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Selected exercises from Kempf's *Algebraic varieties*

Todo list

This document contains solutions to (almost) every exercise in the 1999 printing of Kempf's *Algebraic varieties*. When not overly onerous to typeset, it has been endeavoured to make them complete. These solutions may of course contain errors or typos. Please report any of these to `stefand@math.utoronto.ca`. Some sections contain additional notes or exposition, mostly checking exercises left to the reader in the main text.

update version

1. **Algebraic varieties: definition and existence.**

1.1. **Spaces with functions.**

1.1.1. Let $f\colon X_1 \to X_2$ be continuous. Then if $V \subset X_2$ is open and $g\colon V \to k$ is continuous we have $f^*(g)(x) = (g \circ f)(x)\colon f^{-1}(V) \to k$ is continuous on an open subset, hence regular. Therefore f is a morphism.

1.1.2. The identity map $\mathrm{id}\colon X_1 \to X_1$ is continuous and $\mathrm{id}^*(g) = g$, so it pulls back regular functions to regular functions and is a morphism. If $\varphi\colon X_1 \to X_2$ and $\psi\colon X_2 \to X_3$ are morphisms, then $\psi \circ \varphi$ is continuous. If $f\colon V \subset X_3 \to k$, we have

$$(\psi \circ \varphi)^*(f)(v) = \varphi^*(\psi^*(f))(v)$$

for $v \in (\psi \circ \varphi)^{-1}(V)$ is regular. Therefore spaces with functions form a category.

1.1.3. Let $\iota\colon U \hookrightarrow X$ be inclusion of a subspace. Then $\iota^* g = g \circ \iota = g \restriction_U$ is regular if g is. Therefore ι is a morphism as it is continuous by definition. If $g\colon Y \to U$ is a morphism, by the above $\iota \circ g$ also is. If $\iota \circ g$ is a morphism, let f be regular on U, so that $f \circ \iota$ is regular on X. Then $(\iota \circ g)^*(f)(y) = f(\iota(g(y)) = f \restriction_U (g(y)) = (g^* \iota^* f)(y) = (g^* f)(y)$ is regular. Therefore g is a morphism.

1.1.4. Let $f\colon X \to Y$. Say f is a morphism. We have

$$\begin{tikzcd}
V_\alpha \arrow[r, hook, "\iota_\alpha"] & Y \\
U_\alpha \arrow[u, "f_\alpha"] \arrow[r, hook, "j_\alpha"] & X. \arrow[u, "f"']
\end{tikzcd}$$

Therefore if f is a morphism then $\iota_\alpha \circ f_\alpha$ is, so f_α is. If f_α is a morphism for all α, then let g be regular on X so that $g\restriction_{V_\alpha}$ is regular for all α. Then $f_\alpha^* g\restriction_{V_\alpha}$ is regular for all α. As $f_\alpha = f \circ j_\alpha$ we have

$$f_\alpha^* g\restriction_{V_\alpha} = (f \circ j_\alpha)^* g\restriction_{V\alpha} = j_\alpha^* f^* g\restriction_{V_\alpha},$$

so $f^* g\restriction_{V_\alpha}$ is regular on each U_α and is regular.

1.2. **Varieties.**

1.2.1. We claim $k[\mathbb{P}^1] = k$ for k algebraically closed. If $g(x) = p(x)/q(x)$ has no poles then $q \mid p$ as k is algebraically closed, and g is a polynomial. A polynomial with no pole at ∞ is constant.

1.2.2. Let f be regular on $U = \mathbb{A}^1 \backslash \{x_1, \ldots, x_n\}$. Then $f(x) = \frac{p(x)}{\prod(x-x_i)^{m_i}}$, so $D(f) = \{x \mid f(x) \neq 0\} = \mathbb{A}^1 \setminus \{\xi_1, \ldots, x_r\}$, the roots of p, which is open. Further, $\frac{1}{f} = \frac{\prod(x-x_i)^{m_i}}{p(x)}$ is regular.
If $U = \bigcup U_\alpha = \bigcup(\mathbb{A}^1 \setminus \{x_\alpha^1, \ldots, x_\alpha^{n_\alpha}\}$ and f is regular on U, then $f = \frac{p(x)}{\prod(x-y_i)}$ with the y_i among the x_j^α. Then f is regular on each U_α. Conversely if f_α has no poles in any U_α, f has no poles in U. Therefore $\mathbb{A}^1$ is a space with functions.

1.2.3. Use theorem 1.3.1. If $\mathbb{P}^1$ is affine, by the theorem $\mathbb{P}^1 = \operatorname{Spec} A$ and $A \simeq k[\mathbb{P}^1] = k$. but $\operatorname{Spec} k$ is a point where $\mathbb{P}^1$ is not; it contains at least $0, 1 \in k$ and ∞.

1.2.4. This is uniqueness of representing objects of representable functors: $X \simeq Y$ as affine varieties iff $k[X] \simeq k[Y]$. This uniqueness follows from the Yoneda lemma.

1.3. **The existence of affine varieties.**

1.3.2 todo

1.3.3 We have $0 \in \sqrt{0}$, and if $a, b \in \sqrt{0}$, then $a + b$ is, by the binomial theorem. If $a \in \sqrt{0}$ and $b \in A$, then as our rings are commutative, $(ab)^n = a^n b^n = 0 b^n = 0$ for some n. So $\sqrt{0}$ is an ideal. Clearly if $A = \{0\}$ then $A = \sqrt{0}$. Conversely, if $A = \sqrt{0}$ then for some n, $1^n = 1 = 0$, so A is the zero ring.

1.3.4 Let $A = k[X_1, \ldots, X_n] = K[X_1] \oplus \cdots \oplus k[X_n]$, and because $\operatorname{Hom}(\cdot, \cdot)$ is additive, $\operatorname{Spec} A = \operatorname{Hom}_{\mathbb{K}-\mathbb{A}<\eth}(A, k) = \bigoplus_i \operatorname{Hom}(k[X_i], k) = k^n$. Likewise if $A = k[X_1, \ldots X_n]/(f_1, \ldots, f_m)$ then $\operatorname{Spec} A = \{x \in k^n \mid f_i(x) = 0 \text{ for some } i\}$.

1.3.5 If $\{b_1, \ldots, b_n\}$ generate A, then $A_{(a)}$ is generated by $\frac{b_i}{a}$, so is finitely-generated. It has no nilpotents as if $b^n/a_n = 0$ in $A_{(a)}$ then there is m such that $a^m(b^n a - 0a^n) = b^n a^m = 0$, so that $(ba)^{\max m,n} = 0$ and $ba = 0$. then $b/a = 0$ in the localization.

1.3.6 First, $D(a_1) \cap D(a_2) = D(a_1 a_2)$ because if x is in the intersection, then $x(a_1) \neq 0$ and $x(a_2) \neq 0$, so $x(a_1 a_2) = x(a_1)x(a_2) \neq 0$ because fields are integral domains.
Clearly $\{D(A)\}_{a \in A}$ is a cover, in fact, $D(1) = \operatorname{Spec} A$. It now follows by elementary topology from the intersection property that we have a base.

1.4. **The Nullstellensatz.**

1.4.4 There are several useful versions of Nakayama's lemma. Its first real use will occur in chapter 5, and it will be used to prove results of the flavour "it's enough to check things about morphisms of vector bundles on fibres."

Lemma 1 (Nakayama's lemma for local rings)**.** *Let A be a local ring with maximal ideal I, M a finitely- generated A-module such that $M = I \cdot M$. Then $M = 0$.*

Proof. The proof is by induction on the number of generators. If M is generated by a single element then m then $m = im$ for some $i \in I$, and $i \neq 0$ implies $(1 - i)m = 0$, but $1 - i$ is a unit. In general if $m_1, \ldots, m_n$ be a set of generators. Then we have $m_1 = i_1 x_1$ for $x_1 \in M$ and $i_1 \in I$. Now $x_1 = \sum_j a_j m_j$ for $a_j \in A$, and so $m_1 = \sum_j i_j m_j$ for $i_j \in I$. If $a_1 \neq 0$, then

$(1-i_1)$ is a unit, and we have $(1-i_1)m_1 = \sum_{j\neq 1} i_j m_j$. In either case, M is generated by $n-1$ elements, and we are done by induction. □

Lemma 2 (Nakayama's lemma in **CRing**)**.** *Let A be any commuatative ring with identity, I be an ideal of A, and M be a finitely-generated A-module such that $M = I \cdot M$. Then there is $a \in 1 + I$ such that $aM = 0$.*

See the section 7.2.7 in Vakil's *The Rising Sea* for a proof of several version's of Nakayama's lemma, or §4.1 of Eisenbud's *Commutative Algebra with a View Toward Algebraic Geometry*.

1.4.9 We claim points in a variety are closed. If $x \in \operatorname{Spec} A$ is a point in an affine variety, then by maximality of the ideal $\ker x$, the Nullstellensatz says that $\{x\}$ is closed. Now if $X = \bigcup_{i=1}^n X_i$ is a variety with an affine open cover and $x \in X$, then without loss of generality $x \in X_1$ and is closed in this open set. Therefore there is $C \subset X$ closed such that $\{x\} = C \cap X_1$. Taking unions over the X_i, we can assume that $x \notin X_j$ for $j > 1$. Then $\{x\} = C \cap (\bigcup_{i=2}^n X_i)^c \subset C \cap X_1$ is a closed set in X.

1.5. The rest of the proof of existence of affine varieties/subvarieties.

1.5.1. todo

1.5.2. Affine varieties are functors $k-\mathbf{Alg} \to \mathbf{Set}$ and morphisms are natural transformations. The Yoneda lemma now says that the functor $\operatorname{Spec}\colon k-\mathbf{Alg} \to \mathbf{AffVar}^{\mathrm{op}}$ is fully-faithful, and it is obviously essentially surjective (it is by definition surjective on objects). Therefore we have an equivalence of categories.

1.5.3. We claim a subset $X \subset Y$ of a space with functions is naturally a space with functions, and the inclusion $\iota\colon X \hookrightarrow Y$ is a morphism. If X is a space with functions, let $V \subset Y$ be open, so that

$$\iota^*\colon k[V] \to k[\iota^{-1}(V)] = k[V \cap X]$$
$$f \mapsto f\restriction_{V\cap X}$$

but $V \cap X$ is open in X, and this is precisely what regular functions on X look like according to page 9 of Kempf.

Let $U = \bigcup_\beta U_\beta \subset X$. Say f is regular on U. Then there is $U = \bigcup_\alpha (X \cap V_\alpha)$ such that $f(y) = g_\alpha(y)$ for all $y \in X \cap V_\alpha$. So $U_\beta = \bigcup_\alpha (X \cap V_\alpha \cap U_\beta)$ and if $y \in X \cap V_\alpha \cap U\beta$, then $f(y) = g_\alpha(y)$, so $f\restriction_{U_\beta}$ is regular.

If all $f\restriction_{U_\beta}$ are regular, then for all β, $U_\beta = \bigcup_\alpha (X \cap V_{\alpha,\beta})$ such that $f_\beta(y) = g_{\alpha,\beta}(y)$ on $X \cap V_{\alpha,\beta}$ for regular $g_{\alpha,\beta}$. Then $U = \bigcup_{\alpha,\beta}(X \cap V_{\alpha,\beta}$ and restricted to any $X \cap V_{\alpha,\beta}$, $f = g_{\alpha,\beta}$ is regular, so f is regular on U.

Next we must show $\frac{1}{f}$ is regular on $D(f)$. Let $U = \bigcup_\alpha (X \cap V_\alpha)$ so that we have $f\restriction_{X\cap V_\alpha} = g_\alpha$ and g_α is regular on $V_\alpha \subset Y$. Then

$$D(f) = \bigcup_\alpha (D(g_\alpha) \cap X)$$

is open in X, and on $D(f) \cap X \cap V_\alpha$, $\frac{1}{g_\alpha} = \frac{1}{f\restriction_{X\cap V_\alpha}}$ is regular, and $D(f) = \bigcup_\alpha D(f) \cap V_\alpha \cap X$ is a union of open sets in X on which $\frac{1}{f}$ is regular, so $\frac{1}{f}$ is regular on $D(f) \subset X$.

1.5.5 We show that $\operatorname{Spec}(A/I)$ has the induced structure of a space with functions. That the Zariski topology on $\operatorname{Spec}(A/I)$ is the subspace topology coming from the Zariski topology on $\operatorname{Spec} A$ is the correspondence theorem for ideals (and in particular for prime/maximal ideals).

