**Reine Mathematik**

# On the K-theory of groups with finite decomposition complexity

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vorgelegt von
Daniel Kasprowski
aus Münster
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Westfälische
Wilhelms-Universität
Münster

| | |
|---|---|
| Dekan: | Prof. Dr. Martin Stein |
| Erster Gutachter: | Prof. Dr. Arthur Bartels |
| Zweiter Gutachter: | Prof. Dr. Johannes Ebert |
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## Abstract

In this thesis it is proved that the assembly map in algebraic $K$- and $L$-theory with respect to the family of finite subgroups is injective for groups $G$ that admit a finite dimensional model for $\underline{E}G$ and for which the family $\{H\backslash G\}_{H\leq G \text{ finite}}$ has finite decomposition complexity. Finite decomposition complexity is a generalization of finite asymptotic dimension introduced by Guentner, Tessera and Yu. In particular, the above result applies to finitely generated linear groups over fields with characteristic zero with a finite dimensional model for $\underline{E}G$.

## Zusammenfassung

In dieser Arbeit wird bewiesen, dass die Assembly Abbildung in algebraischer $K$- und $L$-Theorie bezüglich der Familie der endlichen Untergruppen für eine Gruppe $G$ injektiv ist, wenn $G$ ein endliches Model für den klassifizierenden Raum $\underline{E}G$ besitzt und die Familie $\{H\backslash G\}_{H\leq G \text{ endlich}}$ endliche Zerlegungskomplexität hat. Endliche Zerlegungskomplexität ist eine Verallgemeinerung von endlicher asymptotischer Dimension eingeführt von Guentner, Tessera und Yu. Es wird gezeigt, dass obiges Resultat insbesondere auch für endlich erzeugte lineare Gruppen über Körpern der Charakteristik Null gilt, für die ein endliches Model für $\underline{E}G$ existiert.

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# Introduction

Many interesting obstructions in topology lie in the $K$- and $L$-theory of group rings. For example Wall's finiteness obstruction for a finitely-dominated space $X$ with fundamental group $\pi$ lives in $K_0(\mathbb{Z}\pi)$. Another example of this flavor is the next theorem.

**The s-Cobordism Theorem** (Barden-Mazur-Stallings, see [Ker65])**.** *Let $(W, \partial_0 W, \partial_1 W)$ be a smooth $h$-cobordism of dimension at least 6 and with fundamental group $\pi$. Then $W$ is diffeomorphic to a cylinder $\partial_0 W \times [0,1]$ if and only if its Whitehead torsion $\tau(W, \partial_0 W) \in Wh(\pi) = K_1(\mathbb{Z}\pi)/\langle \pm g \mid g \in \pi \rangle$ vanishes.*

Unfortunately, computing the $K$- and $L$-theory of group rings is in general very hard. One approach to simplify their computations is given by the Farrell-Jones conjecture introduced in [FJ93]. For every ring $R$ (with involution in the case of $L$-theory) and every group $G$ there are $G$-homology theories $H_*^G(\_;\mathbb{K}_R)$ and $H_*^G(\_;\mathbb{L}_R)$, such that

$$H_*^G(G/H;\mathbb{K}_R) \cong K_*(R[H]) \text{ and } H_*^G(G/H;\mathbb{L}_R) \cong L_*(R[H])$$

for every subgroup $H \leq G$. In the formulation of Davis and Lück [DL98] the Farrell-Jones conjecture predicts that the assembly maps

$$H_*^G(\underline{\underline{E}}G;\mathbb{K}_R) \to H_*^G(pt;\mathbb{K}_R) \cong K_*(R[G])$$

and

$$H_*^G(\underline{\underline{E}}G;\mathbb{L}_R) \to H_*^G(pt;\mathbb{L}_R) \cong L_*(R[G])$$

are isomorphisms. Here $\underline{\underline{E}}G$ is the classifying space for virtually cyclic subgroups of $G$, see Section 1.2. The left hand side is built only from the $K$- resp. $L$-groups of the group rings $R[H]$ with $H \leq G$ virtually cyclic. If one knows these $K$-groups and the Farrell-Jones conjecture is true for the group $G$, then the computation of $K_*(R[G])$ can be attacked using the Atiyah-Hirzebruch spectral sequence. So far no counterexamples to the conjecture are known and it has been proved for many classes of groups. In [BFJR04, BR05] the Farrell-Jones conjecture is proved for fundamental groups of closed Riemannian manifolds with negative sectional curvature and in [BL12a, BL12b, BLR08a, BLR08b, Weg12] this is generalized to all hyperbolic and CAT(0)-groups.

If the Farrell-Jones conjecture is true for a torsion free group $G$, then $Wh(G) = 0$. In particular, this implies that all $h$-cobordisms of dimension at least 6 with fundamental group $G$ are products. The Farrell-Jones conjecture also implies the Borel conjecture, i.e. that all aspherical manifolds of dimension at least 5 with fundamental group $G$ are topologically rigid. This shows that the conjecture is not only a tool for computations but also implies other important results.

The maps

$$H_*^G(\underline{\mathrm{E}}G;\mathbb{K}_R) \to H_*^G(\underline{\underline{\mathrm{E}}}G;\mathbb{K}_R) \quad \text{and} \quad H_*^G(\underline{\mathrm{E}}G;\mathbb{L}_R) \to H_*^G(\underline{\underline{\mathrm{E}}}G;\mathbb{L}_R)$$

are known to be split injective. Here $\underline{\mathrm{E}}G$ is the classifying space for finite subgroups. Therefore, the Farrell-Jones conjecture implies that the assembly maps

$$H_*^G(\underline{\mathrm{E}}G;\mathbb{K}_R) \to H_*^G(pt;\mathbb{K}_R) \cong K_*(R[G])$$

and

$$H_*^G(\underline{\mathrm{E}}G;\mathbb{L}_R) \to H_*^G(pt;\mathbb{L}_R) \cong L_*(R[G])$$

are split injective. In this thesis we prove for a certain class of groups that these assembly maps are indeed split injective, see the theorem below. The main property we use is that a certain family of quotients of $G$ has finite decomposition complexity (FDC) introduced by Guentner, Tessera and Yu in [GTY13], which is a generalization of finite asymptotic dimension (FAD), see Section 1.4.

The main theorem of this thesis is the following result, see Theorem 3.2.2 and Theorem 3.3.1.

**Theorem.** *Let $R$ be a ring and let $G$ be a discrete group such that $\{H\backslash G\}_{H\in\mathcal{F}in}$ has finite decomposition complexity, where $\mathcal{F}in$ is the family of finite subgroups of $G$. Furthermore, assume that there is a finite dimensional $G$-CW-model for the classifying space for proper $G$-actions $\underline{E}G$. Then the assembly map in algebraic $K$-theory*

$$H_*^G(\underline{E}G;\mathbb{K}_R) \to H_*^G(pt;\mathbb{K}_R) \cong K_*(R[G])$$

*is split injective.*

*If $G$ is as above and $R$ is a ring with involution, such that for every finite subgroup $H \leq G$ there is an $i_0 \in \mathbb{N}$ with $K_{-i}(\mathcal{A}[H]) = 0$ for $i \geq i_0$, then also the assembly map in $L$-theory*

$$H_*^G(\underline{E}G;\mathbb{L}_R) \to H_*^G(pt;\mathbb{L}_R) \cong L_*(R[G])$$

*is split injective.*

Let $R$ be a commutative ring with unit and $G$ a finitely generated subgroup of $GL_n(R)$ with a bound on the size of its finite subgroups. In Theorem 4.1.13 we prove that the family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC. This shows that the assumptions of our main theorem are satisfied for a large class of groups. Some more details on groups satisfying the assumptions of the main theorem can be found at the beginning of Chapter 4.

Knowing that these assembly maps are split injective is much weaker then knowing that the full Farrell-Jones conjecture holds, but there are still some applications. For example if for a group $G$ the above assembly maps are split injective, then the Novikov conjecture, i.e. the homotopy invariance of higher signatures, holds for manifolds with fundamental group $G$. For the groups considered in this thesis the Novikov conjecture was already proved in [GTY12].

The main theorem is a generalization of the results from Bartels-Rosenthal ([BR07b]) and Ramras-Tessera-Yu ([RTY]). In [RTY] the injectivity of the assembly maps is proved under the stronger assumption that $G$ has FDC and a finite model for E$G$. That $G$ has a finite model for E$G$ in particular implies that $G$ is torsion free and therefore also satisfies the assumptions of our main theorem. Due to an error in [BR07b] the proof given there only holds for groups with finite asymptotic dimension and a finite model for $\underline{E}G$. This implies that there is an upper bound on the order of the finite subgroups of $G$ and thus the family $\{H \backslash G\}_{H \in \mathcal{F}in}$ has FDC by Corollary 4.1.3. Both times the strategy of the proof is to first use the geometric condition FDC respectively FAD to show that a certain non-equivariant map is an isomorphism and then show that the assembly map is split injective, using the Descent Principle, see Section 3.1. While the first part of the proof given here is very similar to that in [RTY], for the Descent Principle to hold the finiteness condition on the classifying space is needed. To generalize this to only finite dimensional classifying spaces, instead of working with the usual proper homotopy fixed points, i.e. equivariant maps from $\underline{E}G$, we need to consider maps from $\underline{E}G$ which are in some sense bounded on each degree, see Section 3.1.

In the first chapter we will recall some basic definitions we need in this thesis and also the concept of finite decomposition complexity, which was introduced in [GTY13, GTY12]. We will also recall some facts about algebraic $K$- and $L$-theory and define the assembly maps. In the last section of the first chapter we reformulate the assembly maps in terms of controlled categories. Using controlled categories is so far the most important tool for proving the Farrell-Jones conjecture since it allows to use the geometric properties of the group.

In the second chapter we generalize the results from [RTY] to a version which is equivariant with respect to finite subgroups so that we can allow for groups with torsion.

In the third chapter we start by generalizing the Descent Principle and then use it to show how the result from chapter 2 implies the main theorem.

The last chapter provides some examples of groups satisfying the assumptions of the main theorem. As already mentioned the main class of groups for which we can prove the assumptions is a subclass of linear groups.

In the thesis we will use the following notations and conventions:

- $G$ always denotes a group and all groups are discrete and countable. If not specified otherwise $\mathcal{F}in$ denotes the family of finite subgroups of $G$.

- Metrics in this thesis are allowed to take on the value $\infty$ if not specified otherwise, but metrics on groups are always finite. For a group $G$ a metric $G$-space is a metric space with an isometric $G$-action and a metric ($G$-)family is a family of metric ($G$-)spaces.

