Perverse Sheafs Seminar
Talk 2

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28.03.2024

1 Sheaf Theory

We want to study sheaves of of complex vector spaces on complex varieties endowed with the metric topology. Let's start with a bit of sheaf theory.

Definition 1.1. Let X be a topological space. A **presheaf of $\mathbb{C}$-vector spaces** is a functor $\mathcal{F} : \mathbf{Ouv}_X^{\mathrm{op}} \to \mathbf{Vect}_{\mathbb{C}}$. A **morphism** of presheafs $\phi : \mathcal{F} \to \mathcal{G}$ is a natural transformation of functors. We denote the resulting category of presheafs on X by $\mathbf{Presh}(X)$.
A sheaf $\mathcal{F}$ is a presheaf such that the following holds. Let $U = \bigcup_\alpha U_\alpha$. Then,

1. (Identity) If $s \in \mathcal{F}(U)$ such that $s|_\alpha = 0$ for all α, then $s = 0$.
2. (Gluing) If $s_\alpha \in \mathcal{F}(U_\alpha)$ are given such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all α and β, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_\alpha} = s_\alpha$.

A morphism of sheafs is just the morphism of presheafs. We obtain the category of sheaves on X: $\mathbf{Sh}(X)$.

Example 1.2. (Constant Sheaf) Let M be a vector space. The **constant presheaf** on X with value M, denoted $\underline{M}_{\mathrm{pre}}$, has values

$$\underline{M}_{\mathrm{pre}}(U) = M \tag{1.1}$$

for all open $U \subset X$ and restrictions

$$\mathrm{res}_{V,U} = \mathrm{id}_V \tag{1.2}$$

for all inclusions $U \hookrightarrow V$. This is not a sheaf. Take e.g. $X = \{1, 2\}$ with the discrete topology. Then any sheaf with $\mathcal{F}(\{1\}) = \mathcal{F}(\{2\}) = M$ glues to $\mathcal{F}(X) = M \oplus M$.
The **constant sheaf** on X with value M, denoted $\underline{M}$, is given by

$$\underline{M}(U) = \{\text{locally constant functions } s : U \to M\} \tag{1.3}$$

and

$$\mathrm{res}_{V,U}(s) = (s|_U : U \to M). \tag{1.4}$$

Example 1.3. (Skyscraper Sheaf) Let X be Hausdorff and $x \in X$. The skyscraper sheaf at x with value M, denoted $\underline{M}_x$, is

$$\underline{M}_x(U) = \begin{cases} M, & \text{if } x \in U \\ 0, & \text{else} \end{cases} \tag{1.5}$$

with obvious restriction maps.

In order to better understand sheaves and their morphisms, we look at stalks.

Definition 1.4. Let $\mathcal{F}$ be a presheaf on X and $x \in X$. The **stalk** at x is the vector space

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U). \tag{1.6}$$

For $x \in U$ the image of a section $s \in \mathcal{F}(U)$ under the canonical map $\mathcal{F}(U) \to \mathcal{F}_x$ is denoted s_x and is called the **germ** of s at x. Define the **support** of $\mathcal{F}$ by $\operatorname{supp} \mathcal{F} = \overline{\{x \in X | \mathcal{F}_x \neq 0\}}$ and the support of a section $s \in \mathcal{F}(U)$ by $\operatorname{supp} s = \{x \in U | s_x \neq 0\}$.

In other words, the stalk is

$$\mathcal{F}_x = \{(U, s) | U \ni x \text{ open and } s \in \mathcal{F}(U)\}/\sim, \tag{1.7}$$

where $(U, s) \sim (U', s')$ if and only if there exists an open $V \subset U \cap U'$ such that $x \in V$ and $s|_V = s'|_V$. Note that for any map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get an induced map of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$. Stalks have the following nice properties.

Lemma 1.5. Let $\mathcal{F} \in \mathbf{Sh}(X)$ and $s, t \in \mathcal{F}(U)$. Then $s = t$ if and only if $s_x = t_x$ for all $x \in U$.

Lemma 1.6. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$.

Theorem 1.7. The category of sheaves of complex vector spaces $\mathbf{Sh}(X)$ on any topological space X is abelian.

Before we prove this, let's remind ourselves of a couple of things. Recall that an additive category $\mathbf{A}$ is abelian if the following hold.

1. Every morphism in $\mathbf{A}$ has a kernel and a cokernel.

2. Every monic morphism is a kernel, and every epi is a cokernel.

We need one more standart lemma in order to define the cokernel.