Let $U \subset \operatorname{Spec}(A/I)$ be open and $f \in k[U]$. Then we can cover U with sets $D([h_\alpha])$ for $[h_\alpha] \in A/I$, and we have

$$U = \bigcup_\alpha D([h_\alpha]) = \bigcup_\alpha \operatorname{Spec}(A/I) \cap D(h_\alpha).$$

For each α we have $f|_{D([h_\alpha])} = [g_\alpha]$ for some $[g_\alpha] \in (A/I)_{[h_\alpha]} = A_{h_\alpha}/I_{h_\alpha}$ (equality as rings). This shows that $\operatorname{Spec}(A/I)$ has the induced space with functions structure.

1.5.7 Let $X \subset Y$ be a subspace of a space with functions. We claim $Z \to X$ is a morphism iff $Z \to X \to Y$ is a morphism, for all Z. Let $\iota\colon X \hookrightarrow Y$. Then if $Z \to X$, the composite $Z \to X \hookrightarrow Y$ is a morphism. If $Z \to X \hookrightarrow Y$ is a morphism with $\varphi\colon Z \to X$, let f be regular on $U \subset X$. Then $U = \bigcup_\alpha (X \cap V_\alpha)$ and $f \restriction_{V_\alpha} = g_\alpha$ is regular on $V_\alpha \subset Y$. So

$$(\iota \circ \varphi)^* g_\alpha = (\iota \circ \varphi)^*(f \restriction_{V\alpha}) = \varphi^*(\restriction_{X \cap V_\alpha}) = (\varphi^* f) \restriction_{\varphi^{-1}(X \cap V_\alpha)}$$

is regular on $(\iota \circ \varphi)^* f \restriction_{V_\alpha} = \varphi^{-1}(X \cap V_\alpha)$. Then $\bigcup_\alpha \varphi^{-1}(X \cap V_\alpha) = \varphi^{-1}(U)$ and φ is a morphism because $\varphi^* f$ is regular.

1.6. $\mathbb{A}^n$ and $\mathbb{P}^n$.

1.6.2 If $\mathbb{A}^2 \setminus \{0\}$ was affine, we would have $\mathbb{A}^2 \setminus \{0\} \simeq \mathbb{A}^2$ as affine varieties, because they have the same coordinate algebras by the lemma. Let $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2 \setminus \{0\}$ be the isomorphism, and consider the isomorphism $\varphi^*\colon k[X_1, X_2] \to k[X_1, X_2]$. Considering the natural grading, we see φ^* is determined by

$$X_1 \mapsto a_1 X_1 + b_1 X_2 + c_1 \qquad X_2 \mapsto a_2 X_1 + b_2 X_2 + c_2.$$

We can recover φ by (abusing notation)

$$\varphi(X_1, X_2) = (\varphi^*(X_1), \varphi^*(X_2)) = (a_1 X_1 + b_1 X_2 + c_1, a_2 X_1 + b_2 X_2 + c_2) \neq 0.$$

That φ^* is surjective says that $\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \neq 0$, but this contradicts φ avoiding the origin.

1.6.4 We claim all morphisms $\mathbb{P}^n \to X$ where X is quasi-affine are constant. We have $\mathbb{P}^n \to X \hookrightarrow \mathbb{A}^m$, so it enough to show that all morphisms $\mathbb{P}^n \to \mathbb{A}^m$ are constant $\forall m, n$. For $n = 0$ the answer is obvious. Projecting onto the i-th coordinate of $\mathbb{A}^m$ gives $\mathbb{P}^n \xrightarrow{f} \mathbb{A}^m \to \mathbb{A}^1$ *i.e.* an element of $k[\mathbb{P}^n] = k$. Therefore f is constant.

Remark 1. Compare this to the proof that there an no compact complex submanifolds of $\mathbb{C}^n$.

1.6.5 We claim closed subsets $X \subset \mathbb{P}^n$ are precisely the sets $\{(X_0, \dots, X_n) \mid f_i(X_0, \dots, X_n) = 0 \ \forall i \in I\}$ and some index set I, where the f_i are homogeneous polynomials. Say $\{f_i\}_i$ are homogeneous polynomials and let X be defined by the above. Then

$$\pi^{-1}(X) = \left\{(Y_0, \dots, Y_n) \in \mathbb{A}^{n+1} \setminus \{0\} \,\middle|\, f_i(Y_0, \dots, Y_n) = 0 \ \forall i\right\},$$

and so $\pi^{-1}(X)$ is closed; it is $\bigcap_i \mathrm{zeros}(f_i)$. Conversely, say $X \subset \mathbb{P}^1$ is closed. Then $\pi^{-1}(X)$ is closed by definition and there are polynomials f_i such that

$$\pi^{-1}(X) = \left\{(Y_0, \dots Y_n) \,\middle|\, f_i(Y_0, \dots, Y_n) = 0 \ \forall i, \ \exists \lambda \in k^\times \text{ s.t. } (Y_0, \dots, Y_n) = \lambda(X_0, \dots, X_n)\right\}$$

where (X_i) is a point in X. The fact that $f_i(\lambda \mathbf{y}) = f_i(\mathbf{y}) = 0$ proves the homogeneous components of the f_i vanish on X. Conversely, if these components vanish, f_i vanishes. Therefore (in the notation of the lemma)

$$X = \{x = (X_0, \dots, X_n) \in \mathbb{P}^n \mid f_{i,j}(x) = 0 \ \forall 0 \leq j \leq \deg\ f_i, \ i \in I\}.$$

Lemma 3. *Let k be infinite and f be as above. Then each homogeneous summand of f vanishes on X.*

Proof. Let $\mathbf{x} \in X$ and let f_i denote (in constrast to the above) the degree i (homogeneous) component of f. Then

$$f(\lambda \mathbf{x}) = \lambda^{\deg f} f_{\deg f}(\mathbf{x}) + \cdots + \lambda f_1(\mathbf{x}) + f_0 = 0$$

for all $\lambda \in k$. That is, the polynomial

$$g_{\mathbf{x}}(t) = f_{\deg f}(\mathbf{x}) t^{\deg f} + \cdots + f_1(\mathbf{x}) t + f_0$$

has infinitely many zeros, hence is the zero polynomial. ∎

Remark 2. The $f_{i,j}$ should be exactly the generators of the homogeneous ideal defining X as a subvariety.

2. **The preparation lemma and some consequences.**

2.2 **The Hilbert basis theorem.**

2.2.1 We claim all ideals of a ring A are finitely-generated iff all submodules of finitely-generated A-modules are finitely- generated. Ideals of A are A-submodules of the finitely-generated A-module A, so the second condition implies the first. Conversely, if M is an A-module with one generator, then $M \simeq A/I$ for an ideal $I \subset A$ as A-modules. One checks that A-submodules of A/I are ideals of A/I, hence if $N \subset M$ is a submodule, then by the correspondence theorem, M is finitely-generated, as ideals of A are. Now let M be generated by $\{x_1, \dots, x_n\}$. If $N \subset M$ is a proper submodule than without loss of generality $x_n \notin N$. Therefore $N \subset \langle x_1, \dots, x_{n-1}\rangle$, and by induction the A-module $\langle x_1, \dots, x_{n-1}\rangle$ has all submodules finitely-generated.

2.3 **Irreducible components.**

2.3.6 We claim any topological space X is irreducible iff any non-empty open subset is dense. Let X be irreducible and $U \subset X$ nonempty and open. Then $X = \overline{U} \cup (X \setminus U) = U \cup (X \setminus U)$. Clearly $X \neq X \setminus U$, so $X = \overline{U}$. Conversely, let $X = X_1 \cup X_2$ be a union of closed subsets. If $X \neq X_1$, then $X \setminus X_1$ is nonempty open, hence dense. Therefore $\overline{X \setminus X_1} = \overline{X_2 \setminus X_1} = X$. Then $(X_2 \setminus X_1) \subset X_2$, and so $X \subset X_2$ and $X = X_2$.

2.3.7 Let X be irreducible and $U \subset X$ be open. We claim U is irreducible. Let $V \subset U$ be open, so V is open in X. Then $\overline{V} = X$ and so the closure of V in U is U. By the above, U is irreducible.

2.3.8 Let X be covered by open irreducible subsets. We claim X is connected iff it is irreducible. Say $X = \bigcup_\alpha U_\alpha$ as above and X is connected with $X = X_1 \cup X_2$ for closed X_i. Then $X = \bigcup_\alpha (X_1 \cap U_\alpha) \cup \bigcup_\alpha (X_2 \cap U_\alpha)$. and for each α, $U_\alpha = (X_1 \cap U_\alpha) \cup (X_2 \cap U_\alpha)$. These sets are closed in U_α so either $U\alpha \subset X_1$ or $U_\alpha \subset X_2$. Therefore

$$X_i = \bigcup_{\substack{\alpha \\ U_\alpha \subset X_i}} U_\alpha$$

and the X_i are open and closed. Therefore $X = X_1$ or $X = X_2$. Conversely, irreducible spaces are always connected.

2.3.9 Let $\{U_i\}$ be an open cover of a Noetherian space X. We claim the components of X are precisely the closures of the components of the U_i. Let Y be a component of some $U = U_i$. Then $\overline{Y} \subset X$ is closed and irreducible. If $\overline{Y} \subset Z \subset X$ for a component Z, then

$$Y = \overline{Y} \cap U \subset U \cap Z \subset X,$$

and $U \cap Z$ is irreducible because it's open in Z, and closed in U_i as Z is closed. Therefore $Y = U \cap Z$ so $\overline{Y} = \overline{U \cap Z} = Z$ as open sets are dense in Z. This says Y is an component. Conversely, if Y a component of X, write $U_i = Z_{i,1} \cup \cdots \cup Z_{i,1}$. Then $Y \cap U_i$ is irreducible and closed in U_i. If Y is not a component of U_i, by 2.3.4 without loss of generality $Y \cap U_i \subset Z_{i,1}$. By irreducibility of Y, $\overline{Y \cap U_i} = Y \subset \overline{Z_{i,1}}$, so $Y = \overline{Z_{i,1}}$ and Y is the closure of a component of U_i. If $Y \cap U_i$ is a component of U_i, then $Y = \overline{Y \cap U_i}$ anyway.

2.4 **Affine and finite morphisms.**

2.4.2 (a) We claim the inclusion $i \colon X \hookrightarrow Y$ of a closed subvariety is finite. Let $Y = \bigcup_{i=1}^n U_i$ with $U_i = \operatorname{Spec} A_i$, so that $i^{-1}(U_j) = U_j \cap X$ is a closed subspace of U_j, hence affine. We have

$$k[U_j \cap X] = k[\operatorname{Spec}(A_i/I_i)] = A_i/I_i,$$

where I_i is the ideal of vanishing for $X \cap U_i$. Therefore i is finite, as A_i/I_i is a finitely-generated A-module.

(b) Let f be a regular function on X. We claim $i\colon D(f) \hookrightarrow X$ is an affine morphism. Write $X = \bigcup_{i=1}^n U_i$ via an affine open cover, so that $i^{-1}(U_i) = D(f) \cap U_i = \{x \in X \mid f(x) \neq 0 \ \forall x \in \operatorname{Spec} A_i\}$ is open in $U_i = \operatorname{Spec}(A_i)$. This set is precisely $\operatorname{Spec}((A_i)_{f\restriction_{U_i}})$, where $f \restriction_{U_i}$ is the element of A_i defined by the map ϕ and the regular function $f \restriction_{U_i}$.

2.5 **Dimension.**

2.5.6 Let $X = \bigcup_{i=1}^n U_i$ be a variety with an affine cover. We claim $\dim X = \max_i \dim U_i$. Clearly $\dim U_i \leq \dim X_i$, so $\max \dim U_i \leq \dim X$. Say

$$X_p \subsetneq \cdots X_1 \subsetneq X_0 \subset X$$

is a chain in X. Then $X_i \cap U$ is irreducible, and closed in U for any U such that $U \cap X_p \neq \emptyset$ (by assumption, one of the U_i will do). Thus we have

$$X_p \cap X \subsetneq \cdots \subsetneq X_0 \cap U,$$

and $\dim X \leq \dim U \leq \max_i \dim U_i$, whence equality.

2.5.7 Let $Z \subset X$ be a closed subset of an irreducible variety. We claim $\dim Z = \dim X$ iff $Z = X$. Say $\dim Z = \dim X = p$ but $Z \subsetneq X$. There is therefore a chain

$$Y_p \subsetneq Y_{p-1} \subsetneq \cdots \subsetneq Y_0 \subseteq Z$$

in Z, and as X is irreducible, we get the chain

$$Y_p \subsetneq Y_{p-1} \subsetneq \cdots \subsetneq Y_0 \subsetneq X$$

of length $p+1$ in X. This is a contradiction. The converse is obvious.