- For $r > 0$ and a subspace $Y$ of a metric space $X$ we define

$$Y^r := \{x \in X \mid d(x, Y) < r\}.$$

- For metric spaces $\{(X_i, d_i)\}_{i \in I}$ we define $\coprod_{i \in I} X_i$ to be the formal set-theoretic disjoint union of the spaces $X_i$, i.e. points in $\coprod_{i \in I} X_i$ are pairs $(x, i)$ with $x \in X_i$ and we give this space the metric

$$d(x,i),(y,j)) = \begin{cases} d_i(x,y) & \text{if } i = j \\ \infty & \text{else} \end{cases}$$

- If $I$ is a $G$-set, $X$ a $G$-space and $\{Y_i\}_{i \in I}$ a family of subspaces of $X$ with the property that $y \in Y_i, g \in G$ implies that $gy \in Y_{gi}$, then we always use the following $G$-action on $\coprod_{i \in I} Y_i$:

$$g(y, i) := (gy, gi)$$

  On $\prod_{i \in I} Y_i$ we use the following $G$-action:

$$g(y_i)_{i \in I} := (gy_{g^{-1}i})_{i \in I}$$

  Analogously when $I$ is a $G$-set, $\mathcal{A}$ is a $G$-category and $\{\mathcal{U}_i\}_{i \in I}$ is a family of full subcategories of $\mathcal{A}$ with the property that $A \in \mathcal{U}_i, g \in G$ implies that $gA \in \mathcal{U}_{gi}$, then we use the following $G$-action on $\prod_{i \in I} \mathcal{U}_i$:

$$g(A_i)_{i \in I} := (gA_{g^{-1}i})_{i \in I}$$

# 1 Preliminaries

In this chapter we want to give some basic definitions in particular about $G$-CW complexes and classifying spaces and introduce the concept of finite decomposition complexity. Furthermore, we want to recall several facts about $K$-theory and define the main object of our studies, the assembly map (see Conjecture 1.6.6). In the last section we will reformulate the assembly map using controlled categories.

## 1.1 $G$-CW complexes

The spaces we work with will in most cases be simplicial $G$-CW complexes. For more information on $G$-CW complexes see for instance [Lüc89, Section 1 and 2] and [tD87, Section II.1 and II.2].

**Definition 1.1.1.** Let $G$ be a group. A $G$-CW complex is a $G$-space $X$ together with a filtration $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X$ such that $X \cong \operatorname{colim}_{n\in\mathbb{N}} X_n$ and such that for every $n \in \mathbb{N}$ there exists a collection $\{G_i\}_{i\in I_n}$ of subgroups of $G$ and a $G$-pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G/G_i \times S^n & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/G_i \times D^n & \longrightarrow & X_n \end{array}$$

Equivalently, a $G$-CW complex is a CW complex $X$ with a cellular $G$-action, where an action is cellular if for each open cell $e$ of $X$ and every $g \in G$ the left translation $ge$ is again an open cell of $X$ and if $ge = e$ then $gx = x$ for all $x \in e$. See [tD87, Proposition II.1.15].

A $G$-CW complex is called finite dimensional if the underlying CW complex is finite dimensional, i.e. there exists $n \in \mathbb{N}$ with $X = X_n$.

A $G$-CW complex is called finite if it is built out of finitely many $G$-cells $G/G_i \times D^n$, i.e. it is finite dimensional and each index set $I_n$ is finite.

**Definition 1.1.2.** A simplicial $G$-CW complex is a simplicial complex with a simplicial $G$-action such that for each open simplex $\sigma$ and every $g \in G$ either $g\sigma \cap \sigma = \emptyset$ or $gx = x$ for every $x \in \sigma$.

For every simplicial complex $K$ with a simplicial $G$-action the barycentric subdivision $BK$ is a simplicial $G$-CW complex.

**Definition 1.1.3.** A metric space $(X, d)$ (resp. the metric $d$) is called proper if for every $R > 0$ and every $x \in X$ the closed ball $\overline{B}_R(x) := \{y \in X \mid d(x, y) \leq R\} \subseteq X$ is compact.

We will call a metric $d$ on $X$ finite if $d(x,y) < \infty$ for all $x, y \in X$.

Mostly we will consider simplicial $G$-CW complexes with proper $G$-invariant metrics. To gain such metrics we will use the following.

**Definition 1.1.4** (Simplicial path metric)**.** Let $K$ be a simplicial complex with vertices $v_i$. Define the euclidean metric $d'$ on $K$ by

$$d'(\sum_i x_i v_i, \sum_i y_i v_i) = \sqrt{\sum_i |x_i - y_i|^2}.$$

Define the path metric $d$ on $K$ by

$$d(x,y) = \inf \sum_{i=0}^{N} d'(p_i, p_{i+1})$$

where the infimum is taken over all sequences $x = p_0, .., p_{N+1} = y$ (with arbitrary $N$) such that $p_i$ an $p_{i+1}$ lie in a common simplex of $K$. We define $d(x,y) = \infty$ if no such path exists, i.e. if $x$ and $y$ lie in different path components of $K$.

**Remark 1.1.5.** The path metric restricts to the euclidean metric on each simplex. By [Spa66, Theorem 3.2.8] the euclidean metric generates the weak topology on $K$ if and only if $K$ is locally finite. By [Roe03, Proposition 1.24] the metrics $d$ and $d'$ generate the same topology and the path metric $d$ is proper if $K$ is locally finite. If $G$ acts simplicially on $K$, then $d$ and $d'$ are $G$-invariant.

## 1.2 Classifying spaces

A classifying space of a group $G$ is a contractible $G$-CW complex with a free $G$-action. More generally, we need the definition of classifying spaces for families of subgroups. Recall the following from [Lüc05, Section 1.2] and [tD87, Section I.6].

**Definition 1.2.1.** Let $G$ be a group. A family of subgroups $\mathcal{F}$ of $G$ is a set of subgroups which is closed under conjugation, i.e. $H \in \mathcal{F}, g \in G$ implies $g^{-1}Hg \in \mathcal{F}$, and under finite intersection, i.e. $H, H' \in \mathcal{F}$ implies $H \cap H' \in \mathcal{F}$.

The examples of families of subgroups we will consider are $\{1\}$, the family consisting only of the trivial group, $\mathcal{F}in$, the family of all finite subgroups, and $\mathcal{V}Cyc$, the family of all virtually cyclic subgroups. A group is called virtually cyclic if it contains a cyclic subgroup of finite index.

**Definition 1.2.2.** Let $\mathcal{F}$ be a family of subgroups of $G$. A $G$-$\mathcal{F}$-CW complex is a $G$-CW complex which isotropy groups all belong to $\mathcal{F}$.

**Definition 1.2.3.** Let $G$ be a group and $\mathcal{F}$ a family of subgroups of $G$ then a classifying space $\mathrm{E}_{\mathcal{F}}G$ for the family $\mathcal{F}$ is a terminal object in the $G$-homotopy category of $G$-$\mathcal{F}$-CW complexes, i.e. for every $G$-$\mathcal{F}$-CW complex $X$ there exists up to $G$-homotopy a unique $G$-map $X \to \mathrm{E}_{\mathcal{F}}G$.

**Proposition 1.2.4** ([Lüc05, Theorem 1.9],[tD87, Section I.6]). *For every family of subgroups $\mathcal{F}$ of $G$ there exists a model for $E_{\mathcal{F}}G$. A $G$-$\mathcal{F}$-CW complex $X$ is a model for $E_{\mathcal{F}}G$ if and only if $X^H \simeq *$ for all $H \in \mathcal{F}$.*

For the families of subgroups above we define the following abbreviations:

$$\mathrm{E}G := E_{\{1\}}G$$
$$\underline{\mathrm{E}}G := E_{\mathcal{F}in}G$$
$$\underline{\underline{\mathrm{E}}}G := E_{\mathcal{V}Cyc}G$$

## 1.3 Metric properties of $\underline{\mathrm{E}}$G

Since we want to use metric methods for proving the injectivity of the assembly map we need to know that we can find models for $\underline{\mathrm{E}}G$ with nice metric properties. This is established by the following theorem.

**Theorem 1.3.1.** *If $G$ admits a finite dimensional $G$-CW model for $\underline{\mathrm{E}}G$, then it also admits a finite dimensional simplicial $G$-CW model for $\underline{\mathrm{E}}G$ with a proper $G$-invariant metric.*

Before we can give the proof of this theorem we first need the following facts about $G$-CW complexes and simplicial complexes.

**Theorem 1.3.2** (Simplicial Approximation, [Hat02, Theorem 2C.1]). *If $K$ is a finite simplicial complex and $L$ is an arbitrary simplicial complex, then any map $f : K \to L$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of $K$.*

**Definition 1.3.3** (Simplicial Mapping Cone). Let $f : K \to L$ be a simplicial map. The simplicial mapping cylinder $M(f)$ is the following simplicial complex. Vertices are the disjoint union of the vertices of $L$ and those of the barycentric subdivision $BK$ of $K$. Let $\sigma$ be a simplex of $K$ then for any vertex $y \in L$ the vertices $\sigma \in BK$ and $y$ span a 1-simplex in $M(f)$ if there is a vertex of $\sigma$ that is mapped to $y$ under $f$. A finite set of vertices $x_1, .., x_n \in BK$ and $y_1, .., y_m \in L$ spans an $(n + m - 1)$ simplex in $M(f)$ if $x_1, .., x_n$ span an $(n-1)$-simplex in $BK$, $y_1, .., y_m$ span an $(m-1)$-simplex in $L$ and $x_i, y_j$ span a 1-simplex in $M(f)$ for all $i, j$.

The simplicial mapping cone $C(f)$ is $M(f)$ together with a cone on $BK \subseteq M(f)$.

We can now prove the following.

**Lemma 1.3.4.** *Let $X$ be a finite dimensional $G$-CW-complex with countably many cells, then $X$ is $G$-homotopy equivalent to a countable simplicial $G$-CW complex of the same dimension.*

This fact is well known and for example in [Hat02, 2C.5] a proof for the case $G = \{e\}$ can be found. We will here give this proof in an equivariant fashion.

