $\mathfrak{a}_P \subset \mathfrak{a}_0 \xrightarrow{\sim} \mathfrak{a}_{P'} \subset \mathfrak{a}_0$. Here $\mathfrak{a}_0$ is the analogue of the Lie algebra for a fixed minimal parabolic subgroup P_0 of G.

Example 20. Let $G = \mathrm{GL}_n$, and let $(n_1, \ldots, n_p)$ and $(n'_1, \ldots n'_{p'})$ be partitions of n with corresponding parabolic subgroups P and P'. Then the Weyl set is empty if $p \neq p'$, as the parabolic subalgebras will have different dimensions; this becomes clear after drawing some pictures ($n = 4$ works well) to see how many negative roots must be added to $\mathfrak{b}$ for each size block used. If $p = p'$, then

$$W(\mathfrak{a}_P, \mathfrak{a}_{P'}) = \left\{ w \in \mathfrak{S}_p \,\middle|\, n'_i = n_{w(i)} \right\}.$$

Indeed, unless P and P' have the same number of blocks of the same size, they will not even have the same dimension. One can see this by overlaying the two block patterns and deforming one into another. When an $m \times m$ block is shifted to $m - 1 \times m - 1$, m flipped roots are lost. To remain a partition of n, another block must from $m' \times m'$ to $m' + 1 \times m' + 1$, and for the number of roots to remain the same we must have $m' = m - 1$. Finally, if the block structures are the same, the elements of the Weyl set permute the blocks.

Remark 6. The above claims are true, but the justification is our own and is uncertain.

As with the Weyl group in many other cases, the set $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ being nonempty controls the equivalence of some induced representations. See [1] §7.

References

[1] J. Arthur, *An Introduction to the Trace Formula*, Clay Mathematics Proceedings **4** (2005), 3–263.

[2] A. Borel, *Linear Algebraic Groups*, Second Enlarged Edition, Graduate Texts in Mathematics **126**, Springer-Verlag, New York 1991.

[3] B. Conrad, lecture notes. Available at `https://math.stanford.edu/~conrad/210CPage/`.

[4] W. Fulton, J. Harris, *Representation theory: A First Course*, Graduate Texts in Mathematics **129**, Springer-Verlag, New York 2002.

[5] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics **9**, Springer-Verlag, New York 1972.

[6] J. Kamnitzer, *Representation Theory, geometric Langlands duality and categorification*, lecture notes, 2014. Available at `http://www.math.toronto.edu/~jkamnitz/zurich/zurich.pdf`.

[7] A.W. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton 1986.

[8] I. Loseu, lecture notes, 2015. Available at `https://www.northeastern.edu/iloseu/RTMI.html`.

[9] T.A. Springer, *Linear Algebraic Groups*, Progress in Mathematics **9**, Birkhäuser, Boston 1998.