Lemma 1.8. The inclusion $\mathbf{Sh}(X) \to \mathbf{Presh}(X)$ admits a left adjoint $\mathbf{Presh}(X) \to \mathbf{Sh}(X)$ called sheafification and denoted by $\mathcal{F} \mapsto \mathcal{F}^+$. The canonical $i : \mathcal{F} \to \mathcal{F}^+$ induces an isomorphism on stalks $i_x : \mathcal{F}_x \to \mathcal{F}^+_x$.

We are now in position to proof the above theorem.

Proof. We can explicitely construct kernels and cokernels. The kernel of a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is the sheaf given by

$$(\ker \phi)(U) = \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)). \tag{1.8}$$

And the cokernel is the sheafification of the presheaf which assings

$$U \mapsto \operatorname{coker}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)). \tag{1.9}$$

The universal property of sheafification secures that this does indeed give a cokernel. Similarly one defines the image of a sheaf morphism.
Taking filtered colimits commute with taking limits, so $(\ker \phi)_x \cong \ker(\phi_x)$ and $(\operatorname{coker} \phi)_x \cong \operatorname{coker}(\phi_x)$. With lemma 1.5, this implies that ϕ is a monomorphism (resp. epi) if and only if ϕ_x is a injective (resp. surj.) for all $x \in X$. Now lemma 1.6 guarantees that every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel. □

An important fact is that taking stalks is exact.

Proposition 1.9. A sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ in $\mathbf{Sh}(X)$ is exact if and only if for every $x \in X$, the sequence $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is exact. I.e. The stalk functor $\mathbf{Sh}(X) \to \mathbf{Vect}_{\mathbb{C}}$ is exact.

Now, since we know that $\mathbf{Sh}(X)$ is abelian, we can consider its derived category $D(X) := D(\mathbf{Sh}(X))$.

Definition 1.10. Let $\mathcal{F} \in D(X)$. Define its support by

$$\text{supp } \mathcal{F} = \overline{\bigcup_{i \in \mathbb{Z}} \text{supp } \mathrm{H}^i(\mathcal{F})}. \tag{1.10}$$

Note once again that the above defnition makes sense, because we properly defined kernels, images and cokernels of sheaves.

Definition 1.11. Let $f : X \to Y$ be a continous map.

1. For $\mathcal{G} \in \mathbf{Sh}(Y)$, the **pullback** of $\mathcal{F}$ denoted by $f^*\mathcal{G}$, is the sheafification of the presheaf $f^*_{\text{pre}}(\mathcal{G})$ given by
$$f^*_{\text{pre}}(\mathcal{G})(U) = \varinjlim_{V \subset Y, f(U) \subset V} \mathcal{G}(V). \tag{1.11}$$

2. Let $\mathcal{F} \in \mathbf{Sh}(X)$, the **pushforward** is the sheaf ${}^\circ f_*\mathcal{F}$ given by
$$({}^\circ f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)). \tag{1.12}$$

Recall that these are adjoint to each other, i.e.

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathcal{G}, {}^\circ f_*\mathcal{F}). \tag{1.13}$$

Remark 1.12. The pullback is functorial in the sense that $(g \circ f)^*\mathcal{F} = f^*(g^*\mathcal{F})$ for $f : X \to Y$, $g : Y \to Z$ and $\mathcal{F} \in \mathbf{Sh}(Z)$.

The pullback functor also behaves well with stalks.

Lemma 1.13. There is a canonical bijection of stalks $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$.

The analog for the pushforward is not true.

Lemma 1.14. The pullback functor is exact.

Proof. We already know that

$$0 \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0 \tag{1.14}$$

is exact if and only if

$$0 \to (f^*\mathcal{F})_x \to (f^*\mathcal{G})_x \to (f^*\mathcal{H})_x \to 0 \tag{1.15}$$

is exact. As seen above this is

$$0 \to \mathcal{F}_{f(x)} \to \mathcal{G}_{f(x)} \to \mathcal{H}_{f(x)} \to 0, \tag{1.16}$$

which is of course exact. □

Note that this is exact since we work with sheaves of vector spaces. In general the pullback functor is only right exact.

Example 1.15. (Pushforward) We can obtain the global section $\Gamma(\mathcal{F})$ as the pushforward of $f : X \to \{\text{pt}\}$:

$$^\circ f_* \mathcal{F}(\text{pt}) = \mathcal{F}(f^{-1}(\text{pt})) = \mathcal{F}(X) = \Gamma(\mathcal{F}). \tag{1.17}$$

The skyscraper sheaf can also be described as a pushforward. Namely let $\underline{M}$ be the constant sheaf of the closed subspace $\overline{\{x\}}$ for $x \in X$ and $i : \overline{\{x\}} \to X$ the inclusion. Then the skyscraper sheaf is $\underline{M}_x = {}^\circ i_* \underline{M}$.