*Proof.* We will prove this by an induction over the skeleta $X^{(n)}$ of $X$. Let $Y_0 := X^{(0)}$ be the zero skeleton of $X$. This is a zero dimensional simplicial $G$-complex. Suppose we already have constructed an $n$-dimensional simplicial $G$-complex $Y_n$ with a $G$-homotopy equivalence $f_n : X^{(n)} \to Y_n$. Let $\varphi_\alpha : G/G_\alpha \times S^n \to X^{(n)}$ be the attaching maps of the $n+1$-cells of $X$ and let $e$ denote the neutral element of $G$. By Theorem 1.3.2 the map $f_n \circ \varphi_\alpha(e, \_) : S^n \to (X^{(n)})^{G_\alpha} \to Y_n^{G_\alpha}$ is homotopic to a simplicial map $\hat{\psi}_\alpha : S^n \to Y_n^{G_\alpha}$. This yields a simplicial $G$-equivariant map $\psi_\alpha : G/G_\alpha \times S^n \to Y_n$. Define $Y_{n+1} := C(\coprod \psi_\alpha)$ with the $G$-action permuting the cones, then $f : X^{(n)} \to Y_n$ can be extended to a $G$-homotopy equivalence $f_{n+1} : X^{(n+1)} \to Y_{n+1}$. By this construction we get a simplicial $G$-complex $Y$ which is $G$-homotopy equivalent to $X$, has the same dimension and is also countable. To turn $Y$ in a simplicial $G$-CW complex we might need to use one step of barycentric subdivision. □

**Lemma 1.3.5.** *Let $X$ be a finite dimensional, countable, (simplicial) $G$-$\mathcal{F}in$-CW-complex. Then $X$ is $G$-homotopy equivalent to a locally finite, finite dimensional, countable, (simplicial) $G$-$\mathcal{F}in$-CW complex.*

*Proof.* Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an enumeration of the $G$-cells of $X$. For every $t \geq 0$ define $Y_t := \{\sigma_n \mid n \leq \lfloor t \rfloor\}$ and let $X_t$ be the smallest subcomplex of $X$ containing $Y_t$. Since $Y_t$ contains only finitely many $G$-cells, $X_t$ is finite for every $t \geq 0$ as well.

The mapping telescope $T := \{(x,t) \in X \times [0,\infty) \mid x \in X_t\}$ is a $G$-CW complex and since $X$ has only finite stabilizers, $T$ is locally finite.

The natural projection $p : T \to X$ is bijective on $\pi_0$. Let $x_0 \in X$ and choose $t > 0$ with $x_0 \in X_t$. Let $f : (S^n, pt) \to (X, x_0)$ be a pointed map. There exists $t'$ with $f(S^n) \subseteq X_{t'}$. The inclusion $X_{t'} \times \{t'\} \subseteq T$ gives a map $g' : (S^n, pt) \to (T, (x_0, t'))$ with $p \circ g' = f$. Using the linear path from $(x_0, t)$ to $(x_0, t')$ we get a map $g : (S^n, pt) \to (T, (x_0, t))$ with $p \circ g \simeq f$. Therefore, $p$ is surjective on $\pi_n$. A similar argument shows that $p$ is injective on $\pi_n$ as well.

By the same argument for each subgroup $H \leq G$ the projection of the fixed point spaces $p : T^H \to X^H$ is an isomorphism for all homotopy groups.

Since both $T$ and $X$ are $G$-CW complexes, the map $p$ is a $G$-homotopy equivalence, [tD87, Propostion II.2.7].

If $X$ is simplicial, then there is a simplicial structure on $T$ with vertices $(v, n)$, where $n \in \mathbb{N}$ and $v$ a vertex of $X_n$. □

**Lemma 1.3.6.** *If $X$ is a model for $\underline{E}G$, then $X$ contains a countable $G$-subcomplex which is still a model for $\underline{E}G$.*

*Proof.* Because $G$ has only countably many finite subgroups there exists a countable $G$-subcomplex $X_0 \subseteq X$ with $X_0^H \neq \emptyset$ for all $H \leq G$ finite. Inductively define countable $G$-subcomplexes $X_i \subseteq X$, $i \in \mathbb{N}$, such that $X_{i-1}^H \hookrightarrow X_i^H$ is null homotopic for every finite subgroup $H \leq G$. Those exist because $X^H$ is contractible and they can be chosen as countable complexes since the image of every contraction of $X_{i-1}^H$ in $X^H$ lies in a countable subcomplex. Since $(\bigcup_{i\in\mathbb{N}} X_i)^H = \bigcup_{i\in\mathbb{N}} X_i^H$ and $X_i^H$ is contractible in $X_{i+1}^H$ the subcomplex $(\bigcup_{i\in\mathbb{N}} X_i)^H$ has vanishing homotopy groups, and is therefore contractible. So $\bigcup_{i\in\mathbb{N}} X_i$ is a countable subcomplex of $X$ which is still a model for $\underline{E}G$. □

Combining the three lemmas above we can prove Theorem 1.3.1.

*Proof of Theorem 1.3.1.* By Lemma 1.3.6 there exists a countable, finite dimensional $G$-CW model for $\underline{E}G$. Thus, by Lemma 1.3.4 and Lemma 1.3.5 the group $G$ admits a locally finite, finite dimensional, simplicial $G$-CW model $X$ for $\underline{E}G$. By Remark 1.1.5 the path metric on $X$ has the desired properties. □

## 1.4 Finite decomposistion complexity (FDC)

The main metric property we are interested in is finite decomposition complexity. This is a generalization of finite asymptotic dimension. We start by giving the definition of asymptotic dimension due to Gromov [Gro93].

**Definition 1.4.1.** Let $X$ be a metric space. A decomposition $X = \bigcup_{i\in I} U_i$ is called <u>$r$-disjoint</u>, if $d(U_i, U_j) > r$ for all $i \neq j \in I$. We then denote the decomposition by

$$X = \bigcup^{r-disj} U_i.$$

A metric space $X$ has <u>asymptotic dimension at most $d$</u> if for every $r > 0$ there exist $d+1$ subspaces $U_i$ covering $X$, i.e. $X = \bigcup_{i=0}^{d} U_i$, and $r$-disjoint decompositions

$$U_i = \bigcup_{j\in J_i}^{r-disj} U_{i,j}$$

such that $\sup_{i,j} \operatorname{diam} U_{i,j} < \infty$.

If there exists $d \in \mathbb{N}$ such that $X$ has asymptotic dimension at most $d$ we say that $X$ has <u>finite asymptotic dimension (FAD)</u>.

A space $X$ has asymptotic dimension $d$ if it has asymptotic dimension at most $d$ but not asymptotic dimension at most $d-1$. We denote the asymptotic dimension of $X$ by $\operatorname{asdim} X$.

A family $\{X_\alpha\}_{\alpha\in A}$ has <u>asymptotic dimension at most $d$ uniformly</u> if for every $r > 0$ there exist decompositions

$$X_\alpha = \bigcup_{i=0}^{d} U_i^\alpha, \qquad U_i^\alpha = \bigcup_{j\in J_i^\alpha}^{r-disj} U_{i,j}^\alpha$$

for all $\alpha \in A$ as above such that $\sup_{\alpha,i,j} \operatorname{diam} U_{i,j}^\alpha < \infty$.

**Example 1.4.2.** The space $\mathbb{R}^n$ with the euclidean metric has asymptotic dimension $n$.

Recall the following definition.

**Definition 1.4.3.** A map $f\colon (X,d_X)\to (Y,d_Y)$ between metric spaces is a coarse embedding if there exist proper non-decreasing functions $\rho,\delta\colon [0,\infty)\to[0,\infty)$ such that

$$\delta(d_X(x,x'))\le d_Y(f(x),f(x'))\le \rho(d_X(x,x')),\quad \forall x,x'\in X \text{ with } d_X(x,x')<\infty.$$

A coarse embedding $f\colon X\to Y$ is a coarse equivalence if there exists a coarse embedding $g\colon Y\to X$ such that both compositions have bounded distance to the identity, i.e. there exists $R>0$ such that

$$d_Y(f\circ g(y),y)\le R,\ d_X(g\circ f(x),x)\le R,\quad \forall x\in X, y\in Y.$$

**Lemma 1.4.4.** *If there exists a coarse embedding $f\colon X\to Y$, then $\operatorname{asdim} X\le \operatorname{asdim} Y$. In particular, if $f$ is a coarse equivalence, then $\operatorname{asdim} X=\operatorname{asdim} Y$.*

*Proof.* Let $f\colon (X,d_X)\to(Y,d_Y)$ be a coarse embedding and let $\rho,\delta\colon[0,\infty)\to[0,\infty)$ be proper non-decreasing functions such that

$$\delta(d_X(x,x'))\le d_Y(f(x),f(x'))\le \rho(d_X(x,x')),\quad \forall x,x'\in X \text{ with } d_X(x,x')<\infty.$$

Assume $Y$ has finite asymptotic dimension and define $n:=\operatorname{asdim} Y$. Let $R>0$ be given and choose a cover $Y=\bigcup_{i=0}^n Y_i$ with $\rho(R)+1$-disjoint decompositions

$$Y_i=\bigcup_{j\in J_i}^{\rho(R)+1-disj} Y_{ij},$$

such that $\sup_{i,j}\operatorname{diam} Y_{ij}<\infty$.

Define $X_i:=f^{-1}(Y_i)$ and $X_{ij}:=f^{-1}(Y_{ij})$. Let $\{X_{ijk}\}_{k\in J_{ij}}$ be the decomposition of $X_{ij}$ into infinitely apart subspaces, i.e. $\operatorname{diam} X_{ijk}<\infty$ and $d_X(X_{ijk},X_{ijk'})=\infty$ for $k\neq k'$. For $x\in X_{ij}, x'\in X_{ij'}$ with $j\neq j'$ and $d(x,x')<\infty$ we have

$$\rho(d_X(x,x'))\ge d_Y(f(x),f(x'))\ge\rho(R)+1$$

and since $\rho$ is non-decreasing this implies that the decomposition $X_i=\bigcup_{j\in J_i,k\in J_{ij}} X_{ijk}$ is $R$-disjoint. Furthermore, $\delta(\operatorname{diam} X_{ijk})\le \operatorname{diam} Y_{ij}$ for all $k\in J_{ij}$ which implies $\sup_{ijk}\operatorname{diam} X_{ijk}<\infty$ since $\delta$ is proper. It follows that $\operatorname{asdim} X\le n=\operatorname{asdim} Y$. □

Since asymptotic dimension is a coarse invariant we can define the asymptotic dimension of a group as implied by the following proposition.

**Proposition 1.4.5.** *Let $G$ be a countable discrete group, then $G$ admits a finite, proper, left-invariant metric. Furthermore, for any two such metrics $d,d'$ the identity*

$$\mathrm{id}\colon (G,d)\to(G,d')$$

*is a coarse equivalence. In particular, setting $\operatorname{asdim} G:=\operatorname{asdim}(G,d)$ does not depend on the chosen proper, left-invariant metric.*

*Proof.* Let a enumeration $\{g_n\}$ of the elements of $G$ be given and let $e$ denote the neutral element of $G$. Define

$$l(g) := \min\left\{n_1 + \ldots + n_m \mid n_i \in \mathbb{N}, \exists \epsilon_i \in \{-1,1\} : g = g_{n_1}^{\epsilon_1} \cdot \ldots \cdot g_{n_m}^{\epsilon_m}\right\}.$$

Then $d(g,g') := l(g^{-1}g')$ is a finite, proper, left-invariant metric.