2.6 **Hypersurfaces and the principal ideal theorem.**

2.6.6 (The converse to the principal ideal theorem is false.) Considering $H = \big\{(X,Y,Z) \,\big|\, XY - Z^2 = 0\big\} \subset \mathbb{A}^3$, we see H includes the line $L = \{(X,Y,Z) \mid X = Z = 0\}$. (Carefully, this is noting that $(XY - Z^2) \subset (X, Z)$ and the ideal (X, Z) is the L.) Note $g(X,Y,Z)) = XY - Z^2$ is regular, so H is a hypersurface of with $\dim H = 2$ and $\dim L = 1 = 3-2$. In $k[H] = k[X,Y,Z]/(XY - Z^2)$, $I(L)$ is generated by $\bar{X}$ and $.\bar{Z}$, and this ideal is not principal (consider the natural grading on the quotient).

touch up?

3. Products; separated and complete varieties.

3.1. **Products.**

3.1.2

3.1.4 Let X and Y be quasi-affine. Then for some m, n, we have $X \subset U \subset \mathbb{A}^n$ and $Y \subset V \subset \mathbb{A}^n$. We have $X \times Y \subset U \times V \subset \mathbb{A}^n$, and $U \times V$ is open in product $\mathbb{A}^n \times \mathbb{A}^m$. Thus $X \times Y$ is locally closed,and $X \times Y$ is quasi-affine.

3.1.5 We claim $\mathbb{A}^n \simeq \mathbb{A}^1 \times \cdots \mathbb{A}^1$. It is enough to show that $k[X] \otimes_k k[Y] \simeq k[X,Y]$. Define $k[X] \times k[Y] \to k[XY]$ by $(P,Q) \mapsto PQ$. This is bilinear, so induces a map out of the tensor product. It is easy to see this map is a bijection.

3.1.6 We claim $\dim(X \times Y) = \dim X + \dim Y$ for varieties X and Y. If this holds for affine varieties, then by the proof of 3.1.3, $\dim X_i \times Y_j = \dim X_i + \dim Y_j$, so

$$\dim(X \times Y) = \max_{i,j} \dim(X_i \times Y_j) = \max_i \dim X_i + \max_j \dim Y_j = \dim X + \dim Y$$

by the last section. For affine varieties, we must show that there is a finite surjective morphism $X \twoheadrightarrow \mathbb{A}^{\dim X}$, so that $X \times Y \twoheadrightarrow \mathbb{A}^{\dim X + \dim Y}$, after which we appeal to the claim for affine spaces. The claim for affine spaces follows from 3.1.5.

To prove the claim, note by the Noether normalization lemma, we can find $k[X_1, \ldots, X_{\dim X}] \hookrightarrow k[X]$ an integral extension, so that $X \twoheadrightarrow \mathbb{A}^{\dim X}$ is a surjective finite morphism. Therefore $X \times Y \twoheadrightarrow \mathbb{A}^{n+m}$ is also surjective of finite type, and so $\dim(X \times Y) = \dim \mathbb{A}^{n+m} = n + m$ by 3.1.2 and 2.5.2 taken together.

If X and Y are irreducible and that $X \times Y = Z_1 \cup Z_2$. We have morphisms $\iota_y \colon X \to (X \times Y) := Z$ sending $x \mapsto (x, y)$, so $X_i := \bigcap_{y \in Y} \iota_y^{-1}(Z_i) \subset X$ is closed. Now, $\{x\} \times Y$ is irreducible, and for all x we have $x \times Y = ((x \times Y) \cap Z_1) \cup ((x \times Y) \cap Z_2)$. Therefore $x \times Y \subset Z_1$ or Z_2. That is, $x \times Y \subset Z_1 \cup Z_2$, whence without loss of generality $X = X_1$. Then $Z = Z_1$.

3.2. Products of projective varieties.

3.3. Graphs of morphisms and seperatedness

3.3.1. Obviously $(f, 1_Y)^{-1}(\Delta_Y)^{-1} = \{(x, y) \mid f(x) = y\}$, which is the claim.

3.3.3 The Segre embedding is given explicitly as

$$S \colon ([X_0 : X_1 : \cdots : X_n], [Y_0 : \cdots : Y_n] \to [X_0Y_0 : X_0Y_1 : \cdots : X_1Y_0 : \cdots : X_iY_j : \cdots : X_nY_n]$$

and so when $(X) = (Y)$, $X_i = Y_i$ and $X_iY_j = Y_iX_j$. Therefore $\Delta \mathbb{P}^n = S(\mathbb{P}^n \times \mathbb{P}^n) \cap \{Z_{ij} = Z_{ji}\}$ in $\mathbb{P}^{nm+n+m}$.

3.3.6 Let $f, g \colon X \to Y$ be morphisms, where Y is separated. If $f = g$ on an open dense set, $f = g$ everywhere. Let U be open dense in X and consider $\varphi \colon X \to Y \times Y$ given by $x \mapsto (f(x), g(x))$. Then $\varphi(U) \subset \Delta_{Y \times Y}$ and φ is continuous, so

$$\varphi(\bar{U}) = \varphi(X) \subset \overline{\varphi(U)} \subset \overline{\Delta_Y} = \Delta_Y,$$

and $f = g$ on all of X.

3.3.7 Let $h \colon O \to Y$, where $O \subset X$ is dense open, and Y is separated. We claim there is a maximal open set to which h extends as a morphism, and that this extension is unique. Build the usual partial order by "extends" on pairs (U, f) where $U \supset O$ is open and f extends h. Given a chain, taking the union gives an upper bound. By Zorn's lemma, there is a maximal element (U, f). Thus h has at least one maximal extension. Given another one (U, f'), we have $f' = h = f$ on O which is dense in particular in U, so $f = f'$ is unique.

3.5 Cones and projective varieties.

3.5.2 Mimicking the proof of the claim in the proof of lemma 1.6.1, one shows that $D(f)$ is isomorphic to $\mathrm{Spec}(k[C(X)]_{(f)})_{\text{degree } 0}$.

3.7 Complete varieties.

3.7.1 Let X be complete and $Z \subset X$ be closed. Then

$$\Delta_Z = \Delta_X \cap (Z \times Z)$$

as before and Z is separated. Given closed $Z' \subset Z \times Y$, note that $Z \times Y \subset X \times Y$ is closed, so that $Z' \subset X \times Y$ is closed. Thus $\pi_Y(Z')$ is closed in Y, and this is equal to the projection from $Z \times Y$.

Say X is complete and Y is separated, and $\varphi \colon X \to Y$ is a morphism. First we show that $\varphi(X)$ is separated. By lemma 3.3.2, $\mathrm{graph}(\varphi \times \varphi)$ is closed in $X \times Y \times Y$, because $Y \times Y$ is separated (which holds by the same lemma). As X is complete, $\pi_{Y \times Y}(\mathrm{graph}(\varphi \times \varphi)) = \Delta_{\varphi(X)} \subset Y \times Y$ is closed in $Y \times Y$, and by the same logic, $\varphi(X)$ is closed in Y. Therefore $\Delta_{\varphi(X)} \subset \varphi(X) \times \varphi(X)$ is closed.

Now let W be any variety and let $Z \subset \varphi(X) \times W$ be closed. Consider the morphism

$$\psi = \varphi \times \mathrm{id} \colon X \times W \to \varphi(X) \times W.$$

Note that $\psi^{-1}(Z)$ is closed in the source, and that $\pi_W(\psi^{-1}(Z)) = \pi_W(Z)$ is closed in W.

Remark 3. Being separated is a Hausdorff-like condition, and being complete is therefore a compact-like condition.

3.7.3 Let $X = U \cap C \subset \mathbb{A}^n$ be a quasi-affine complete variety. Then we claim X is a finite set. It is enough to show that irreducible quasi-affine complete varieties are points and then use that varieties are Noetherian spaces. Consider a coordinate X_i on $\mathbb{A}^n$. Then if $k[C] = k[\mathbb{A}^n]/I$, consider $[X_i] \restriction_U$, which is a regular function on an irreducible complete variety, hence is constant. This holds for each i, so the components of X are points.

Remark 4. This also is reminiscent of the classification of embedded compact complex submanifolds of $\mathbb{C}^n$.

Remark 5. Abelian varieties, defined in the next section of Kempf, are irreducible complete algebraic groups, in particular, varieties. This gives a source of examples of algebraic groups which are not linear, by the last exercise.

3.10 **Blowup of $\mathbb{A}^n$ at the origin.**

3.10.1 (a) We claim $Z = \{(p, \ell) \mid p \in \ell\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ is a closed subvariety. Note that $Z^c = \{(p, \ell) \mid p \notin \ell\} \subset (\mathbb{A}^n \setminus \{0\}) \times \mathbb{P}^{n-1}$ is the preimage of $\Delta^c_{\mathbb{P}^{n-1}}$ under the map $f\colon \mathbb{A}^n \setminus \{0\}) \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ sending $(p, \ell) \mapsto ([p], \ell)$. As $\mathbb{P}^{n-1}$ is separated, Z^c is open in $(\mathbb{A}^n \setminus \{0\}) \times \mathbb{P}^{n-1}$, hence is open in $\mathbb{A}^n \times \mathbb{P}^{n-1}$. Therefore Z is closed.

(b) We claim the projection $\pi_{\mathbb{A}^n}\colon Z \to \mathbb{A}^n$ is birational. Let $V = \mathbb{A}^n \setminus \{0\}$ and $U = Z \cap (\mathbb{A}^n \setminus \{0\} \times \mathbb{P}^{n-1}$. Then both these sets are open and dense in their respective supersets. If $p \neq 0$, then p and 0 determine a unique line in $\mathbb{A}^n$, so $\pi_{\mathbb{A}^n}$ is an isomorphism on between U and V. The inverse maps $p \mapsto (p, [p])$.

(c) We claim $(Z, \pi_{\mathbb{P}^{n-1}})$ is a locally trivial line bundle over $\mathbb{P}^{n-1}$. Note that Z is actually the tautological bundle over $\mathbb{P}^{n-1}$, so it is locally trivial: take U in Z such that orthogonal projection is an isomorphism from lines in U to $\mathbb{A}^1$ *i.e.* U is defined by nonvanishing of one of the coordinates X_i on $\mathbb{A}^n$. For example in $\mathbb{A}^2_{\mathbb{R}}$ if $\ell = \{x = y\}$ and we project onto the x-axis, the neighbourhood of U on which $\pi_{\mathbb{P}^1_{\mathbb{R}})}$ is trivial $D(y) \subset \mathbb{P}^2_{\mathbb{R}}$ *i.e.* we remove only the vertical line.

(d) The exceptional divisor $E = \pi^{-1}_{\mathbb{A}^n}(\{0\}0$ is the image of the zero section of the bundle in (c). So locally Z looks like $\mathbb{A}^1 \times D(X_i)$ for some coordinate X_i of $\mathbb{A}^n$. Then locally E is defined by the equation $X_i = 0$. (Note $D(X_i) \subset \mathbb{P}^{n-1}$ translates to the condition that $X_i \not\equiv 0$, and not $X_i \neq 0$.)

4. Sheaves.

4.1. The definition of presheaves and sheaves.

4.1.4 Let $X = U \sqcup V$ with the topology $\{\emptyset, U, V, X\}$, where U and V are nonempty sets. Define $\mathcal{F}(U) = \{*\}$ and $\mathcal{F}(V) = \{*\}$, and $\mathcal{F}(X) = \{a, b\}$ with $a \neq b$ such that $a \restriction U = *$ and $b \restriction U = *$. There is no smaller neighbourhood of X. Therefore $a_x = b_x$, and likewise with points in V. But $a \neq b$ in $\mathcal{F}(X)$. Therefore $\mathcal{F}$ is not decent. Presheaf axiom (b) is vacuous, and (a) holds by construction.

4.1.5 Let $\mathcal{F}$ be the presheaf of bounded functions on $\mathbb{R}$. Clearly $\mathcal{F}$ satisfies the uniqueness sheaf axiom; functions that agree pointwise agree. Setting $f_n(x) = x\chi_{[n,n+1]}(x)$, each f_n is bounded, and they satisfying the patching condition, but the function that they assemble to is $f(x) = x$, which is not bounded on $\mathbb{R}$.