Example 1.16. (Pullback) In a similar manner we can obtain stalks as the pullback of the map $f : \{\text{pt}\} \to X, \text{pt} \;\mapsto x$:

$$(f^* \mathcal{F})(\text{pt}) = \varinjlim_{U \ni f(\text{pt})} \mathcal{F}(U) = \varinjlim_{U \ni x} \mathcal{F}(U) = \mathcal{F}_x. \tag{1.18}$$

The restriction of a sheaf $\mathcal{F}$ to a subspace $Y \subset X$ is denoted by $\mathcal{F}|_Y$ and is given as the sheafification of the assignment

$$U \mapsto \varinjlim_{U \subset V \subset X} \mathcal{F}(V). \tag{1.19}$$

But this is just the pullback along the inclusion $i : Y \to X$.
Let's compute the pullback along $f : X \to Y$ of a constant sheaf $\underline{M}_Y$. We claim that $f^* \underline{M}_Y \cong \underline{M}_X$.
Let $p : Y \to \{\text{pt}\}$ be the projection. Then $\underline{M}_Y = p^* M$, where we take M to be the sheaf with value M on pt. By functoriality

$$f^* \underline{M}_Y \cong f^*(p^* M) \cong (p \circ f)^*(M) \cong \underline{M}_X, \tag{1.20}$$

where the last isomorphism holds since $p \circ f$ is just the projection of X to a point.

Proposition 1.17. The category $\mathbf{Sh}(X)$ has enough injectives.

Proof. Let $\mathcal{F}$ be a sheaf. For each point $x \in X$ the stalk $\mathcal{F}_x$ is a vector space and therefore injective. Define the sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \prod_{x \in U} \mathcal{F}_x. \tag{1.21}$$

Then the obvious map $\theta : \mathcal{F} \to \mathcal{I}$ defined by maps $s \mapsto \prod_{x \in U} s_x$ for $s \in \mathcal{F}(U)$ is injective by lemma 1.5. Note that we can equivalently describe $\mathcal{I}$ as the product over all $x \in X$ of skyscraper sheaves $\underline{\mathcal{F}_x}_x$. A skyscraper sheaf $\underline{M}_x$ is injective because there is a natural isomorphism $\text{Hom}_{\mathbb{C}}(\mathcal{G}_x, M) \cong \text{Hom}_{\mathbf{Sh}(X)}(\mathcal{G}, \underline{M}_x)$. Every vector space is injective, so $\text{Hom}_{\mathbf{Sh}(X)}(-, \underline{M}_x)$ is exact. □

Lemma 1.18. The pushforward functor is left exact.

Proof. This follows from the fact that the sections functor $\Gamma(V, -)$ is left exact. I.e. for every $V \subset Y$, the sequence

$$0 \to \mathcal{F}(f^{-1}(V)) \to \mathcal{G}(f^{-1}(V)) \to \mathcal{H}(f^{-1}(V)) \tag{1.22}$$

is exact. □

Remark 1.19. We know from last talk that all left exact functors have derived functors. Hence, by the above lemmas, the pullback and pushforward functors have derived functors. From now on, the symbols f^* and f_* denote the derived functors! Also note that as $\mathbf{Sh}(X)$ has enough injectives, we get the derived **Hom** functor:

$$R\mathrm{Hom} : D^-(X)^{\mathrm{op}} \times D^+(X) \to D^+(\mathbf{Vect}_{\mathbb{C}}). \tag{1.23}$$

We will need the following two propositions.

Proposition 1.20. For $A \in D^-(X)$ and $B \in D^+(X)$, there is a natural isomorphism

$$\mathrm{Hom}_{D(X)}(A, B) \cong H^0(R\mathrm{Hom}(A, B)), \tag{1.24}$$

and

$$\mathrm{Ext}^n_{\mathbf{Sh}(X)}(A, B) \cong H^n(R\mathrm{Hom}(A, B)) \tag{1.25}$$

for $A, B \in \mathbf{Sh}(X)$.

Proposition 1.21. Let $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{C}$ be left exact functors with adapted classes $\mathbf{Q} \subset \mathbf{A}$ and $\mathbf{P} \subset \mathbf{B}$ such that

$$F(\mathbf{Q}) \subset \mathbf{P}. \tag{1.26}$$

Then the natural map $R(F \circ G) \to RF \circ RG$ is an isomorphism.