For the second part we can assume that $G$ is infinite. For each $r > 0$ define

$$\rho(r) := \sup\left\{d'(e,g) \mid g \in G, d(e,g) \leq r\right\}.$$

This function is proper since the metric $d'$ is proper and $G$ is infinite. Furthermore, $d'(g,g') = d'(e,g^{-1}g') \leq \rho(d(e,g^{-1}g')) = \rho(d(g,g'))$. Define

$$\delta(r) := \sup\left\{L \mid \sup\{d(e,g) \mid d'(e,g) \leq L\} < r\right\}.$$

This function is proper since the metric $d'$ is proper and we get

$$d'(g,g') = d'(e,g^{-1}g') \geq \delta(d(e,g^{-1}g')) = \delta(d(g,g')).$$

□

**Example 1.4.6.** The euclidean metric on $\mathbb{Z}^n$ is proper and left-invariant. The inclusion $\mathbb{Z}^n \to \mathbb{R}^n$ is a coarse equivalence. Therefore, $\operatorname{asdim} \mathbb{Z}^n = n$.

The group $\mathbb{Z}^\infty := \bigoplus_{n\in\mathbb{N}} \mathbb{Z}^n$ has the following proper left-invariant metric

$$d((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) := \sum_{n\in\mathbb{N}} n|a_n - b_n|.$$

Note that it is important to scale the metric in the $n$-th component by $n$ to obtain a proper metric. Taking the induced metric on $\mathbb{Z}^\infty$ as a subspace of $\mathbb{R}^\infty := \bigoplus_{n\in\mathbb{N}} \mathbb{R}$ with the euclidean metric would not give a proper metric on $\mathbb{Z}^\infty$. With this proper metric $\mathbb{Z}^\infty$ is not coarsely equivalent to $\mathbb{R}^\infty$.

Since $\mathbb{Z}^\infty$ contains $\mathbb{Z}^n$ for every $n$ it has infinite asymptotic dimension. But still it has a kind of two step finite asymptotic dimension as we will explain now.

First, let $n \in \mathbb{N}$ be given. We can decompose $\mathbb{Z}^\infty$ as

$$\mathbb{Z}^\infty = \bigcup_{i\in I}^{n-disj} g_i + \mathbb{Z}^n;$$

where $\{g_i\}_{i\in I}$ is a representation system of the cosets for the subgroup $\mathbb{Z}^n \leq \mathbb{Z}^\infty$. This decomposition is $n$-disjoint by our choice of metric on $\mathbb{Z}^\infty$.

The subspaces $g_i + \mathbb{Z}^n$ are all isometric and have asymptotic dimension $n$. Therefore, the family $\{g_i + \mathbb{Z}^n\}_{i\in I}$ has asymptotic dimension $n$ uniformly. Note that the asymptotic dimension of the family depends on the chosen number $n \in \mathbb{N}$.

This example motivates the following definition of finite decomposition complexity. Finite decomposition complexity was first introduced in [GTY13, 2.1.3].

**Definition 1.4.7.** Let $r > 0$. A metric family $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ r-decomposes over a class of metric families $\mathfrak{D}$ if for every $\alpha \in A$ there exists a decomposition $X_\alpha = U_\alpha^r \cup V_\alpha^r$ and $r$-disjoint decompositions

$$U_\alpha^r = \bigcup_{i \in I(r,\alpha)}^{r-disj} U_{\alpha,i}^r, \qquad V_\alpha^r = \bigcup_{j \in J(r,\alpha)}^{r-disj} V_{\alpha,j}^r$$

such that the families $\{U_{\alpha,i}^r\}_{\alpha \in A, i \in I(r,\alpha)}$ and $\{V_{\alpha,j}^r\}_{\alpha \in A, j \in J(r,\alpha)}$ lie in $\mathfrak{D}$. A metric family $\mathcal{X}$ decomposes over $\mathfrak{D}$ if it $r$-decomposes over $\mathfrak{D}$ for all $r > 0$.

Let $\mathfrak{B}$ denote the class of bounded families, i.e. $\mathcal{X} \in \mathfrak{B}$ if there exists $R > 0$ such that $\operatorname{diam} X < R$ for all $X \in \mathcal{X}$. We set $\mathfrak{D}_0 = \mathfrak{B}$ and given a successor ordinal $\gamma + 1$ we define $\mathfrak{D}_{\gamma+1}$ to be the class of all metric families which decompose over $\mathfrak{D}_\gamma$. For a limit ordinal $\gamma$ we define

$$\mathfrak{D}_\gamma = \bigcup_{\beta < \gamma} \mathfrak{D}_\beta.$$

A metric family $\mathcal{X}$ has finite decomposition complexity (FDC) if $\mathcal{X} \in \mathfrak{D}_\gamma$ for some ordinal $\gamma$.

A metric space $X$ has FDC if the family $\{X\}$ consisting only of $X$ has FDC.

**Remark 1.4.8.** It is easy to see that every metric space lying in $\mathfrak{D}_n$ with $n \in \mathbb{N}$ has asymptotic dimension at most $2^n$. By [GTY13, Thoorem 4.1] the converse holds, i.e. every metric space with finite asymptotic dimension lies in $\mathfrak{D}_n$ for some $n \in \mathbb{N}$. By the same proof every metric family with finite asymptotic dimension uniformly lies in $\mathfrak{D}_n$ for some $n \in \mathbb{N}$.

**Example 1.4.9.**

(a) By Remark 1.4.8 our above decomposition of $\mathbb{Z}^\infty = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ proves that $\mathbb{Z}^\infty$ has FDC.

(b) Let $R$ be a commutative ring with unit. By [GTY13, Theorem 5.2.2] every countable subgroup of $GL_n(R)$ has FDC.

(c) By [GTY13, Theorem 5.1.2] all elementary amenable groups have FDC.

**Example 1.4.10.** The space $\mathbb{R}^\infty = \bigoplus_{n \in \mathbb{N}} \mathbb{R}$ with metric

$$d((x_n),(y_n)) := \sum_{n \in \mathbb{N}} |x_n - y_n|$$

does not have FDC. We will prove this by contradiction and assume $\mathbb{R}^\infty$ has FDC. Applying the definition of FDC with parameter $r = 1$ in all steps we get a sequence $\mathcal{U}^n$, $n \in \mathbb{N}$ of metric families with

- $\mathcal{U}^0 := \{\mathbb{R}^\infty\}$;
- there are decompositions $U = V_0 \cup V_1$ for all $n \in \mathbb{N}, U \in \mathcal{U}^n$ such that

$$V_i = \bigcup_{j \in I_i}^{1-disj} V_i^j$$

  and $V_i^j \in \mathcal{U}^{n+1}$;
- there exists $N \in \mathbb{N}$ with $\sup_{U \in \mathcal{U}^N} \operatorname{diam} U < \infty$.

Defining $r\mathcal{U}^n$ to be the family consisting of all spaces $rU := \{ru \mid u \in U\}$ with $U \in \mathcal{U}^n$ and letting $N$ be as above, we obtain a sequence of families, such that

- $r\mathcal{U}^0 := \{\mathbb{R}^\infty\}$;
- there are decompositions $rU = rV_0 \cup rV_1$ for all $n \in \mathbb{N}, rU \in \mathcal{U}^n$ such that

$$rV_i = \bigcup_{j \in I_i}^{r-disj} rV_i^j$$

  and $rV_i^j \in r\mathcal{U}^{n+1}$;
- $\sup_{rU \in r\mathcal{U}^N} \operatorname{diam} rU < \infty$.

Using this decomposition implies that $\mathbb{R}^\infty \in \mathfrak{D}_N$ and thus we have $\operatorname{asdim} \mathbb{R}^\infty \leq 2^N$ by Remark 1.4.8. But $\operatorname{asdim} \mathbb{R}^\infty = \infty$, since for every $n \in \mathbb{N}$ the space $\mathbb{R}^n$ embeds into $\mathbb{R}^\infty$.

**Definition 1.4.11.** By [GTY13, Coarse Invariance 3.1.3] FDC is a coarse invariant and therefore we say a group $G$ has FDC if it has FDC with any (and thus every) finite, proper, left-invariant metric.

In the next section we will give a reminder on the definitions of coarse invariance for metric families and recall inheritance properties for FDC from [GTY13].

## 1.5 Inheritance properties of FDC

To formulate the inheritance properties of FDC we need the following elementary concepts from coarse geometry. For more information on coarse geometry see for example [Roe03] or [GTY13, Section 3].

**Definition 1.5.1.** A map $F\colon \mathcal{X} = \{X_i\} \to \mathcal{Y} = \{Y_j\}$ between metric families consists of maps $f : X_i \to Y_j$ such that every $X_i \in \mathcal{X}$ is the domain of at least one map of the family. It is

- uniformly expansive if there exists a non-decreasing function
$$\rho\colon [0,\infty) \to [0,\infty)$$
such that for every $(f\colon X_i \to Y_j) \in F, x, y \in X_i$ with $d(x,y) < \infty$
$$d(f(x), f(y)) \leq \rho(d(x,y)), \tag{1.5.2}$$

- effectively proper if there exists a proper non-decreasing function
$$\delta\colon [0,\infty) \to [0,\infty)$$
such that for every $(f\colon X_i \to Y_j) \in F, x, y \in X_i$ with $d(x,y) < \infty$
$$\delta(d(x,y)) \leq d(f(x), f(y)), \tag{1.5.3}$$

- a coarse embedding if it is both uniformly expansive and effectively proper,

- a coarse equivalence if it is a coarse embedding and there exists a coarse embedding $G : \mathcal{Y} \to \mathcal{X}$ such that there is a constant $C > 0$ with $d(x, g \circ f(x)), d(y, f \circ g(y)) \leq C$ for all $x \in X_i, y \in Y_j, (f\colon X_i \to Y_j) \in F, (g\colon Y_j \to X_i) \in G$.

A map $f\colon X \to Y$ between metric spaces is metrically coarse if it is uniformly expansive and proper. If $X$ is proper and the metric on $X$ is finite, then $f$ is metrically coarse if it is a coarse embedding.

A continuous, metrically coarse homotopy between proper continuous maps is called a metric homotopy.

**Remark 1.5.4.** In the case where the metric spaces have a finite metric these definitions coincide with the common definitions but we allow effectively proper maps to map points with infinite distance close together. This allows us that metric spaces $X = \coprod_{i\in I} X_i$ and metric families $\{X_i\}_{i\in I}$ can be regarded in the same way.

**Definition 1.5.5.** A family of subspaces $\mathcal{Z}$ of a metric family $\mathcal{Y}$ is a family of metric spaces such that for every $Z \in \mathcal{Z}$ there exists a fixed $Y \in \mathcal{Y}$ with $Z \subseteq Y$. Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a map of metric families. We define $F^{-1}(\mathcal{Z}) := \{f^{-1}(Z) \mid f \in F, f\colon X \to Y, Z \subseteq Y\}$.