4.1.6 Let $\mathcal{F}$ be the holomorphic functions on a domain $U \subset \mathbb{C}$ such that $z\frac{df}{dz} = 1$. Clearly $\mathcal{F}$ is a presheaf under restriction of functions, with uniqueness as U is connected. If $U = \bigcup_\alpha U_\alpha$ is a cover with the patching condition satisfied, then the f_α glue to a function on U which is holomorphic as this is a local property. If $z_0 \in U$, then $z_0 \in U_\alpha$ for some α and on U_α,

$$z \lim_{w \to 0} \frac{f(w) - f(z_0)}{w - z_0} = z \restriction_{U_\alpha} \lim_{\substack{w \to 0 \\ w \in U_\alpha}} \frac{f_\alpha(w) - f(z_0)}{w - z_0} = 1.$$

Therefore on $U_\alpha \cap U_\beta$, $z \restriction_{U_\alpha} \frac{df_\alpha}{dz} = 1 = z \restriction_{U_\beta} \frac{df_\beta}{dz}$, so by gluing and uniqueness applied to the holomorphic functions $z \restriction_{U_\gamma} \frac{df_\alpha}{dz}$, $z\frac{df}{dz} = 1$ on U.

4.1.7 For $\mathcal{F}$ from 4.1.6, we claim $\mathcal{F}_0$ is empty, and all the other stalks are (non-canonically) isomorphic to $\mathbb{C}$. Indeed, there are no holomorphic functions such that $z\frac{df}{dz} = 1$ on a set containing $z = 0$, so the stalk at 0 is empty. Away from zero, we claim that $f \mapsto f(w)$ descends to a bijection $\mathcal{F}_w \to \mathbb{C}$. Indeed, the holomorphic functions are their own Taylor series, and the differential equation means that $f^{(n)} = (-1)^{n-1}z^{-n}$. Therefore in a small neighbourhood of w, the Taylor differ only in the leading term, *i.e.* are determined totally by $f(w)$.

4.1.8 Let σ and τ be sections in $\mathcal{F}(U)$ for decent $\mathcal{F}$. We claim $V := \{x \in U \,|\, \sigma_x = \tau_x\}$ is open. Notice that V is just the complement of the support of $\sigma - \tau$, and the support is closed. Therefore V is open.

4.1.9 We collect some examples of sheaves where V is closed. A vacuous example is $\mathcal{F}$ from 4.1.7: it follow from 4.1.8 that V is open in U, and we know that $f_w = g_w$ iff $f(w) = g(w)$, so if $f \neq g$, V must be empty. Hence it is closed. If $f = g$ it is all of U, which is closed (in U) as well.

4.2. **The construction of sheaves.** The sheafification procedure given in Kempf if unusual, so I will use the usual presentation in this section. Just about nowhere are the details written down, so I will write them down now.

Proposition 1 (Sheafification.)**.** *Let $\mathcal{F}$ be a presheaf. Then there is a unique sheaf $\mathcal{F}^\#$ with a morphism $\theta\mathcal{F} \to \mathcal{F}^\#$ of presheaves such that for any morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of presheaves with $\mathcal{G}$ a sheaf, there is a unique morphism of sheaves $\tilde{\alpha}$ such that $\alpha = \tilde{\alpha} \circ \theta$. If $\mathcal{F}$ is a sheaf already, $\mathcal{F}^\# \simeq \mathcal{F}$. A sheaf and its sheafification have the same stacks, and θ induces an isomorphism of them.*

Proof. Uniqueness follows as the sheafification solves a universal mapping problem. This implies the last sentence. For existence, put

$$\mathcal{F}^\#(U) = \left\{ f\colon U \to \coprod_{x \in U} \mathcal{F}_x \,\middle|\, \forall x \in U \;\exists V \ni x \text{ and } s \in \mathcal{F}(V) \text{ s.t. } f(y) = s_y \;\forall y \in V \right\}.$$

Note that uniqueness is obvious; the sections of $\mathcal{F}^\#$ are functions. Likewise they glue automatically; we need only check the glued function satisfies the criterion still. Say f is glued from functions f_α. Then if $x \in U$, $x \in U_\alpha$ for some α and $f(x) = f_\alpha(x)$. Therefore there is a smaller neighbourhood W of x such that $f(y) = f_\alpha(y) = s_y$ for a section s in $\mathcal{F}(W)$. This shows gluing holds.

Define $\theta(U)(s) = x \mapsto s_x$. We claim evaluation $f \mapsto f(x) \in \mathcal{F}_x$ gives an isomorphism $\mathcal{F}^\#_x \to \mathcal{F}_x$. Clearly if f and g in $\mathcal{F}^\#(U)$ are in the same germ, they agree at x. Therefore the evaluation is well-defined. Clearly evaluation is surjective; if $s_x \in \mathcal{F}_x$ and $s \in \mathcal{F}(V)$ take $f\colon y \mapsto s_y$ in $\mathcal{F}^\#(V)$. For injectivity, if $f(x) = g(x)$ then on a small neighbourhood V of x, $f(y) = s^f_y$ and $g(y) = s^g_y$ for some sections s^f and s^g such that $s^f_x = s^g_x$, so these sections agree on a small neighbourhood, and f and g on a small neighbourhood. This also shows θ induces an isomorphism on stalks.

Here is the key insight in verifying the universal property: if $\mathcal{G}$ is a sheaf, we have now shown that $\mathcal{G} \simeq \mathcal{G}^\#$, via θ. Therefore is is enough to get a morphism into $\mathcal{G}^\#$. Given $\alpha(U)$, we get maps $\{\alpha_x\}_{x \in U}$, and let $\tilde{\alpha}(U)(f)$ be the composite

$$U \xrightarrow{f} \coprod_{x \in U} \mathcal{F}_x \longrightarrow \coprod_{x \in U} \mathcal{G}_x.$$

For all $x \in U$ there is V and $s \in \mathcal{F}(V)$ such that $f(y) = s_y$, so that $\tilde{\alpha}(U)(f)(x) = (\alpha(V)(s))_x$ and $\alpha(V)(s) \in \mathcal{G}(V)$. If $f = \theta(U)(s)\colon x \mapsto s_x$, then $\tilde{\alpha}(U)(f)(x) = \alpha_x(s_x)$ and $\alpha(U)(s) \in \mathcal{G}(U)$ is in this language $x \mapsto (\alpha(U)(s))_x$. Therefore the triangle commutes. $\square$

4.2.4 Let $\mathcal{F}(U) = k$with identity restriction maps. We claim this sheaf is isomorphic to the sheaf of constant k-valued functions. This is obvious. Further, we claim the sheafification $\mathcal{F}^\#$ is the (sheaf) of locally constant k-valued functions on X. Uniqueness obviously holds for $\mathcal{F}^{\#}$, and if a family of sections satisfies the patching condition, we can *define* a locally constant function using them, in the obvious way. Therefore $\mathcal{F}^{\#}$ is a sheaf. Let $\mathcal{G}$ be any sheaf with a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$. Given a locally constant function f we get constant functions f_i on connected components U_i of U. Define $\tilde{\alpha}(U)(f)$ to be unique element of $\mathcal{G}$ that restricts to $\alpha(U_i)(f_i)$ on $U_i \subset U$. Note this section exists as the gluing hypothesis is vacuous; connected components do not intersect.

4.2.5 This is tautological, if $\mathcal{G}$ is decent.

4.2.6 This contained in the proposition above.

4.3. Abelian sheaves and flabby sheaves.

4.3.4 We claim the presheaf $\mathcal{F}/\mathcal{G}$ is always decent. It is enough to show that if $\tau_y = 0$ for all $y \in V$, then $\tau = 0 \in \mathcal{F}(V)/\mathcal{G}(V)$. We have $W_y \ni y$ open in V such that $\tau\restriction_{W_y} = 0$. Thus we have $\tau\restriction_{W_y} \in \mathcal{G}(V)$ for all $y \in V$. Therefore $\tau \in \mathcal{G}(V)$ and τ is 0 in the quotient.

4.3.5 (a) It is obvious that $d\colon C^\infty \to \Omega$ is a group homomorphism for every open U; $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are linear, and d commutes with restrictions as partial derivatives are determined locally.

(b) Any section in $I\Omega(U)$ is gotten by gluing elements from $d(C^\infty(V))$ for some $V \hookrightarrow U$ by definition. Therefore the elements of $I\Omega(U)$ are locally of the form df for $f \in C^\infty(V)$.

(c) Let $\mathbb{R}^{\#}$ be the sheaf of locally constant functions. Then

$$0 \longrightarrow \mathbb{R}^{\#} \longrightarrow C^\infty \xrightarrow{d} I\Omega \longrightarrow 0$$

is exact. The only thing to note is that $df = 0$ implies only that f is locally constant (note that U need not be connected).

(d) See examples 4.7 and 4.26 in Lee's *Differential Manifolds* for a 1-form that is locally exact but not exact.

4.3.6 Given a diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d] & \mathcal{G} \arrow[d] \\ \mathcal{F}' \arrow[r, hook] & \mathcal{G}', \end{tikzcd}$$

we extend to a diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \text{pre-}(\mathcal{F}/\mathcal{G}) \arrow[r] \arrow[d] & \mathcal{G}/\mathcal{F} \arrow[d] \\ \mathcal{F}' \arrow[r, hook] & \mathcal{G}' \arrow[r] & \text{pre-}(\mathcal{G}'/\mathcal{F}') \arrow[r] & \mathcal{G}/\mathcal{F}'. \end{tikzcd}$$

For any open U, we naturally get morphisms of presheaves into the quotient presheaves, and the horizontal morphisms into the quotient sheaves exist by definition. Therefore we have a morphism of presheaves pre-$(\mathcal{G}/\mathcal{F}0 =) \to \mathcal{G}'/\mathcal{F}'$, and so we get the last vertical morphism in the category of sheaves out of $\mathcal{G}/\mathcal{F}$ via the universal property.

4.3.7 We complete an exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d, hook] & \mathcal{F}_2 \arrow[r] \arrow[d, hook] & \mathcal{F}_3 \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{G}_1 \arrow[r, hook] & \mathcal{G}_2 \arrow[r] & \mathcal{G}_3 \arrow[r] & 0 \end{tikzcd}$$

of abelian sheaves to an exact diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{F}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[r] \arrow[d] & \mathcal{G}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}_1/\mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{G}_2/\mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{G}_3/\mathcal{F}_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}
$$

We get the morphisms of sheaves from exercise 4.3.6, and we can check exactness on stalks. But it is well-known that this procedure in **Ab** produces an exact diagram.

Proposition 2. *stalk-exactness equivalence*

Proof. todo

∎

5. **Sheaves in algebraic geometry.**

5.1. **Sheaves of rings and modules.**

5.1.1. Let $\mathcal{M}$ be a sheaf of $\mathcal{A}$-modules. We have a sheaf $\operatorname{Sym}^n \mathcal{M}$ defined to be the sheafification of the presheaf $U \mapsto \operatorname{Sym}^n_{\mathcal{A}(U)} \mathcal{M}(U)$. If $\mathcal{M}$ is locally free, then let $X = \bigcup_\alpha U_\alpha$ be an open cover such that $\mathcal{M} \restriction_{U_\alpha} \simeq \mathcal{A} \restriction_{U_\alpha}^{\oplus I_\alpha}$. Note that in this case $\simeq^n_{\mathcal{A}U_\alpha} \mathcal{M}(U_\alpha)$ and its restrictions are all direct sums of (sections of) $Aa \restriction_{U_\alpha}$. Therefore the presheaf Sym^n agrees with its sheafification when restricted to U_α, and is a locally free sheaf. The same goes through for the exterior algebra.

5.1.2. Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of $\mathcal{A}$-modules. Define $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U)$. The restriction maps for this presheaf work as follows. If $V \hookrightarrow U$, then we have $\mathcal{A}(U) \to \mathcal{A}(V)$ and we get a map of $\mathcal{A}(U)$-modules

$$\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U) \to \mathcal{M}(V) \otimes_{\mathcal{A}(V)} \mathcal{N}(V)$$

induced from the $\mathcal{A}(U)$-bilinear map $\mathcal{M}(U) \times \mathcal{N}(U) \to \mathcal{M}(V) \otimes_{\mathcal{A}(V)} \mathcal{M}(V)$ sending $(m, n) \mapsto m \restriction_V \otimes n \restriction_V$. The bilinearity follows from the compatibility of scalar multiplication and restriction.