Theorem 1.22. Let $f : X \to Y$ be a continous map. For $\mathcal{F} \in D^-(Y)$ and $\mathcal{G} \in D^+(X)$, there are natural isomorphisms

$$R\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}), \tag{1.27}$$

and

$$\mathrm{Hom}_{D(X)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{D(Y)}(\mathcal{F}, f_*\mathcal{G}). \tag{1.28}$$

Proof. We can replace $\mathcal{G}$ by an injective resolution, because the category of sheaves has enough injectives. We know that the adjointness holds in the abelian category of sheaves. Then proposition 1.21 gives the first claim. Note that, using proposition 1.20, the first statement immediately implies the second. □

Remark 1.23. Generalising the theorem above, suppose we have $\mathcal{F} \in D^+(X)$ and left exact functors F, G, F', G' and we wish to prove

$$RF(RG(\mathcal{F})) \cong RF'(RG'(\mathcal{F})). \tag{1.29}$$

The plan involves two steps. First prove the statement at the level of abelian categories, i.e. there is a natural isomorphism

$$F(G(\mathcal{F})) \cong F'(G'(\mathcal{F})), \tag{1.30}$$

where $\mathcal{F}$. This implies for $\mathcal{F} \in D^+(X)$:

$$R(F \circ G)(\mathcal{F}) \cong R(F' \circ G')(\mathcal{F}). \tag{1.31}$$

Now we need to find some adapted class $\mathbf{P}$ for G and G' that fulfills the requirements of proposition 1.21. This proposition then gives us the above statement.
However, not all functors take injective sheafs to injective sheafs. We will now see another class.

Definition 1.24. A sheaf $\mathcal{F}$ is **flabby** if all restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ are surjective.

As expected, these turn out to be an adapted class for both the pullback and the push-forward functor.

Proposition 1.25. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then:

1. For $\mathcal{F} \in D(Z)$, there is a natural isomorphism $(g \circ f)^*\mathcal{F} \cong g^* f^* \mathcal{F}$.
2. For $\mathcal{F} \in D(X)$, there is a natural isomorpmhism $g_* f_* \mathcal{F} \cong (g \circ f)_* \mathcal{F}$.

Proof. We prove this following the outline of the above remark. In the abelian category case both statements follow immediately from the definitions. The pullback is exact, so (1) of course holds. For (2), we note, directly from the definition of the pushforward, that f_* sends flabby sheafs to flabby sheafs. Then (2) follows from proposition 1.21. □

Lemma 1.26. Let X be a topological space and $x \in X$. For $\mathcal{F} \in D^+(X)$, there is a natural isomorphism

$$\mathrm{H}^k(\mathcal{F}_x) \cong \varinjlim_{U \ni x} \mathrm{H}^k(R\Gamma(\mathcal{F}|_U)). \tag{1.32}$$

Example 1.27. (and exercise) Let $j : \mathbb{C}^\times \to \mathbb{C}$ be the inclusion. Then $(j_* \underline{\mathbb{C}}_{\mathbb{C}^\times})|_0 \cong R\Gamma(\underline{\mathbb{C}}_{\mathbb{C}^\times})$.
Using the above lemma, we have

$$\mathrm{H}^k((j_* \underline{\mathbb{C}}_{\mathbb{C}^\times})|_0) \cong \varinjlim_{U \ni 0} \mathbf{H}^k(U, (j_* \underline{\mathbb{C}}_{\mathbb{C}^\times})|_U), \tag{1.33}$$

where we adopt Achar's notation for hypercohomolgy, i.e. $\mathbf{H}^k(X, \mathcal{F}) = \mathrm{H}^k(R\Gamma(\mathcal{F}))$. Now notice that the restriction to open U is the pullback along the inclusion $i : U \to \mathbb{C}$. We have the commutative diagram,

$$\begin{array}{ccc} j^{-1}(U) & \xrightarrow{g} & \mathbb{C}^\times \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j} \\ U & \xrightarrow{i} & \mathbb{C}, \end{array}$$

with obvious maps. Thus the sheaf $j_* \underline{\mathbb{C}}_{\mathbb{C}^\times})|_U$ is the pushforward along f of the pullback along g. But the pullback of the constant sheaf is constant. We obtain

$$\mathbf{H}^k(U, (j_* \underline{\mathbb{C}}_{\mathbb{C}^\times})|_U) \cong \mathbf{H}^k(U, (f_* \underline{\mathbb{C}}_U)). \tag{1.34}$$

Note that we can assume U to be some disk centered at 0 in the above colimit. Therefore, with $f^{-1}(U) \simeq \mathbb{S}^1$ and using theorem 1.1.18 from Achar,

$$\mathbf{H}^k(U, (f_* \underline{\mathbb{C}}_U)) \cong \mathbf{H}^k_{sing}(\mathbb{S}^1, \mathbb{C}) \cong \mathbf{H}^k(\mathbb{C}^\times, \mathbb{C}). \tag{1.35}$$

Hence, the cohomology is

$$\mathbf{H}^k_{sing}(\mathbb{S}^1, \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } k = 0, 1 \\ 0, & \text{else} \end{cases} . \tag{1.36}$$