Recall the following inheritance properties of FDC from [GTY13].

**Lemma 1.5.6** (Coarse Invariance [GTY13, 3.1.3])**.** *Let $\mathcal{X}, \mathcal{Y}$ be metric families. If there is a coarse embedding from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{Y}$ has FDC, then so does $\mathcal{X}$. In particular if $\mathcal{X}$ has FDC, then any family of subspaces of $\mathcal{X}$ has FDC.*

**Theorem 1.5.7** (Fibering [GTY13, 3.1.4])**.** *Let $\mathcal{X}, \mathcal{Y}$ be metric families and $F\colon \mathcal{X} \to \mathcal{Y}$ uniformly expansive. Assume $\mathcal{Y}$ has FDC and for every bounded family of subspaces $\mathcal{Z}$ of $\mathcal{Y}$ the family $F^{-1}(\mathcal{Z})$ has FDC. Then $\mathcal{X}$ has FDC.*

**Remark 1.5.8.** Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a map of metric families. The families $F^{-1}(\mathcal{Z})$ for every bounded family of subspaces $\mathcal{Z}$ of $\mathcal{Y}$ have FDC if and only if for every $R > 0$ the families $F^{-1}(\{B_R(y) \subseteq Y \mid y \in Y, Y \in \mathcal{Y}\})$ have FDC.

**Theorem 1.5.9** (Finite Union [GTY13, 3.1.7])**.** *Let $X$ be a metric space. Let $\mathcal{X}_1, .., \mathcal{X}_k$ be families of subspaces of $X$ that have FDC. We define $\mathcal{X}$ to be the family consisting of all unions $\bigcup_{i=1}^k X_i$ with $X_i \in \mathcal{X}_i$. Then $\mathcal{X}$ has FDC.*

The last theorem is proved in [GTY13] only for metric spaces not for metric families. But the same proof holds for metric families using that if two families $\mathcal{U}, \mathcal{V}$ have FDC, then the union $\mathcal{U} \cup \mathcal{V}$ has FDC as well.

Furthermore, all metrics in [GTY13] are finite, but the same proofs hold in the setting where infinite metrics are allowed. This follows from the fact that for the purpose of decomposing the space $X = \coprod_{i\in I} X_i$ can be treated in the same way as the family $\mathcal{X} = \{X_i\}_{i\in I}$.

## 1.6 K- and L-theory

As mentioned in the introduction the $K$- and $L$-theory of group rings $R[G]$ plays an important role in topology. There is a functor $\mathbb{K}(= \mathbb{K}^{-\infty})$ from small additive categories to (non-connective) spectra such that $\pi_n\mathbb{K}(\mathcal{P}_R) \cong K_n(R)$, where $\mathcal{P}_R$ is the category of finitely generated projective $R$-modules, or to be more precise a small skeleton of this category. This functor is constructed in [PW85]. There also is a functor $\mathbb{L}(= \mathbb{L}^{-\infty})$ from small additive categories with involution to (non-connective) spectra such that $\pi_n\mathbb{L}(\mathcal{P}_R) = L_n(R)$ for any ring $R$ with involution. This functor is constructed in [Ran92]. In the following $\mathbb{K}$ and $\mathbb{L}$ will always denote the functors to non-connective spectra. We will state some more properties of these functors at the end of this section. We denote the homotopy groups $\pi_n$ of $\mathbb{K}(\mathcal{A})$ and $\mathbb{L}(\mathcal{A})$ by $K_n(\mathcal{A})$ and $L_n(\mathcal{A})$ respectively. It is well known that both functors $\mathbb{K}$ and $\mathbb{L}$ commute with taking fixed points, i.e. if $\mathcal{A}$ is an additive $G$-category, then the natural maps

$$\mathbb{K}(\mathcal{A}^G) \to \mathbb{K}(\mathcal{A})^G \quad \text{and} \quad \mathbb{L}(\mathcal{A}^G) \to \mathbb{L}(\mathcal{A})^G$$

are weak equivalences.

To define the assembly maps we need the following definition.

**Definition 1.6.1.** Let $G$ be a group. A <u>$G$-homology theory</u> is a functor $\mathbb{H}^G$ from the category of $G$-CW pairs to spectra such that the following axioms hold:

(a) (Homotopy invariance) If $f$ is a $G$-homotopy equivalence, then $\mathbb{H}^G(f)$ is a weak equivalence.

(b) (Pair sequence) For any $G$-CW pair $(X, A)$ the inclusions $A \to X$ and $X \to (X, A)$ induce a homotopy fibration sequence of spectra:

$$\mathbb{H}^G(A) \to \mathbb{H}^G(X) \to \mathbb{H}^G(X, A)$$

(c) (Excision) For a $G$-CW pair $(X, A)$ and a cellular map $f : A \to B$ the map $F : (X, A) \to (X \cup_f B, B)$ induces a weak homotopy equivalence $\mathbb{H}^G(f)$.

**Definition 1.6.2.** The orbit category $OrG$ of a group $G$ has the $G$-sets $G/H$, where $H \leq G$ is a subgroup, as objects and has all $G$-maps between the objects as morphisms.

From any functor $F\colon OrG \to \mathfrak{Spectra}$ a $G$-homology theory $\mathbb{F}$ can be constructed via

$$\mathbb{F}(X) := \mathrm{Map}_G(\_, X_+) \wedge_{OrG} F,$$

see [DL98, Section 4].

This functor has the property that $\mathbb{F}(G/H) \simeq F(G/H)$. We will denote its homotopy groups by

$$H_n^G(\_; F) := \pi_n \mathbb{F}(X).$$

**Definition 1.6.3.** Let $\mathcal{A}$ be an additive $G$-category, $H \leq G$ a subgroup. We define the additive category $\mathcal{A}[H]$ as the additive category with objects being the objects of $\mathcal{A}$ and morphisms $\varphi\colon A \to B$ being collections $\{\varphi_h\}_{h \in H}$ of morphisms $\varphi_h : A \to h^{-1}B$ of $\mathcal{A}$, such that $\varphi_h = 0$ for almost every $h \in H$. Addition of morphisms is defined componentwise and composition of morphisms $\varphi, \varphi'$ is defined as

$$(\varphi \circ \varphi')_h := \sum_{kl=h} l^{-1}(\varphi_k) \circ \varphi'_l.$$

**Example 1.6.4.** Let $G$ act on a ring $R$ and let $R_tG$ denote the group ring twisted by this action. Furthermore, let $\mathcal{F}(R)$ denote a small model for the $G$-category of finitely generated free $R$-modules. Then the category $\mathcal{F}(R)[G]$ is equivalent to $\mathcal{F}(R_tG)$, see [BR07a, Proposition 6.7].

In [BR07a, Definition 3.1] for every additive $G$-category $\mathcal{A}$ a functor

$$\mathbb{K}_{\mathcal{A}}\colon OrG \to \mathfrak{Spectra}$$

with the property $\mathbb{K}_{\mathcal{A}}(G/H) \simeq \mathbb{K}(\mathcal{A}[H])$ is defined. As mentioned above these give rise to $G$-homology theories. We also have a functor $\mathbb{L}_{\mathcal{A}}\colon OrG \to \mathfrak{Spectra}$ for any additive $G$-category $\mathcal{A}$ with involution, see [BR07a, Section 5].

The $K$-theoretic assembly map with respect to a family of subgroups $\mathcal{F}$ and an additive $G$-category $\mathcal{A}$ is the map

$$H_*^G(E_{\mathcal{F}}G; \mathbb{K}_{\mathcal{A}}) \to H_*^G(G/G; \mathbb{K}_{\mathcal{A}}) \cong K_*(\mathcal{A}[G]),$$

induced by the map $E_{\mathcal{F}}G \to G/G$. The $L$-theoretic assembly map for an additive $G$-category $\mathcal{A}$ with involution is the map

$$H_*^G(E_{\mathcal{F}}G; \mathbb{L}_{\mathcal{A}}) \to H_*^G(G/G; \mathbb{L}_{\mathcal{A}}) \cong L_*(\mathcal{A}[G]).$$

**Conjecture 1.6.5** (Farrell-Jones conjecture)**.** *For every group $G$, every additive $G$-category $\mathcal{A}$ (with involution) and every $n \in \mathbb{N}$ the assembly maps*

$$H_n^G(\underline{\underline{E}}G; \mathbb{K}_\mathcal{A}) \to H_n^G(pt; \mathbb{K}_\mathcal{A}) \cong K_n(\mathcal{A}[G])$$

*and*

$$H_n^G(\underline{\underline{E}}G; \mathbb{L}_\mathcal{A}) \to H_n^G(pt; \mathbb{L}_\mathcal{A}) \cong L_n(\mathcal{A}[G])$$

*are isomorphisms.*

The maps

$$H_n^G(\underline{E}G; \mathbb{K}_\mathcal{A}) \to H_n^G(\underline{\underline{E}}G; \mathbb{K}_\mathcal{A}) \text{ and } H_n^G(\underline{E}G; \mathbb{L}_\mathcal{A}) \to H_n^G(\underline{\underline{E}}G; \mathbb{L}_\mathcal{A})$$

are split injective, see [Bar03a, Theorem 1.3]. Therefore the Farrell-Jones conjecture implies the following conjecture.

**Conjecture 1.6.6.** *For every group $G$, every additive $G$-category (with involution) and every $n \in \mathbb{N}$ the assembly maps*

$$H_n^G(\underline{E}G; \mathbb{K}_\mathcal{A}) \to H_n^G(pt; \mathbb{K}_\mathcal{A}) \cong K_n(\mathcal{A}[G])$$

*and*

$$H_n^G(\underline{E}G; \mathbb{L}_\mathcal{A}) \to H_n^G(pt; \mathbb{L}_\mathcal{A}) \cong L_n(\mathcal{A}[G])$$

*are split injective.*

In this thesis only the later conjecture is considered. For the reminder of this section we will recall some properties of the functors $\mathbb{K}$ and $\mathbb{L}$, which we will need later.

**Definition 1.6.7** ([CP95, Definition 1.27])**.** Let $\mathcal{U}$ be a full subcategory of an additive category $\mathcal{A}$. We say that $\mathcal{A}$ is $\mathcal{U}$-filtered if every object $A \in \mathcal{A}$ has a family of decompositions $\{A \cong E_i \oplus A_i\}$ called filtration with $E_i \in \mathcal{U}, A_i \in \mathcal{A}$, such that

(a) for each $A \in \mathcal{A}$ the decomposition forms a directed poset under the partial order $E_i \oplus A_i \leq E_j \oplus A_j$ whenever the canonical maps $E_i \to A \to A_j$ and $A_j \to A \to E_i$ are trivial,

(b) for every $A \in \mathcal{A}, U \in \mathcal{U}$ every map $f : A \to U$ factors as $A \cong E_i \oplus A_i \to E_i \to U$ for some $i$,

(c) for every $A \in \mathcal{A}, U \in \mathcal{U}$ every map $f : U \to A$ factors as $U \to E_i \to E_i \oplus A_i \cong A$ for some $i$,

(d) for each $A, B \in \mathcal{A}$ the filtration of $A \oplus B$ is equivalent to the sum of the filtrations $\{A = E_i \oplus A_i\}$ and $\{B = F_j \oplus B_j\}$, i.e. to $\{E_i \oplus F_j \oplus A_i \oplus B_j\}$.