Define $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})(U) = \operatorname{Hom}(\mathcal{M} \restriction_U, \mathcal{N} \restriction_U)$, where the right-hand side is morphisms of sheaves of $\mathcal{A} \restriction_U$-modules. This will be a sheaf whenever the target is a sheaf; $\mathcal{M}$ need only be a presheaf. The restriction maps are defined simply by, if $V \hookrightarrow U$, then send

$$
\begin{tikzcd}
\operatorname{Hom}(\mathcal{N} \restriction_U, \mathcal{M} \restriction_U) \arrow[r] & \operatorname{Hom}(\mathcal{N} \restriction_V, \mathcal{M}_\restriction V) \\
\alpha \arrow[r, mapsto] & \alpha \restriction_V .
\end{tikzcd}
$$

Say that $\alpha \in \mathcal{H}om_{\mathcal{A}}(\mathcal{N}, \mathcal{M})(U)$, $U = \bigcup_\alpha U_\alpha$ such that $\alpha \restriction_{U_\alpha} = 0 \ \forall \alpha$. Let $n \in \mathcal{N}(U)$. Then we see that, for all α,

$$(\varphi(U)(n)) \restriction_{U_\alpha} = \varphi(U_\alpha)(n \restriction_{U_\alpha}) = 0$$

for all n. So $\varphi(n) = 0 \in \mathcal{M}(U)$. Taking $V = \bigcup_\alpha (V \cap U_\alpha)$ for all $V \hookrightarrow U$ open, we see that $\varphi = 0$ in the hom sheaf. For gluing, say U is as above and we have a family of φ_α such that $\varphi_\alpha \restriction_{U_{\alpha\beta}} = \varphi_\beta \restriction_{U_{\alpha\beta}}$. If $V = \bigcup_\alpha (V \cap U_\alpha)$ is an open cover of V, and if $n \in \mathcal{N}(V)$, we have

$$(\varphi_\alpha(n)) \restriction_{V \cap U_\alpha \cap U_\beta} \varphi_\alpha \restriction_{V \cap U_\alpha} (n \restriction_{V \cap U_\alpha \cap U_\beta}) = \varphi_\beta \restriction_{V \cap U_\alpha} (n \restriction_{V \cap U_\alpha \cap U_\beta}) = (\varphi_\beta(n)) \restriction_{V \cap U_\alpha \cap U_\beta}$$

so that $\{\varphi_\alpha(n)\}$ glues in $\mathcal{M}(V)$. We must check that it glues to a morphism of $\mathcal{A}(V)$-modules, but this follows easily. If $r \in \mathcal{A}(V)$ then for α, we have $(r\varphi(n)) \restriction_{U_\alpha \cap V} = r \restriction_{U_\alpha \cap V} \varphi(n) \restriction_{V \cap U_\alpha}$, while also $(\varphi(rn)) \restriction_{V \cap U_\alpha} = \varphi \restriction_{U_\alpha \cap V} ((rn) \restriction_{U_\alpha \cap V}) = r \restriction_{U \cap \cap V} \varphi \restriction_{U_\alpha \cap V} (r \restriction_{U_\alpha \cap V})$. By uniqueness in the target sheaf we have $\varphi(rn) = r\varphi(n)$. The other properties are verified with similar arguments.

5.1.3. Let U be a set in an open cover of X such that $\mathcal{M}|_U \simeq \mathcal{O}_U^{\oplus n}$. Then $(\bigwedge^n \mathcal{O}_U^{\oplus n})(V) \simeq \mathcal{O}_U(V)$ for all $V \subset U$ as $\mathcal{O}_U$-modules. Therefore $\det \mathcal{M}$ is invertible. Recall that $\bigwedge(M \oplus N) = \bigwedge(M) \otimes_A \bigwedge(N)$ for free A-modules M and N over a commutative ring A. The given short exact sequence is "locally split" as $\mathcal{O}_U^{\oplus n}$ is a free $\mathcal{O}_U$-module and so all Ext groups are trivial. Therefore locally $\mathcal{M}_2 \simeq \mathcal{M}_1 \oplus \mathcal{M}_3$ and so $\det(\mathcal{M}_2) \simeq \det \mathcal{M}_1 \otimes_{\mathcal{A}} \det \mathcal{M}_3$ because this equality holds as sheaves on the base of topology given by affine opens, for example.

5.3 **Coherent sheaves.**

5.3.2 We claim a quasi coherent $\mathcal{O}_X$-submodule $\mathcal{M}$ of a coherent $\mathcal{O}_X$-module $\mathcal{N}$ for a variety X is actually coherent. Without loss of generality $X = \operatorname{Spec} A$ is affine where A is Noetherian. By proposition 5.2.2, $\mathcal{M} \simeq \tilde{M}$ for an A-module M, and likewise we get $N \supset M$. By proposition 5.3.1, N is finitely-generated and it is enough to show the same of M. But A is Noetherian.

The vector bundles $\mathcal{O}_{\mathbb{P}^n}(m)$. We will expand here on why $\mathcal{O}_{\mathbb{P}^n}(m) \restriction_{D(X_i)} = X_i^m \mathcal{O}_{\mathbb{P}^n} \restriction_{D(X_i)}$. It is enough to note that if $f \in \mathcal{O}_{\mathbb{P}^n}(m)(U \cap D(f))$, then by definition, f is a homogeneous regular function of degree m on $\pi^{-1}(U \cap D(f)) \subset \mathbb{A}^{n+1}$, and so is $\frac{1}{X_i}$ where X_i is viewed as a regular function on $\mathcal{A}^{n+1}$. Therefore $1f = \frac{X_i^m}{X_i^m} f = X_i^m (\frac{f}{X_i^m})$ is still homogeneous of degree m, and $\frac{f}{X_i^m}$ is regular on $\pi^{-1}(U \cap D(X_i))$.

5.4 **Quasicoherent sheaves on projective varieties.**

5.4.2 If $M = \bigoplus_{n=1}^\infty M_n$ is a graded $k[C(X)]$-module, then we have $\tilde{M} \simeq \tilde{N}$ as quasicoherent sheaves on X, where $N = \bigoplus_{n=n_0}^\infty$, since the localizations $M_{(f)}$ and $N_{(f)}$ are isomorphic for any homogeneous function f: we have

$$\frac{m}{f^d} = \frac{f^N m}{f^{N+d}}$$

for any $N \in \mathbb{N}$ and so can always find representatives in higher degrees. In particular the zeroth-degree part of this graded localized module is zero and therefore the associated sheaf is zero restricted to every $D(f)$, and these cover X. But it will not be true that $M \simeq N$ as graded $k[C(X)]$-modules. In particular, if M has finitely-many nonzero direct summands, then $\tilde{M} = 0$.

Remark 6. If we apply this construction to the *Serre twists* of the $k[C(x)]$-module $k[C(x)]$, where we define a shifted grading by $k[C(X)](m)_i = k[C(X)]_{i+m}$, we get by definition the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$.

Example 1 (A sheaf that is not quasicoherent). This is example is taken from an MSE answer by Georges Elencwajg, who in turn extracted the example from `https://stacks.math.columbia.edu/tag/01B1`. The definitions and lemmas in §5.1 of Liu are of great help.

We will exhibit a sheaf $\mathcal{F}$ and a point over which $\mathcal{F}$ cannot be generated by sections. This will show that besides not being quasicoherent, there is no surjection $\mathcal{O}_X^{(I)} \twoheadrightarrow \mathcal{F}$.

Let $X = \mathbb{R}$ and $j\colon (0, \infty) \hookrightarrow X$ be the open inclusion. Consider $\mathcal{F} = i_! \underline{\mathbb{Z}}_{(0,\infty)}$. On any connected neighbourhood $U \ni 0$ we have $\mathcal{F}(U) = \{s \in \underline{\mathbb{Z}}(U \cap (0, \infty \mid \operatorname{supp} s \text{ is closed in } U\}$, but unless $s = 0$ $\operatorname{supp} s = U \cap (0, \infty)$ is never closed in U, so $\mathcal{F}(U) = 0$ for all such U. But $j_!$ is extension by zero on stalks, so $j_! \underline{\mathbb{Z}}$ has many nonzero stalks to the right of 0, and therefore $\mathcal{F}$ cannot be generated by sections over U. By lemma 5.1.3 (p.175) of Liu, there cannot be a surjection of the form above.

5.5 **Invertible sheaves**

Proof of lemma 5.5.1 The section σ exists for any U because $\mathcal{L}$ is invertible, and note that any open set U is dense. Let $x \in X$ and let U be a a neighbourhood of x such that $\mathcal{L} \restriction_U \simeq \mathcal{O}_X \restriction_U$. Then σ_x is a unit in $\mathcal{L}_x \simeq \mathrm{O}_{X,x}$, and multiplication $\mathcal{I}_x \to \mathcal{L}_x$ by σ_x is an isomorphism.

Divisors and invertible sheaves of fractional ideals The proof that $\mathcal{I}_D$ is invertible for an irreducible divisor D is the proof of a local converse to the Principal Ideal Theorem from §2.6.

Proof of lemma 5.5.4. To say that $\mathcal{I}_E$ is locally the image of $\mathrm{O}_{\mathbb{P}^n}(-e)$ under multiplication by f, we use that, by the second-last sentence of the proof of the first claim in the proof of the last theorem, locally (f) (or $\tilde{f}$ in that notation) is a basis of $\mathcal{I}_E$. Then use the argument from the second paragraph of the proof of lemma 5.5.1. Note that $\mathcal{I}_E$ is being defined as a subsheaf of $\mathbf{Rat}(\mathbb{P}^n)$. The phrase "thus as $\mathcal{I}_E$ generates Pic" I found confusing: what is pertinent is that it's just been shown that for any E, $\mathcal{I}_E \simeq \mathrm{O}_{\mathbb{P}^n}(-e)$ for an integer $-e$, and the irreducible divisors generate IFI which surjects onto Pic. Thus we see that the elements of Pic are at most the twisting sheaves, which have no isomorphisms between themselves by the next part of the proof.

Remark 7. Note that $\mathbb{P}^1$ is the flag variety for the algebraic group SL_2, and that the cocharacter lattice of SL_2 is $\mathbb{Z}$.

5.7 Morphisms to projective space and affine morphisms

5.7.2 todo

5.7.3 Let X be affine. Then we claim $\mathcal{O}_X$ is very ample. Let $X = \bigcup_{i=1}^n D(f_i)$ be an open cover of X. On $D(f_i)$, define

$$\varphi_i \colon x \mapsto [f_1(x)/f_i(x) : \cdots : 1 : \cdots : f_n(x)/f_i(x)],$$

where we recall that strictly speaking, $f(x) := x(f)$. On $D(f_i) \cap D(f_j) = D(f_i f_j)$, multiplication of each coordinate by $f_i(x)/f_j(x)$ shows $\varphi_i = \varphi_j$ on $D(f_i) \cap D(f_j) = D(f_i f_j)$. Therefore these morphisms glue to a global morphism $\varphi \colon X \to \mathbb{P}^n$. It is easy to see that X is isomorphic to $\varphi(X)$.

5.7.4

5.7.5

5.7.6 As $\mathcal{L}$ is invertible and X is projective, $\mathcal{L} \simeq \mathcal{O}_X(m)$ for some m, in the notation of p.60. Then $\mathcal{O}_X(m)^{\otimes n} = \mathcal{O}_X(nm)$, so we use corollary 5.4.3 to get a surjection of the kind needed to apply lemma 5.7.1.

6. Smooth varieties and morphisms

6.1 The Zariski cotangent space and smoothness

6.1.1 If f in $\mathcal{O}_{X,x}$ is the germ of a constant function, then $f - f(x) = 0$ even before the quotient. Property (b) follows from linearity of the evaluation map $\mathcal{O}_{X,x} \to k$ and the quotient map $\mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2$. For the Liebnitz rule, let f and g be germs in $\mathcal{O}_{X,x}$. Then

$$\begin{aligned} fg - f(x)g(x) - f(x)g + f(x)g(x) - f(x)g + f(x)g(x) &= fg - f(x)g - g(x)f + f(x)g(x) \\ &= (f - f(x))(g - g(x)) \in \mathfrak{m}_x^2 \end{aligned}$$

Therefore the two sides of the Liebnitz rule are equal in the quotient.
Furthermore, let $\psi \colon \mathcal{O}_{X,x} \to W$ be k-linear and obey (a),(b),(c). Consider the diagram

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "\psi"] \arrow[d, "\phi"] & W \\ \mathfrak{m}_x \arrow[ur, "\psi'"] \arrow[r] & \mathfrak{m}_x/\mathfrak{m}_x^2. \arrow[u, "\lambda"'] \end{tikzcd}$$

We have $\psi(f) = \psi(f - f(x) \cdot 1)$ by (a), thus inducing ψ'. Here $\phi(f) = f - f(x)$. We must show that ψ' dies on $\mathfrak{m}_x^2$. Let $f = f_1 f_2 \in \mathfrak{m}_x^2$. Then $\psi(f_1 f_2) = f_1(x)\psi(f_2) + f_2(x)\psi(f_1) = 0$, and $\phi(f_1 f_2) = f_1 f_2 - f_1(x) - f_2(x) = f_1 f_2$. Therefore $\psi'(f_1 f_2) = \psi(f_1 f_2) = 0$. Thus ψ' factors through the quotient, yielding λ. Notice the ψ followed by the quotient map is exactly $d(-)|_x$.