The quotient $\mathcal{A}/\mathcal{U}$ is defined as the category having the same objects as $\mathcal{A}$ and morphism equivalence classes of morphism of $\mathcal{A}$ where $f, g : A \to B$ are equivalent if $f - g$ factors through some $U \in \mathcal{U}$. We then call $\mathcal{U} \to \mathcal{A} \to \mathcal{A}/\mathcal{U}$ (or $\mathcal{U} \to \mathcal{A}$) a Karoubi filtration.

**Definition 1.6.8.** An additive category $\mathcal{A}$ is called flasque if there exists a functor $F\colon \mathcal{A} \to \mathcal{A}$ such that $F \oplus \mathrm{id}_{\mathcal{A}}$ and $F$ are naturally isomorphic.

The following facts are for example summarized in [BL12a, Theorem 5.1].

**Theorem 1.6.9.** *Let $\mathcal{A}$ be an additive $G$-category (with involution).*

*(a) If $\mathcal{A}$ is flasque, then $\mathbb{K}(\mathcal{A})$ and $\mathbb{L}(\mathcal{A})$ are weakly contractible.*

*(b) If $\mathcal{A}$ is $\mathcal{U}$-filtered, then*

$$\mathbb{K}(\mathcal{U}) \to \mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{A}/\mathcal{U})$$

*and*

$$\mathbb{L}(\mathcal{U}) \to \mathbb{L}(\mathcal{A}) \to \mathbb{L}(\mathcal{A}/\mathcal{U})$$

*are homotopy fibration sequences.*

*(c) If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of additive categories (with involution), then $\mathbb{K}(\varphi)$ and $\mathbb{L}(\varphi)$ are weak equivalences.*

*(d) If $\mathcal{A} = \mathrm{colim}_i\, \mathcal{A}_i$ is a colimit of additive categories (with involution) over a directed system, then the natural maps $\mathrm{colim}_i\, \mathbb{K}(\mathcal{A}_i) \to \mathbb{K}(\mathcal{A})$ and $\mathrm{colim}_i\, \mathbb{L}(\mathcal{A}_i) \to \mathbb{L}(\mathcal{A})$ are weak equivalences.*

**Theorem 1.6.10** ([Car95],[CP95, Section 5])**.** *Let $\mathcal{A}_i, i \in I$ be additive categories. Then the natural map $\mathbb{K}(\prod_{i\in I} \mathcal{A}_i) \to \prod_{i\in I} \mathbb{K}\mathcal{A}_i$ is a weak equivalence.*

*If $\mathcal{A}_i, i \in I$ are additive categories with involution such that there exists $j \in \mathbb{N}$ with $K_n(\mathcal{A}_i) = 0$ for all $n \leq -j$ and $i \in I$, then also the natural map $\mathbb{L}(\prod_{i\in I} \mathcal{A}_i) \to \prod_{i\in I} \mathbb{L}\mathcal{A}_i$ is a weak equivalence.*

In the next section we want to reformulate the assembly maps using controlled algebra. From now on we will focus on algebraic $K$-theory, but most proofs hold the same way for $L$-theory when considering additive categories with involution, see Section 3.3 for the statement of the main theorem in the $L$-theory case and how the proofs have to be adapted.

## 1.7 Controlled algebra

In a controlled category the objects are families of objects of a category $\mathcal{A}$ indexed over a space $X$. This allows to measure the "length" of morphisms in $X$ and to construct different categories by specifying control conditions. At the end of this section we will see how the assembly map can be described in terms of controlled categories. The definition of a geometric module in this thesis is a slight variation of the definitions in [BR07b] and [RTY]. The first definition of geometric groups appeared in [CH69] and of geometric modules in [Qui79] and [Qui82]. The first definition of continuous control is in [ACFP94].

For the following definitions let $X$ be a metric space, $G$ a group and $\mathcal{A}$ a small additive $G$-category.

**Definition 1.7.1.** Let $Z := G \times X \times [0,1)$. A geometric $\mathcal{A}$-module $M$ over $X$ is given by a sequence of objects $(M_z)_{z \in Z}$ in $\mathcal{A}$, subject to the following conditions:

(a) The image of $\operatorname{supp}(M) = \{z \in Z \mid M_z \neq 0\}$ under the projection

$$p : Z \to X \times [0,1)$$

is locally finite, i.e. for every $(x,t) \in X \times [0,1)$ there exists a neighborhood $U$ such that $U \cap p(\operatorname{supp}(M))$ is finite.

(b) For every $x \in X, t \in [0,1)$ the set $\operatorname{supp}(M) \cap (G \times \{x\} \times \{t\})$ is finite.

A morphism $\varphi\colon M \to N$ between geometric $\mathcal{A}$-modules $M, N$ is a sequence

$$(\varphi_{x,y}\colon M_y \to N_x)_{(x,y) \in Z^2}$$

of morphisms in $\mathcal{A}$, subject to the following conditions:

(a) $\varphi$ is continuously controlled at 1, i.e. for each $x \in X$ and each neighborhood $U$ of $(x,1)$ in $X \times [0,1]$ there exists a neighborhood $V$ of $(x,1)$ in $X \times [0,1]$ such that for all $g, g' \in G, v \in V, y \notin U$, $\varphi_{(g,v),(g',y)} = \varphi_{(g',y),(g,v)} = 0$.

(b) For every $z \in Z$ the set $\{z' \in Z \mid \varphi_{z,z'} \neq 0 \text{ or } \varphi_{z',z} \neq 0\}$ is finite.

(c) $\varphi$ is $R$-bounded for some $R > 0$, i.e. $\varphi_{(g,x,t),(g',x',t')} = 0$ for all $g, g' \in G, x, x' \in X$, $t, t' \in [0,1)$ with $d(x,x') > R$. Then the infimum of these $R$ is called the propagation of $\varphi$.

Let $\mathcal{A}_G(X)$ denote the category of geometric $\mathcal{A}$-modules over $X$ and their morphisms. The composition of morphisms is given by matrix multiplication. $\mathcal{A}_G(X)$ is an additive category with pointwise addition.

**Remark 1.7.2.** Let $\mathcal{A}_c(X) \subseteq \mathcal{A}_G(X)$ be the full additive subcategory with objects having support in $\{e\} \times X \times [0,1)$. This coincides with the definition of $\mathcal{A}_c(X)$ in [RTY]. The inclusion $\mathcal{A}_c(X) \hookrightarrow \mathcal{A}_G(X)$ is an equivalence because of condition (b) on the objects of $\mathcal{A}_G(X)$.

**Definition 1.7.3.** If a subgroup $H \leq G$ acts on $X$ by isometries, then $H$ acts on the category $\mathcal{A}_G(X)$ by $(hM)_{(g,x,t)} := h(M_{(h^{-1}g,h^{-1}x,t)})$ and the corresponding action on the morphisms. Let $\mathcal{A}_G^H(X)$ be the corresponding $H$-fixed point category.

**Definition 1.7.4.** If $G$ acts on $X$ by isometries, let $\mathcal{A}_{G-c}(X)$ be the full subcategory of $\mathcal{A}_G(X)$ with objects having support in $G \times GK \times [0,1)$ for some compact subspace $K \subseteq X$. This is equivalent to the category $\operatorname{colim}_{K \subseteq X cp.} \mathcal{A}_G(GK)$. Let $\mathcal{A}_{G-c}^H(X)$ be the $H$-fixed point category as above.
Furthermore, let $\mathcal{A}_{G-c}^H(X)_0$ and $\mathcal{A}_G^H(X)_0$ be the full subcategory of $\mathcal{A}_{G-c}^H(X)$ respectively $\mathcal{A}_G^H(X)$ with the following condition on the support of the objects:

- For every object $M$ the limit points of the image of $\mathrm{supp}(M)$ under the projection $Z \to X \times [0,1)$ are disjoint from $X \times \{1\}$.

The inclusions of $\mathcal{A}^H_{G-c}(X)_0$ into $\mathcal{A}^H_{G-c}(X)$ and $\mathcal{A}^H_G(X)_0$ into $\mathcal{A}^H_G(X)$ are Karoubi filtrations, see [BR07b, (5.12)].
Define $\mathcal{A}^H_{G-c}(X)^\infty$ and $\mathcal{A}^H_G(X)^\infty$ to be the quotients of $\mathcal{A}^H_{G-c}(X)$ by $\mathcal{A}^H_{G-c}(X)_0$ and $\mathcal{A}^H_G(X)$ by $\mathcal{A}^H_G(X)_0$ respectively.

**Proposition 1.7.5** ([BR07b, (5.15)])**.** *The categories defined above are functorial in $X$ for continuous (equivariant) metrically coarse maps $f : X \to Y$. If two such maps are equivariantly metrically homotopic (see Definition 1.5.1), then they induce equivariant weakly homotopic maps*

$$\mathbb{K}(\mathcal{A}_{G-c}(X)) \to \mathbb{K}(\mathcal{A}_{G-c}(Y))$$

*of the $K$-theory spectra.*

**Remark 1.7.6.** Let $G$ act on $X$ by isometries. Then condition (c) on the morphisms is automatically satisfied for the category $\mathcal{A}^G_{G-c}(X)$. This is the only condition that depends on the metric of $X$. For this reason it makes sense to define the category $\mathcal{A}^G_{G-c}(X)$ for any topological $G$-space $X$ and it is functorial in $X$ for continuous maps. The same holds for $\mathcal{A}^G_{G-c}(X)_0$ and $\mathcal{A}^G_{G-c}(X)^\infty$.