Remark 8. Not only is $T_x X$ affine and has dimension equal to $\dim \operatorname{Cot}_x(X)$, it is isomorphic to $\mathbb{A}^{\dim \operatorname{Cot}_x(X)}$ because $\operatorname{Sym} \operatorname{Cot}_x(X)$ is isomorphic to the polynomial algebra in $\dim \operatorname{Cot}_x(X)$ indeterminates.

6.2 Tangent cones

6.2.5 todo

6.3 The sheaf of differentials

6.3.4 We find the Zariski cotangent space of $C = \{x^2 = y^3\}$ at all its points. It will be one-dimensional at all points except the origin, where it will have dimension two. A drawing in the heuristic case of $\mathbb{R}^2$ is very helpful. By lemma 6.1.2 (c), we have $\operatorname{Cot}_x(C) = n_x/n_x^2$. We have $A = k[x,y]/(x^2 - y^3)$, so any regular function can be written $xf(y) + g(y)$ for $f, g \in k[y]$. Thus $n_{(0,0)}$ is made up of functions where $g(y) \in yk[y]$. Multiplying gives that functions in $n_{(0,0)}^2$ obey $f \in yk[y]$ and $g \in y^2 k[y]$. Thus $n_{(0,0)}/n_{(0,0)}^2 = \{\alpha x + \beta y\}$ has dimension two. It is in fact just $\mathbb{A}^2$.

At $(a, b) \neq (0, 0)$, the ideal corresponding to (a, b) is $(x - a, y - b)$ in A. We use the Taylor series (polynomial) of $f = y^3 - x^2 = 0 \in A$. Then

$$f = 0 = 3b^2(y - b) + 6b(y - b)^2 + 6(y - b)^3 - 2a(x - a) - 2(x - a)^2.$$

Thus $df = 0 = 3b^2(y - b) - 2a(x - a)$ just picks out the linear terms. Therefore $y - b = (2a/3b^2)(x - a)$ in $\operatorname{Cot}_{(a,b)}(C)$. Thus we have $n_{(a,b)}/n_{(a,b)}^2 = ((x - a, y - b))/n_{(a,b)}^2$ is one-dimensional; $x - a$ is a basis.

Remark 9. See Class 21 of Vakil's Foundations of Algebraic geometry (notes available online) for other helpful examples using Taylor series in exactly this way.

6.3.5 Let $f(X_1, \ldots, X_n)$ be regular on $\mathbb{A}^n$. We want to show that $df = \sum_{i=1}^n \frac{\partial f}{\partial X_i} dX_i$, where the partial derivatives are calculated as in calculus. Without loss of generality f is a monomial. The claim is tautological for $f = X_i$ a coordinate function. From this the power rule follows by induction. We now induct on the number of different X_is that appear in f. We have

$$\begin{aligned} df &= d(X_1^{\alpha_1} \cdots X_{n-1}^{\alpha_{n-1}}) X_n^{\alpha_n} = X_n^{\alpha_n} d(X_1^{\alpha_1} \cdots X_{n-1}^{\alpha_{n-1}}) + \alpha_n X_n^{\alpha_n - 1} dX_n (X_1^{\alpha_1} \cdots X_{n-1}^{\alpha_{n-1}}) \\ &= X_n^{\alpha_n} \sum_{j=1}^{n-1} \alpha_j X_1^{\alpha_1} \cdots X_j^{\alpha_j - 1} \cdots X_{n-1}^{\alpha_{n-1} - 1} + \alpha_n X_1^{\alpha_1} \cdots X_{n-1}^{\alpha_{n-1}} X_n^{\alpha_n - 1} dX_n \\ &= \sum_{i=1}^n \frac{\partial f}{\partial X_i} dX_i. \end{aligned}$$

This proves the claim.

6.3.6 Let X, Y be affine. We claim $\Omega[X \times Y] = \Omega[X] \otimes_{k[Y]} k[X \times Y] \oplus \Omega[Y] \otimes_{k[Y]} k[X \times Y$. Concretely, we must show any differential is $f(y)\omega_X + g(x)\omega_Y$. Note that $\Omega[X \times Y]$ is generated by differentials of global regular functions as a $k[X] \otimes_k k[Y]$-module. Therefore is suffices to show that such differentials are of the above form. Indeed, $k[X] \otimes_k k[Y] \simeq k[X \times Y]$ with the isomorphism given by multiplication. Thus if $h(x, y) = f(x)g(y)$, we have $dh = f(x)dg + g(y)df$. We must show that $dg \in \Omega[Y]$, which is obvious on a geometric level but requires working through some identifications to explain carefully. We claim we have

$$\begin{tikzcd} k[Y] \arrow[r, "d_Y"] \arrow[d, hook] & \Omega[Y] = I_Y/I_Y^2 \arrow[d, hook] \\ k[X \times Y] \arrow[r, "d_{X \times Y}"] & \Omega[X \times Y] = I_{X \times Y}/I_{X \times Y}^2 \end{tikzcd}$$

We have $I_Y \subset I_{X\times Y}$ under $k[Y] \hookrightarrow k[X] \otimes_k k[Y]$ whence $I_Y^2 \subset I_{X\times Y}^2$ so the inclusion of ideals descends to the quotient giving the right vertical map. We will be done if we show it is an injection. If $g \in I_Y$ and $g(y_1) - g(y_2) \in I_{X\times Y}^2$, then $g(y_1) - g(y_2) = \sum h_1(x,y)h_2(x,y)$ with each $h_i \in I_{X\times Y}$. The left-hand side is a pure function of y, so it is actually in I_Y^2. Thus the right vertical map is an injection and $d_{X\times Y} g \in \Omega[Y]$ naturally. The same goes for f and we are done.

6.3.7 (a) We claim any regular function on C can be written $f(Y) + Xg(Y)$ for polynomials f, g. Regular functions are elements of $k[X,Y]/(X^2+Y^2=1)k[X,Y]$, so that $X^2 = 1 - Y^2$ and any polynomial has a representative that is at most linear in X.

(b) Using (a), as regular functions on $\mathbb{A}^2$, the differential of any regular function on C can be written

$$dh = d(f(Y) + Xg(Y)) = g(Y)dX + \left(\frac{\partial f}{\partial Y} + X\frac{\partial g}{\partial Y}\right) dY.$$

6.3.8 (a) $D(x)$ and $D(y)$ covering C is equivalent to $(0,0)$ not lying on the circle C, which holds in any field at all.

(b) Taking the differential of the equation defining $C \subset \mathbb{A}^2$ gives $2xdx + 2ydy = 0 \iff xdx = -ydy$ when $\operatorname{char} k \neq 2$. Thus on $D(xy)$ we have $\frac{dx}{y} = -\frac{dy}{x}$.

(c) The hint and the fact that the two sought restrictions are equal on $D(xy)$, which is C minus four points, helps a lot. Pick $\omega = ydx - xdy$ and use (b) to check ω restricts as needed.

6.3.9 Again $D(x) \cup D(y) = C$, and on $D(xy)$ we have $\frac{dx}{y^2} = -\frac{dy}{x^2}$. For this case we should require that $\operatorname{char} k \neq 3$. The differential $\omega = ydx - xdy$ now restricts to $\frac{dx}{y^2}$ on $D(y)$ and to $-\frac{dy}{x^2}$ on $D(x)$.

6.3.11 We claim the mentioned subset is the maximal open subset such that $\Omega_X \restriction_U$ is locally free. We use corollary 5.3.4 (c): the map $x \mapsto \dim_k(\Omega_X \restriction_x)$ is constant with value m iff Ω_X is locally free of rank m. On U, $x \mapsto \dim_k(\Omega_X \restriction_x) = \dim_x X$ is constant, so Ω_X is locally free on U. If $V \supset U$ is open with $\Omega_X \restriction_V$ locally free, the dimension function must be constant with value $\dim_x X$ on V. But then X is smooth at all points of V, so $V = U$.

6.3.14 Differentiating the action of G on itself by translation gives isomorphisms between the tangent spaces at every point of G. In particular, because G is smooth at one point, it is smooth everywhere.

7. Curves

7.1 Introduction to curves

See also Hartshorne Ch. 1 §6.

7.2 Valuation criteria

Remark 10. The first paragraph of the proof of proposition 7.2.2, that separatedness implies unique extensions, can be replaced with exercise 3.3.6.

7.2.4 todo

Definition 1. A finite union of locally closed subsets is a *constructible* subset.

Remark 11. See also 7.4.C in Vakil for a criterion for this problem for Noetherian schemes.

7.4 Coherent sheaves on smooth curves

For 4.4.3, we can use that $\dim_k(\mathcal{F}|_x)$ must be constant with value $\operatorname{rank}(\mathcal{F})$ to see that $\mathcal{F}$ is locally free.

7.4.4 See Dummit and Foote or Atiyah-Macdonald for the structure theory of modules over a discrete valuation domain, which is just a specialization of the result for modules over a PID. In particular we always have $M = R^{\oplus n} \oplus \bigoplus_i (R/(\pi^{n_1}))$ for R-modules M with π the uniformizer for finitely- generated modules. Therefore on stalks we have

$$0 \mapsto \bigoplus_i (\mathcal{O}_{C,c}/(\pi^{n_1})) \to \mathcal{O}_{C,c}^{\oplus m} \oplus \bigoplus_i (\mathcal{O}_{C,c}/(\pi^{n_1})) \to \mathcal{O}_{C,c}^{\oplus m} \to 0$$

on stalks. There is an obvious retraction here, but it isn't obvious to me why it induces a surjection $\mathcal{F} \twoheadrightarrow \mathcal{F}_{\mathrm{tors}}$. Note also that away from the finite support of the torsion part, exactness shows $\mathcal{F} \to \mathcal{F}'$ is an isomorphism.

7.5 Morphisms between smooth complete curves

7.5.6 The notation $f^{-1}(0-\infty)$ must mean the divisor $f^{-1}(\{0\}) - f^{-1}(\{\infty\})$. Thus if $\tilde{f}$ has one pole of order one, it has exactly one zero of order one. The same holds for $\tilde{f} - \alpha$ for all $\alpha \in k$, as the poles stay the same. Therefore $\tilde{f}$ is a bijection and an isomorphism.

7.6 Special morphisms between curves

Definition 2. The *trace* or *field trace* of a finite field extension L/K is the map

$$\mathrm{tr}\colon L \to K$$

given by

$$\alpha \mapsto \mathrm{tr}(x \mapsto \alpha x),\ x \in L,$$

because L is a finite-dimensional K-vector space and multiplication by α is K-linear.

7.6.1 todo

8. Cohomology and the Riemann-Roch theorem

There are few excercises in this chapter, but some room to expand on certain proofs.

8.1 The definition of cohomology

The resolution $D^\bullet(\mathcal{F})$ constructed in the first paragraph is the *Godement resolution*. One way to see that it is functorial is that $D(\mathcal{F})$ is just $\iota_*\iota^*\mathcal{F}$, where $\iota\colon X_{\mathrm{disc}} \to X$ is the map from X_{disc}, the set X with the discrete topology, given by $\iota(x) = x$. Then the resolution is

$$0 \to \mathcal{F} \xrightarrow{\epsilon} D(\mathcal{F}) \xrightarrow{d^0} \iota_*\iota^* \operatorname{coker}\epsilon \xrightarrow{d^1} \iota * \iota^* \operatorname{coker} d^0 \to \cdots$$

A morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ clearly induces a morphism $D(\mathcal{F}) \to D(\mathcal{G})$, giving the solid morphisms in

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "\epsilon_{\mathcal{F}}"] \arrow[d, "\phi"] & \iota_*\iota^*\mathcal{F} \arrow[r] \arrow[d, "\iota_*\iota^*\phi"] & \iota_*\iota^* \operatorname{coker}\epsilon_{\mathcal{F}} \arrow[r] \arrow[d, dashed] & \cdots \\
0 \arrow[r] & \mathcal{G} \arrow[r, "\epsilon_{\mathcal{G}}"] & \iota_*\iota^*\mathcal{G} \arrow[r] & \iota_*\iota^* \operatorname{coker}\epsilon_{\mathcal{G}} \arrow[r] & \cdots
\end{tikzcd}$$

The dashed morphism exists thanks to the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\epsilon_{\mathcal{F}}"] \arrow[d, "\phi"] & \iota_*\iota^*\mathcal{F} \arrow[r] \arrow[d, "\iota_*\iota^*\phi"] & \operatorname{coker}\epsilon_{\mathcal{F}} \arrow[d] \\
\mathcal{G} \arrow[r, "\epsilon_{\mathcal{G}}"] & \iota_*\iota^*\mathcal{G} \arrow[r] & \operatorname{coker}\epsilon_{\mathcal{G}}
\end{tikzcd}$$

in which the last vertical morphism is induced directly from the universal property of cokernels. Repeating this process shows functoriality of the Godement resolution.