**Proposition 1.7.7.** *Let $f : X \to Y$ be a homotopy equivalence between topological $G$-spaces then the induced map*

$$f_* : \mathbb{K}(\mathcal{A}^G_{G-c}(X)) \to \mathbb{K}(\mathcal{A}^G_{G-c}(Y))$$

*is a weak homotopy equivalence. The same is true for $\mathcal{A}^G_{G-c}(\_)_0$ and $\mathcal{A}^G_{G-c}(\_)^\infty$.*

*Proof.* [BFJR04, Proposition 5.5] states this for the category $\mathcal{A}^G_{G-c}(\_)^\infty$. But in the proof it is shown for $\mathcal{A}^G_{G-c}(\_)$ as well. Then it follows for $\mathcal{A}^G_{G-c}(\_)_0$ by comparing the long exact sequences

$$\begin{array}{ccccccccc}
\ldots \to & K_n\mathcal{A}^G_{G-c}(X)_0 & \to & K_n\mathcal{A}^G_{G-c}(X) & \to & K_n\mathcal{A}^G_{G-c}(X)^\infty & \to & K_{n-1}\mathcal{A}^G_{G-c}(X)_0 & \to \ldots \\
& \downarrow f_* & & \downarrow f_* \cong & & \downarrow f_* \cong & & \downarrow f_* & \\
\ldots \to & K_n\mathcal{A}^G_{G-c}(Y)_0 & \to & K_n\mathcal{A}^G_{G-c}(Y) & \to & K_n\mathcal{A}^G_{G-c}(Y)^\infty & \to & K_{n-1}\mathcal{A}^G_{G-c}(Y)_0 & \to \ldots
\end{array}$$

and using the 5-lemma. ∎

As already mentioned the assembly map can be interpreted as a map of controlled categories, namely we have the following proposition.

**Proposition 1.7.8.** *The category $\mathcal{A}^G_{G-c}(\underline{E}G)_0$ is equivalent to the category $\mathcal{A}[G]$, see Definition 1.6.3. Furthermore, the boundary map*

$$K_n(\mathcal{A}^G_{G-c}(\underline{E}G)^\infty) \to K_{n-1}(\mathcal{A}^G_{G-c}(\underline{E}G)_0)$$

*is equivalent to the assembly map*

$$H_n^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to H_n^G(pt;\mathbb{K}_{\mathcal{A}}) \cong K_n(\mathcal{A}[G]).$$

*Proof.* This is proved in [BFJR04, Section 6] in the case where the $G$-action on $\mathcal{A}$ is trivial, but the same proof still holds in the general case. □

In the proof of our main theorem we often need to work with products of controlled categories. In order to have good control conditions we can only work with those morphisms which have the same length in each degree. To make this precise we begin with recalling the definition of filtered categories.

**Definition 1.7.9** ([PW85, Definition 1.1])**.** An additive category $\mathcal{A}$ is said to be filtered if there is an increasing filtration

$$F_0(A,B) \subseteq F_1(A,B) \subseteq \ldots \subseteq F_n(A,B) \subseteq \ldots$$

of $\hom(A,B)$ for every pair of objects $A,B \in \mathcal{A}$. Each $F_i(A,B)$ has to be an additive subgroup of $\hom(A,B)$ and we must have $\bigcup_{i\in\mathbb{N}} F_i(A,B) = \hom(A,B)$. We require the zero and identity maps to be in the zeroth filtration degree and for $f \in F_i(A,B)$ and $g \in F_j(B,C)$ the composition $g \circ f$ to be in $F_{i+j}(A,C)$. If $f \in F_i(A,B)$, we say that $f$ has (filtration) degree $i$.

**Example 1.7.10.** The categories $\mathcal{A}_G(X)$ and $\mathcal{A}_c(X)$ are filtered by defining a morphism $f$ to be of degree $n$ if it is $n$-bounded.

**Definition 1.7.11.** For filtered additive categories $\{\mathcal{A}_i\}_{i\in I}$ we define the bounded product $\prod_{i\in I}^{bd}\mathcal{A}_i$ to be the subcategory of $\prod_{i\in I}\mathcal{A}_i$ containing all objects and those morphisms $\varphi = \{\varphi_i\}_{i\in I}$ such that there exists $n \in \mathbb{N}$ with degree $\varphi_i \le n$ for all $i \in I$.

**Proposition 1.7.12.** *Let $\mathcal{B}_i$ be $\mathcal{A}_i$ filtered such that for the decomposition $\{B_i \cong B_{i,j_i} \oplus E_{i,j_i}\}_{j_i\in J_i}$ with $E_{i,j_i} \in \mathcal{A}_i$ the projections $B_i \to E_{i,j_i}, B_{i,j_i}$ and inclusions $E_{i,j_i}, B_{i,j_i} \to B_i$ have degree zero for every $B_i \in \mathcal{B}_i$. Then $\prod_{i\in I}^{bd}\mathcal{B}_i$ is $\prod_{i\in I}^{bd}\mathcal{A}_i$-filtered and the quotient is isomorphic to $\prod^{bd}\mathcal{B}_i/\mathcal{A}_i$, where a morphism $[g] \in \mathcal{B}_i/\mathcal{A}_i$ has degree $k$ if there exist $f \in \mathcal{B}_i, f \in [g]$ with degree $k$.*

*Proof.* Since inclusions and projections the have degree zero, a decomposition of $(B_i) \in \prod_{i\in I}^{bd}\mathcal{B}_i$ is given by a sequence of decomposition of the $B_i$. For $(f_i)\colon (B_i) \to (A_i)$ in $\prod_{i\in I}^{bd}\mathcal{B}_i$ with $(A_i) \in \prod_{i\in I}^{bd}\mathcal{A}_i$ every $f_i$ factors as $B_i \to E_{i,j_i} \xrightarrow{f_i'} A_i$. Since the inclusion $E_{i,j_i} \to B_i$ has degree zero, the map $f_i'$ has the same degree as $f_i$ and the map $(f_i)$ factors as $(B_i) \to (E_{i,j_i}) \xrightarrow{f_i'} (A_i)$. Analogously condition (c) for a Karoubi filtration holds. This implies that $\prod_{i\in I}^{bd}\mathcal{B}_i$ is indeed $\prod_{i\in I}^{bd}\mathcal{A}_i$-filtered.

The categories $\prod_{i\in I}^{bd}\mathcal{B}_i/\prod_{i\in I}^{bd}\mathcal{A}_i$ and $\prod_{i\in I}^{bd}\mathcal{B}_i/\mathcal{A}_i$ both have the same objects. And the natural map

$$F : \prod_{i\in I}^{bd}\mathcal{B}_i/\prod_{i\in I}^{bd}\mathcal{A}_i \to \prod_{i\in I}^{bd}\mathcal{B}_i/\mathcal{A}_i$$

is surjective on morphisms. Let $((f_i)\colon (B_i) \to (B_i')) \in \prod_{i\in I}^{bd} \mathcal{B}_i$ be a morphism, such that $f_i$ factors over $\mathcal{A}_i$ for every $i \in I$. This implies by condition (b) of Definition 1.6.7, that $f_i$ factors as $B_i \to E_{i,j_i} \xrightarrow{f_i'} B_i'$ and again $f_i$ and $f_i'$ have the same degree. This implies that $(f_i)$ factors as $(B_i) \to (E_{i,j_i}) \xrightarrow{(f_i')} (B_i')$ and thus $[(f_i)] = 0$ in $\prod_{i\in I}^{bd} \mathcal{B}_i / \prod_{i\in I}^{bd} \mathcal{A}_i$. Therefore, $F$ is also injective on morphisms. □

**Proposition 1.7.13.** *There is a $\mathcal{A}_{G-c}(X)_0$-filtration of $\mathcal{A}_{G-c}(X)$ such that all the projections and inclusions have degree zero and for every family $\{X_i\}$ the quotient $\prod^{bd} \mathcal{A}_{G-c}(X_i)^\infty$ is isomorphic to $\prod \mathcal{A}_{G-c}(X_i)^\infty$.*

*Proof.* Let $p\colon G \times X \times [0,1) \to X \times [0,1)$ denote the projection. Let $\mathcal{Z}$ denote the family of all subsets $Z \subseteq G \times X \times [0,1)$ such that $p(z)$ has no limit points at $X \times \{1\}$. Then every $M \in \mathcal{A}_{G-c}(X)$ is $\mathcal{A}_{G-c}(X)_0$ filtered by $\{M \cong M|_Z \oplus M|_{G\times X\times[0,1)\setminus Z}\}$. This shows the first part of the proposition. The second follows if we can show that every $f \in \mathcal{A}_{G-c}(X)^\infty$ has a representative of degree 1. Let $\varphi$ be a representative of $f$. For every $x \in X$ let $U_x := B_{1/2}(x) \times [0,1] \subseteq X \times [0,1]$. Since $\varphi$ is continuously controlled at 1, there exists a neighborhood $V_x \subseteq U_x$ of $(x,1) \in X \times [0,1]$ such that $\varphi_{(g',y),(g,v)} = 0$ for all $g, g' \in G, v \in V_x, y \notin U_x$. Define $V := \bigcup_{x\in X} V_x$. Then $M|_{G\times X\times[0,1]\setminus G\times V}$ is an object in $\mathcal{A}_{G-c}(X)_0$ and therefore the morphism $\varphi'\colon M \to N$ defined by

$$\varphi'_{(g',y),(g,v)} = \begin{cases} \varphi_{(g',y),(g,v)} & v \in V \\ 0 & \text{else} \end{cases}$$

also represents $f$.

$\varphi'$ is 1-bounded, since $\varphi'_{(g',y),(g,v)} \neq 0$ implies $v \in V_x$ and $y \in U_x$ for some $x \in X$. Therefore, $d(\mathrm{pr}_X(v), \mathrm{pr}_X(y)) < 1$, where $\mathrm{pr}_X\colon X \times [0,1] \to X$ is the projection. □

**Proposition 1.7.14.** *For a metric space $X = \coprod_{i\in I} X_i$ we have an isomorphism*

$$\mathcal{A}_G(X) \cong \prod_{i\in I}^{bd} \mathcal{A}_G(X_i).$$

*If $X$ is a metric $G$-space, this isomorphism is $G$-equivariant.*

*Proof.* Since morphisms have bounded propagation they can not map from one component of the coproduct to another. Therefore, we get a functor $F\colon \mathcal{A}_G(X) \to \prod_{i\in I} \mathcal{A}_G(X_i)$ by mapping $M \in \mathcal{A}_G(X)$ to $(M|_{X_i})_{i\in I} \in \prod_{i\in I} \mathcal{A}_G(X_i)$, where $M|_{X_i}$ denotes the restriction of $M$ to $G \times X_i \times [0,1)$. Because morphisms in $\mathcal{A}_G(X)$ have bounded propagation the functor $F$ has image in the bounded product. The functor $F\colon \mathcal{A}_G(X) \to \prod_{i\in I}^{bd} \mathcal{A}_G(X_i)$ is a bijection on objects and morphisms and hence an isomorphism. If $X$ is a metric $G$-space, this functor is equivariant. □

# 2 The K-theory of equivariant metric families with FDC

In this chapter we will generalize a result from [RTY] about the vanishing of a certain controlled category. First we need the following definition.