Given a short exact sequence of sheaves, each row in the corresponding morphism of complexes described on p.99 is exact on global (in fact, any) sections because the $D^i(\mathcal{F})$ are flabby, by lemma 4.3.2.

Proof of lemma 8.1.1 By lemma 4.3.2, we know that $D(\mathcal{F})/\mathcal{F}$ is flabby. Therefore

$$0 \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X, D(\mathcal{F})) \longrightarrow H^0(X, D(\mathcal{F})/\mathcal{F}) \longrightarrow 0$$

is exact. Recalling $\mathcal{F} = C^0(\mathcal{F})$ and $D(\mathcal{F}) = D^0(\mathcal{F})$, proceeding inductively we have that each $C^i(\mathcal{F})$ is flabby (note $D^i(\mathcal{F})$ is flabby by construction), and so therefore

$$0 \longrightarrow H^0(X, C^i(\mathcal{F})) \longrightarrow H^0(X, D^i(\mathcal{F})) \longrightarrow H^0(X, C^{i+1}(\mathcal{F})) \longrightarrow 0$$

is exact. The complex $\Gamma(X, D^\bullet(\mathcal{F}))$ is therefore built from short exact sequences (just like the Godement resolution):

$$\begin{tikzcd}
 & 0 \arrow[r] & \Gamma(X, D^0(\mathcal{F})/\mathcal{F}) \arrow[r] \arrow[dr] & 0 & \cdots & \\
0 \arrow[r] & \Gamma(X, \mathcal{F}) \arrow[r] & \Gamma(X, D^0(\mathcal{F})) \arrow[r] \arrow[u] & \Gamma(X, D^1(\mathcal{F})) \arrow[r] \arrow[d] & \Gamma(X, D^2(\mathcal{F})) \arrow[r] \arrow[u] & \cdots \\
 & & 0 \arrow[r] & \Gamma(X, D^1(\mathcal{F})/C^1(\mathcal{F})) \arrow[r] \arrow[ur] & 0 &
\end{tikzcd}$$

and so is exact. Here is an observation given in proof of lemma 8.1.2: Given a resolution

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\epsilon"] & A^0 \arrow[r, "\rho"] \arrow[d, "\psi"] & A^1 \arrow[r] & A^2 \arrow[r] & \cdots \\
 & & \operatorname{coker}(\epsilon) \arrow[r] \arrow[ur, hook] & 0 & &
\end{tikzcd}$$

applying a left-exact functor then gives the kinked exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(F) \arrow[r] & \Gamma(A^0) \arrow[r, "\Gamma(\rho)"] \arrow[d, "\Gamma(\psi)"] & \Gamma(A^1) \\
 & & \Gamma(\operatorname{coker} \epsilon) \arrow[ur, hook] &
\end{tikzcd}.$$

Note though that $\ker \Gamma(\rho) = \ker \Gamma(\psi) = \operatorname{im} \Gamma(\epsilon)$, because Γ preserves injections because it is left-exact. Therefore we have an exact sequence

$$0 \longrightarrow \Gamma(F) \longrightarrow \Gamma(A^0) \longrightarrow \Gamma(A^1).$$

This is one reason one calculates derived functors by throwing away $\Gamma(F)$, the complex would be exact at this place anyway.

Proof of lemma 8.1.2 The second left-exact sequence given in the proof arises in the way expounded upon above. The first follows from the long exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}^0) \longrightarrow H^0(X, \mathcal{G}) \xrightarrow{\delta} H^1(X, \mathcal{F}) = 0 \longrightarrow \cdots,$$

thus in particular

$$H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \xrightarrow{\delta} H^1(X, \mathcal{F}) \longrightarrow 0 \tag{1}$$

is exact.
Then it follows that $\ker(\Gamma(X, \mathcal{F}^1) \to \Gamma(X, \mathcal{F}^2)) = \Gamma(X, \mathcal{G})$ and $\operatorname{im}(\Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{G})) = \operatorname{im}(\Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1))$, so that $\Gamma(X, \mathcal{G})/($image of $\Gamma(X, \mathcal{F}^0)$ is isomorphic to $H^1(X, \mathcal{F})$. For $i > 1$,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{G} \longrightarrow 0$$

gives a long exact sequence in cohomology

$$0 = H^{i-1}(X, \mathcal{F}^0) \longrightarrow H^{i-1}(X, \mathcal{G}) \xrightarrow{\delta} H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F}^0) = 0 \longrightarrow \cdots$$

and therefore δ is an isomorphism. By induction (note that $\Gamma(X, \mathcal{G})$ is the zeroth cohomology group of $\mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$) we have $H^{i-1}(X, \mathcal{G}) \simeq H^{i-1}(\mathcal{F}^1 \to \mathcal{F}^2 \to \cdots)$.

Proof of lemma 8.1.3 By functoriality of the Godement resolution, we get exact sequences of directed systems of sheaves

$$0 \longrightarrow \mathcal{F}_i \longrightarrow D^0(\mathcal{F}_i)_i \longrightarrow D^1(\mathcal{F}_i)_i \longrightarrow \cdots$$

so by A.2.1, we get a resolution

$$0 \longrightarrow \operatorname{colim}_i \mathcal{F}_i \longrightarrow \operatorname{colim}_i D^0(\mathcal{F}_i)_i \longrightarrow \operatorname{colim}_i D^1(\mathcal{F}_i)_i \longrightarrow \cdots,$$

and each D^i term is flabby. Thus we can use this resolution to compute $H^*(X, \operatorname{colim}_i \mathcal{F}_i)$. By lemma 4.4.3 the presheaf colimit is a sheaf, and so we have $\operatorname{colim}_i \Gamma(X, D^j(\mathcal{F}_i)) \simeq \Gamma(X, \operatorname{colim}_i D^j(\mathcal{F}_i))$, again using A.2.1. Now

$$\begin{aligned} H^k(X, \operatorname*{colim}_i \mathcal{F}_i) &\simeq H^k(\operatorname*{colim}_i \Gamma(X, D^0(\mathcal{F}_i)_i) \to \operatorname*{colim}_i \Gamma(X, D^1(\mathcal{F}_i)_i \to \cdots)) \\ &\simeq \operatorname*{colim}_i H^k(\Gamma(X, D^0(\mathcal{F}_i)) \to \Gamma(X, D^1(\mathcal{F}_i) \to \cdots)) \\ &= \operatorname*{colim}_i H^k(X, \mathcal{F}_i). \end{aligned}$$

The following corollary is used but not stated anywhere explicitly

Corollary 1 (Corollary of lemma 8.1.3)**.** *On a Noetherian topological space, cohomology commutes with arbitrary direct sums.*

It is not in general true that cohomology, or even taking global sections, will commute with colimits (note that Γ will always commute with limits, it has a left adjoint that is the constant sheaf functor). In this case cohomology commutes with colimits because global sections do thanks to lemma 4.4.3, which uses that the underlying space is Noetherian in a key way.

8.2 **Cohomology of affines** The opening remark must be a typo in which $\mathcal{F}$ should be assumed flabby. Indeed, if we take $U = X$ then the conclusion is that ${}_U\mathcal{F} = \mathcal{F}$ is flabby.

Remark 12. It is an important point that being a section of $\mathcal{G}/\mathcal{F}$ does not mean coming from a section of $\mathcal{G}$. It is $H^1(X, \mathcal{F})$ that measures the failure of this to happen, as we see from the exact sequence (1); equivalently, it measures how badly a surjective morphism of sheaves fails to be surjective on sections.

Proof of lemma 8.2.1 The first paragraph uses the idea in the above remark. For $i > 1$, the assumption should read "let W be a member of $\mathcal{V}$," and "proposition" should be replaced with "lemma" after (b) on p. 101.

Proof of Serre's theorem There are few typos in this proof. Below is a restructuring of it.

The $\{D(f)\}$ give a basis like in the proposition. Suppose that we know $H^j(X, \mathcal{F}) = 0$ for all $0 < j < i$ and all quasicoherent sheaves $\mathcal{F}$. We will show that $H^i(X, \mathcal{F}) = 0$. By the lemma, for any $\alpha \in H^i(X, \mathcal{F})$,we can find $U_1, \ldots U_d$ distinguished opens such that the image of α in $H^i(X, {}_{U_j}\mathcal{F})$ is zero for $j \in \{1, \ldots, d\}$. We have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_j {}_{U_j}\mathcal{F} \longrightarrow \mathcal{G} = \text{coker} \longrightarrow 0.$$

The long exact sequence in cohomology says that that $\alpha = \delta(\beta)$ (this uses that cohomology commutes with finite direct sums!) for $\beta \in H^{i-1}(X, \mathcal{G})$. But $\mathcal{G}$ is also quasi-coherent, so by our induction hypothesis, $\beta = 0$. For the base case $i = 1$, use that $\Gamma(X, -)$ is exact for affine X.

Remark 13. The corollary says morally that affine morphisms preserve cohomology because Serre's theorem tell us that their fibres can't contribute any cohomology.

The proof of Serre's theorem will go through whenever we have a base of topology like in the proposition, and can establish the base case $H^1 = 0$ for all members of some class of sheaves. As H^1 is about whether local surjectivity comes from global surjectivity, one way to get Serre-type theorems is for partitions of unity to exist on X. See, for example this blog post: `https://sbseminar.wordpress.com/2010/02/02/when-fine-just-aint-enough/`.

This theorem will appear as theorem 10.2.1.

Theorem 1 (A converse to Serre's theorem)**.** *Let X be a quasi-compact scheme such that $H^1(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves of ideals $\mathcal{F} \subset \mathcal{O}_X$. Then X is affine.*

See `https://stacks.math.columbia.edu/tag/01XE` for the proof. Note that the proof bears some similarity to the proof above, including requiring the $H^1 = 0$ hypothesis (although only for special $\mathcal{O}_X$-modules), but essentially aims to show that X has a partition of unity consisting of global functions, from which it follows from lemma 27.27.3 that X is affine.

8.3 Higher direct images

Proof of lemma 8.3.1 We have $R^i f_* \mathcal{F} = (V \mapsto H^i(f^{-1}(V), \mathcal{F})^{\#}$, and for V affine open, Serre's theorem applies for all $i > 0$. The affine opens form a base of topology, so $R^i f_* \mathcal{F}$ is zero on a base, hence is the zero sheaf. Here are some elementary facts which are used, but not spelled out, in this chapter and next.

Lemma 4. *Let $f\colon X \to Y$ be a continuous map of topological spaces. If $\mathcal{F} \in \mathrm{Sh}(X)$ is flabby, then $f_* \mathcal{F}$ is flabby.*

Proof. For all opens $U \subset V$ in Y, the restriction $f_*\mathcal{F}(V) \to f_*\mathcal{F}(U)$ is the restriction $\mathcal{F}(f^{-1}(V)) \twoheadrightarrow \mathcal{F}(f^{-1}(U))$. $\square$

Lemma 5. *Let $f\colon X \hookrightarrow Y$ be a closed inclusion. (Or, replacing f_* with $f_!$, a locally-closed inclusion). Then $H^i(X, \mathcal{F}) \simeq H^i(Y, f_*\mathcal{F})$ for all i.*

Proof. Let $0 \to \mathcal{F} \to \mathcal{G}^\bullet$ be a flabby resolution of $\mathcal{F}$. Then $0 \to f_*\mathcal{F} \to f_*\mathcal{G}^\bullet$ is a complex of flabby sheaves on Y by the above lemma, and exact because f_* is extension by zero on stalks. By the resolution principle the cohomology groups are isomorphic. $\square$

9. General cohomology

9.1 Cohomology of $\mathbb{A}^n \setminus \{0\}$ and $\mathbb{P}^n$

9.2 Cech cohomology

There is a typo in the definition of the differential $\check{\delta}$ (which also appears with typos as $\hat{\delta}$ and just δ). The correct definition should be $\check{\delta}(\alpha) = \beta$, where

$$\beta_{i_0<i_1<\cdots<i_{n+1}} = \sum_{j=0}^{n} (-1)^j \alpha_{i_0<i_1<\cdots<i_{j-1}<i_{j+1}<\cdots<i_{n+1}}|_{U_{i_0}\cap\cdots\cap U_{i_j}\cap\cdots\cap U_{i_{n+1}}}.$$

The agreement of Cech cohomology with the cohomology of the derived functors of $\Gamma(X, -)$ for separated varieties uses separation in a key way: otherwise an intersection of affine opens might not be affine.

Corollary 9.2.3 tells us that after finding an affine open cover of a separated variety, we get a bound on what degree cohomology can appear. Note this bound is not sharp: A line bundle $\mathcal{O}_{\mathbb{P}^1}(r)$ on $\mathbb{P}^1$ has no second cohomology, but as $\mathbb{P}^1$ is not affine, any affine open cover (say, the standard one) will have at least two elements.

There is a generalization and sharpening of corollary 9.2.3:

Theorem 2 (Grothendieck)**.** *Let X be a Noetherian topological space of dimension n. Then for all $i > n$ and sheaves $\mathcal{F}$ of abelian groups on X, we have $H^i(X, \mathcal{F}) = 0$.*

9.3 Cohomology of projective varieties

Proof of theorem 9.3.1 The short exact sequence arises from corollary 5.4.3, followed after twisting by $(-n)$ in the notation of corollary 5.4.3, and $\mathcal{G}$ is coherent because it's a quasicoherent subsheaf of a coherent sheaf. The statement about closed embeddings is lemma 5 in these notes.

There seems to be a typo in the proof: we adapt it to proceed via descending induction. As projective varieties are seperated, we know by 9.2.3 that all the cohomology groups are zero-dimensional for large enough i. Suppose the claim is true for i and all coherent $\mathcal{G}$. We show the claim for $i-1$ and all coherent $\mathcal{F}$. The long exact sequence reads

$$\bigoplus H^{i-2}(\mathcal{O}_{\mathbb{P}^n}(p)) = 0 \to H^{i-2}(X,\mathcal{F}) \xrightarrow{\sim} H^{i-1}(X,\mathcal{G}) \to \bigoplus H^{i-1}(X,\mathcal{O}_{\mathbb{P}^n}(p)) = 0 \to \cdots$$
$$\to \bigoplus H^{n-1}(X,\mathcal{O}_{\mathbb{P}^n}(p)) = 0 \to H^{n-1}(X,\mathcal{F}) \xrightarrow{\delta} H^{n}(X,\mathcal{G}) \to \bigoplus H^{n}(X,\mathcal{O}_{\mathbb{P}^n}(p)) \to H^{n}(X,\mathcal{F}) \to 0$$

Therefore $H^{n-1}(X,\mathcal{F})$ is finite-dimensional as it injects into $H^n(X,\mathcal{G})$, and $H^n(X,\mathcal{F})$ is finite-dimensional as its surjected by a finite-dimensional vector space. For the lower cohomology groups we use the isomorphisms given by the long exact sequence.

9.3.2 We prove $\chi(\mathcal{O}_{\mathbb{P}^n}(r)) = \frac{(n+r)!}{n!r!}$. By corollary 9.1.2, $\mathcal{O}_{\mathbb{P}^n}(r)$ will have either zeroth or n-th cohomology, but not both. It will have nonzero zeroth cohomology iff $r \geq 0$, which is the only case the claim makes sense for. In this case $H^0(X,\mathcal{O}_{\mathbb{P}^n}(r))$ is the space of degree r homogeneous polynomials in $n+1$ variables, and a basis is given by the monomials. The number of monomials is the number of choices of r objects from $n+1$ objects with replacement, hence is $\binom{n+1+r-1}{r}$.

Proof of theorem 9.3.3 There is a typo in the proof of this theorem. The short exact sequence should read

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_{\text{finite}}(\pi^*_{\mathbb{P}^n}\mathcal{O}_{\mathbb{P}^n}(p)) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Again, the long exact sequence in cohomology and descending induction tells us that $R^if_*\mathcal{F} \simeq R^{i+1}f_*\mathcal{G}$, and as remarked in Hartshorne, the long exact sequence in cohomology is a long exact sequence of $\mathcal{O}_Y$-modules, so this isomorphism proves coherence.

The surjection inducing the short exact sequence comes from an adaptation of corollary 5.4.3, where in the language of that corollary, we now need the number of generators of of the corresponding module over $k[X_1,\dots,X_n]\otimes_k A = A[X_1,\dots,X_n]$.

9.4 **Direct images of flat sheaves** Lemma 9.4.4 finally gives us an algebraic criterion to tell whether a coherent sheaf is a vector bundle: the corresponding finitely-generated $k[X]$-module must be a flat $k[X]$-module.

Proof of lemma 9.4.4 The long exact sequence (which continues to the left, as $-\otimes_{\mathcal{O}_{X,y}}\mathcal{O}_{X,y}/\mathrm{m}_y$ is right-exact) should read, if $M = \Gamma(X,\mathcal{G})$,

$$0 \longrightarrow \mathrm{Tor}^1(L,\mathcal{O}_U|_y) = 0 \longrightarrow M\otimes_{\mathcal{O}_{X,y}}\mathcal{O}_{X,y}/\mathfrak{m}_y \longrightarrow \bigoplus \mathcal{O}_U|_y \xrightarrow{\psi} \tilde{L}|_y \longrightarrow 0$$

Thus $\mathcal{G}|_V = 0$ for some neighbourhood $V \ni y$. The leftmost zero in the long exact sequence appears because $\bigoplus \mathcal{O}_U|_y$ is a free, in particular flat, $\mathcal{O}_U|_y$-module.

10 Applications

A. Appendix

A.1 Localization

A.1.1 Let $\psi\colon A \to A_S$ be the map sending $a \mapsto a/1$ in the localization at S. We claim $\ker\psi = \{a \in A \mid \exists s \in S \text{ s.t. } sa = 0\}$. If there is such s, then $a/1 = as/s = 0/s = 0$. Conversely if $a/1 = 0$ $a/1 = 0/s$ and by definition this means there is $s \in S$ such that $sa = 0$.

A.1.2 We claim that if $\psi\colon A \to B$ is a morphism of rings, then we have a triangle

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d] & B \\ B_S \arrow[ur, "\bar{\psi}"] & \end{tikzcd}$$

iff $\psi(S) \subset B^\times$. If we have such a triangle, clearly we have $\psi(S) \subset B^\times$ as $\bar{\psi}(s)$ is a unit in B for every $s \in S$. Conversely define $\bar{psi}(a/s) = \psi(a)\psi(s)^{-1}$. This assignment is unique if the triangle is to commute. Finally, $\bar{\psi}$ is well-defined as $a/s = b/t$ implies there is $u \in S$ such that $u(ta - sb) = 0$ whence $\psi(t)\psi(a) - \psi(s)\psi(b) = 0$ and $\psi(a) = \psi(t)^{-1}\psi(s)\psi(b)$. Now

$$\frac{a}{s} \mapsto \psi(a)\psi(s)^{-1} = \psi(t)\psi(b) = \bar{\psi}(b/t)$$

as required.

A.1.3 We claim $A_S \simeq A[X_s]_{s\in S}/(sX_s - 1 = 0) =: \tilde{A}$ as A-algebras. Define $\varphi\colon a \to \tilde{A}$ by $a \mapsto aX_1 = a \cdot (1X_1) \simeq a$. We show that the target satisfies the universal property from A.1.2. Let $\psi\colon A \to B$ with $\psi(S) \subset B^\times$. Then $\bar{\psi}\colon \tilde{A} \to B$ is determined by $X_s \mapsto \psi(s)^{-1}$ and the diagram commutes. Conversely, this is the only assignment that allows the triangle to commute. Given such a commutative triangle, we have $\bar{\psi}(xX_s) = \psi(s)\psi(X_s) = \bar{\psi}(1) = 1$, so $\psi(s)$ must have been a unit. Therefore $A_S \simeq \tilde{A}$ in **CRing**. The map ψ is clearly A-linear, and we are done.

A.1.4 Let P be a prime ideal. Then we claim A_P is local with maximal ideal $\psi(P)A_P$. It is enough to show that every element not in $\psi(P)A_P$ is a unit. If $a \notin \psi(P)A_P$, then $a = b/s$ with $s \in S$ and $b \notin P$. Therefore $b \in S = A \setminus P$. Thus $a^{-1} = s/b$ exists.

A.1.5 We claim $A_{(f)} = 0$ iff f is nilpotent. One direction is obvious: if $1/f = 0/f$ then there is N such that $f^N = 0$ by definition of $=$ in the localization. This also shows the converse.

A.1.6 We claim $\alpha\colon M \otimes_A A_S \to M_S$ sending $m \mapsto m/1$ is an isomorphism of A_S-modules. The map α sends $m \otimes (a/s) \mapsto (a/s)m/1$ and is clearly surjective. If

$$\sum_{i=1}^n m_i \otimes \frac{a_i}{s_i} \mapsto \sum_{i=1}^n a_i \frac{m_i}{s_i} = 0,$$

then clearing denominators we have have, if $s = s_1 \cdots s_n$,

$$\sum_{i=1}^n m_i \otimes \frac{a_i s}{s_i} \mapsto s\sum_{i=1}^n m_i \otimes \frac{a_i}{s_i} = 0.$$

Therefore

$$s' \sum_{i=1}^n a_i \frac{s}{s_i} m_i = 0$$

in M for some $s' \in S$.

$$1\sum_{i=1}^n m_i \otimes \frac{a_i}{s_i} = \frac{s'}{s'}\frac{s}{s}\sum_{i=1}^n m_i \otimes \frac{a_i}{s_i} = \frac{1}{s's}\sum_{i=1}^n 1 \otimes s's_i\frac{s}{s_i} = \frac{1}{s's}\left(1 \otimes \sum_{i=1}^n s's_i\frac{s}{s_i}\right) = 0$$

and so α is injective.

A.1.7 Localization of A-modules is exact. If we have

$$0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0$$

then consider

$$0 \longrightarrow M_{1,S} \xrightarrow{\bar{\varphi}} M_{2,S} \xrightarrow{\bar{\psi}} M_{3,S} \longrightarrow 0.$$

The map $\bar{\varphi}$ sends $m/s \mapsto \varphi(m)/s$ and if there is $u \in S$ such that $u\varphi(m) = 0$ in M_2, we have $um = 0$ in M_1, whence $m/s = 0$ in $M_{1,S}$. We have that $\bar{\psi}$ is surjective because ψ is. For exactness at $M_{2,S}$, if $m = \bar{\varphi}(n)$, then $m = m_1/s_1 = \varphi(n_1)/s_2$ and so $\bar{\psi}(m) = \psi(m)/s = \psi(\varphi(n_1))/s_2 = 0$. Conversely, if $\bar{\psi}(n/s) = 0 = \psi(n)/s$, then we see that $u\psi(n) = 0$ in M_3 for some $u \in S$. Thus $un = \varphi(n')$ so that $n/s = \varphi(un')/s = \bar{\varphi}(un'/s)$.

A.2 **Direct limits.**

Remark 14. Warning! "Direct limits" are colimits, not limits! We will use the notation colim here instead of Kempf's notation. Note in fact that all colimits here are filtered.

A.2.1 Let $0 \to (M_u)_u \to (N_u)_u \to (P_u)_u \to 0$ be exact. Then we claim $\mapsto \operatorname{colim} N \to \operatorname{colim} M \to \operatorname{colim} P \to 0$ is exact. Note these morphisms exist thanks to the universal property. The proof is a diagram chase: Call α the morphism $N \to P$. If $g \in \operatorname{colim} P$, then there is some $u \in U$ and $g' \in P_u$ such that $g = s_{P,u}(g')$, and there is $f' \in N_u$ mapping down to g' by the original exactness. The obvious square commutes, and g is mapped onto by $f = s_{N,u}(f')$. If $[f] \in \ker(\operatorname{colim} N \operatorname{colim} P)$, then $[f] = s_{N,u}(f)$ for some u, f, and $\alpha_u \colon N_u \to P_u$ sends f to an element in the kernel of $s_{P,u}$. Therefore there is u_1 such that $r^u_{u_1}(\alpha_u(f)) = 0$, whence $r^u_{u_1}(f) = 0 \in \ker \alpha_u$. By exactness here, there is $f' \in M_u$ mapping down to $r^u_{u_1}(f)$. The image of f' in $\operatorname{colim} N$ is equal to $[f]$. For exactness at M is similar.

A.2.2 The tensor product preserves colimits, as it has a right adjoint, and so by A.1.6 it is enough to show the claim for the diagram of modules $\frac{1}{f^n}A$ in $A - \mathbf{mod}$ and then apply the endofunctor $M \otimes_A -$. But it is obvious that $\operatorname{colim}(\frac{1}{f^n}A) = A_{(f)}$.