**Definition 2.0.1.** A metric space $X$ has bounded geometry if for each $R > 0$ there exists $N \in \mathbb{N}$ such that for all $x \in X$ the ball $B_R(x)$ contains at most $N$ points. A metric family $\mathcal{X}$ has bounded geometry uniformly if for each $R > 0$ there exists $N \in \mathbb{N}$ such that for all $X \in \mathcal{X}, x \in X$ the ball $B_R(x)$ contains at most $N$ points.

For the injectivity result of [RTY] the following theorem is the most important tool.

**Theorem 2.0.2** ([RTY, Theorem 6.4])**.** *If $X$ is a bounded geometry metric space with finite decomposition complexity, then for each $n \in \mathbb{Z}$ we have*

$$\operatorname*{colim}_{s\to\infty} K_n(\mathcal{A}_c(P_s X)) = 0,$$

*where $P_s X$ is the Rips complex of $X$ and the colimit is taken with respect to the maps induced by the inclusions $P_s X \hookrightarrow P_{s'} X$ for $s' \geq s$.*

To allow for groups with torsion in our main theorem we need to generalize the above theorem to a version which is equivariant with respect to finite subgroups. We first recall the definition of the Rips complex and then give an equivariant definition of FDC. In the last section of this chapter we show how the proof of [RTY, Theorem 6.4] can be generalized.

## 2.1 The Rips complex

**Definition 2.1.1.** Given a metric space $X$, a subspace $Y \subseteq X$ and a number $s > 0$, the Rips complex $P_s Y$ is the simplicial complex with vertex set $Y$ and with a simplex $\langle x_0, \dots, x_n\rangle$ whenever $d(x_i, x_j) \leq s$ for all $i, j \in \{0, ..., n\}$. We equip $P_s Y$ with the metric induced by the simplicial path metric on $P_s X$. For $s' \geq s$ let $i_{ss'} \colon P_s Y \hookrightarrow P_{s'} Y$ denote the inclusion.

Given a subspace $Z \subseteq X$, a family $\mathbb{W}$ of subspaces of $X$ and $0 < s < s'$, the relative rips complex $P_{s,s'}(Z, \mathbb{W})$ is the subcomplex of $P_{s'} X$ consisting of those simplices $\langle x_0, \dots, x_n\rangle$ satisfying at least one of the following conditions:

- $x_0, .., x_n \in Z$ and $d(x_i, x_j) \leq s$ for all $i, j$ or

- $x_0, .., x_n \in W$ for some $W \in \mathbb{W}$.

Note that in the second case $d(x_i, x_j) \leq s'$ since $\langle x_0, \dots, x_n \rangle$ is a simplex of $P_{s'}X$. We equip $P_{s,s'}(Z, \mathbb{W})$ with the metric induced by the simplicial path metric on $P_{s,s'}(X, \mathbb{W})$. It is crucial for the following arguments, that we do not use the metric induced from $P_{s'}X$. The relative Rips complex was introduced in [GTY12]. For subspaces $Y, W \subseteq X$ we define $P_{s,s'}(Y, W) := P_{s,s'}(Y, \{W\})$.

**Remark 2.1.2.** Note that if $X$ is a metric space with bounded geometry, then the (relative) Rips complex is finite dimensional and locally finite. For this reason we will always use metric spaces with bounded geometry in the following arguments.

We recall the following comparisons between distances in a bounded geometry metric space $X$ and the Rips complex $P_s(X)$ or the relative Rips complex from [RTY]. They will be crucial for the decomposition arguments in Section 2.3.

**Lemma 2.1.3** ([RTY, Lemma 5.3])**.** *Let $W \subseteq X$ be metric spaces, and assume $X$ has bounded geometry. Given $s' \geq s > 0$, let $(P_{s'}W)^t$ denote the $t$-neighborhood of $P_{s'}W$ inside $P_{s,s'}(X, W)$. Then for all $x \in X \cap (P_{s'}W)^t$ (where $X$ is viewed as the 0-skeleton of $P_{s,s'}(X, W)$), we have*

$$d(x, W) \leq (t+1)C(s, X)s,$$

*where $C(s, X) := (2\sqrt{2}+1)^{\dim P_s X - 1}$ (if $\dim P_s X = 0$, we define $C(s, X) := 1$). It follows that inside the simplicial complex $P_{s'}X$ we have inclusions*

$$(P_{s'}W)^t \subseteq P_{s,s'}\left(W^{(t+2)C(s,X)s}, W\right) \subseteq P_{s'}\left(W^{(t+2)C(s,X)s}\right),$$

*where on the left the neighborhood is still taken with respect to the simplicial path metric on $P_{s,s'}(X, W)$. Additionally, for any $U \subseteq X$, we have inclusions*

$$\left(P_{s,s'}(U, W)\right)^t \subseteq (P_{s'}(U \cup W))^t \subseteq P_{s'}\left((U \cup W)^{(t+2)C(s,X)s}\right),$$

*where the first neighborhood is taken inside $P_{s,s'}(X, W)$ and the second is taken inside $P_{s,s'}(X, U \cup W)$.*

**Lemma 2.1.4** ([RTY, Lemma 5.4])**.** *Let $(X, d)$ be a bounded geometry metric space, with subspaces $X_1, X_2 \subseteq X$ and let $\mathbb{W}_1, \mathbb{W}_2$ be families of subspaces of $X$. For $i = 1, 2$, let*

$$W_i = \bigcup \mathbb{W}_i = \{x \in X \mid x \in W \text{ for some } W \in \mathbb{W}_i\}$$

*denote the union of the subspaces in $\mathbb{W}_i$. Set $\mathbb{W} = \mathbb{W}_1 \cup \mathbb{W}_2$, and let $d'$ denote the simplicial path metric on $P_{s,s'}(X, \mathbb{W})$ for some fixed $s, s' > 0$. Setting $V_i = X_i \cup W_i$ and $P_i = P_{s,s'}(X_i, \mathbb{W}_i)$ we have*

$$d(V_1, V_2) \leq (d'(P_1, P_2) + 2)C(s, X)s,$$

*where $C(s, X) := (2\sqrt{2}+1)^{\dim P_s X - 1}$ (if $\dim P_s X = 0$, we define $C(s, X) := 1$).*

## 2.2 Equivariant FDC

We will now give a definition of finite decomposition complexity for families of metric spaces with group actions.

**Definition 2.2.1.** An equivariant metric family is a family $\{(X_\alpha, G_\alpha)\}_{\alpha\in A}$ where $G_\alpha$ is a group and $X_\alpha$ is a metric $G_\alpha$-space.

**Definition 2.2.2.** An equivariant metric family $\mathcal{X} = \{(X_\alpha, G_\alpha)\}_{\alpha\in A}$ decomposes over a class of equivariant metric families $\mathfrak{D}$ if for every $r > 0$ and every $\alpha \in A$ there exists a decomposition $X_\alpha = U_\alpha^r \cup V_\alpha^r$ into $G_\alpha$-invariant subspaces and $r$-disjoint decompositions

$$U_\alpha^r = \bigcup_{i\in I(r,\alpha)}^{r-disj} U_{\alpha,i}^r \quad \text{and} \quad V_\alpha^r = \bigcup_{j\in J(r,\alpha)}^{r-disj} V_{\alpha,j}^r,$$

such that $G_\alpha$ acts on $I(r,\alpha)$ and $J(r,\alpha)$ and for every $g \in G_\alpha$ we have $gU_{\alpha,i}^r = U_{\alpha,gi}^r$ and $gV_{\alpha,j}^r = V_{\alpha,gj}^r$. Furthermore, the families

$$\left\{\left(\coprod_{i\in I(r,\alpha)} U_{\alpha,i}^r, G_\alpha\right)\right\}_{\alpha\in A} \quad \text{and} \quad \left\{\left(\coprod_{j\in J(r,\alpha)} V_{\alpha,j}^r, G_\alpha\right)\right\}_{\alpha\in A}$$

have to lie in $\mathfrak{D}$.

Notice that the underlying sets of $U_\alpha^r$ and $\coprod_{i\in I(r,\alpha)} U_{\alpha,i}^r$ are canonically isomorphic and in this sense the $G_\alpha$-action on $\coprod_{i\in I(r,\alpha)} U_{\alpha,i}^r$ is the same as the action on $U_\alpha^r$, only the metric has changed.

**Definition 2.2.3.** An equivariant metric family $\mathcal{X}$ is called semi-bounded, if there exists $R > 0$ such that for all $(X, G) \in \mathcal{X}, x, y \in X$ we have $d(x, y) < R$ or $d(x, y) = \infty$.

Let $e\mathfrak{B}$ denote the class of semi-bounded equivariant families. We set $e\mathfrak{D}_0 = e\mathfrak{B}$ and given a successor ordinal $\gamma + 1$ we define $e\mathfrak{D}_{\gamma+1}$ to be the class of all equivariant metric families which decompose over $e\mathcal{D}_\gamma$. For a limit ordinal $\gamma$ we define

$$e\mathfrak{D}_\gamma = \bigcup_{\beta<\gamma} e\mathfrak{D}_\beta.$$

An equivariant metric family $\mathcal{X}$ has finite decomposition complexity (FDC) if $\mathcal{X} \in e\mathfrak{D}_\gamma$ for some ordinal $\gamma$.

Note that the equivariant metric family $\{(X_\alpha, \{e\})\}_{\alpha\in A}$ has FDC if and only if the metric family $\{X_\alpha\}_{\alpha\in A}$ has FDC.

## 2.3 An equivariant vanishing result

In this section we prove the following generalization of [RTY, Theorem 6.4].

**Theorem 2.3.1.** *Let $\mathcal{X} = \{(X_\alpha, G_\alpha)\}_{\alpha\in A}$ be an equivariant family with FDC, and let the family $\{X_\alpha\}_{\alpha\in A}$ have bounded geometry uniformly, then*

$$\operatorname*{colim}_s K_n\left(\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}_{G_\alpha}(P_s X_\alpha)\right) = 0$$

*for all $n \in \mathbb{Z}$, where the colimit is taken over the maps induces by the inclusion of the the respective Rips complexes.*

**Remark 2.3.2.** Assume that in the situation of Theorem 2.3.1 all groups $G_\alpha$ are subgroups of a group $G$. Then the inclusions $\mathcal{A}_{G_\alpha}(P_s X_\alpha) \to \mathcal{A}_G(P_s X_\alpha)$ are equivalences of $G_\alpha$-categories, compare Remark 1.7.2. In particular the induced map

$$\mathcal{A}^{G_\alpha}_{G_\alpha}(P_s X_\alpha) \to \mathcal{A}^{G_\alpha}_{G}(P_s X_\alpha)$$

on fixed points is an equivalence.

The idea of the proof of Theorem 2.3.1 is an induction over the decomposition complexity of $\mathcal{X}$. First we will decompose $\mathcal{A}^{G_\alpha}_{G_\alpha}(P_s X_\alpha)$ into $\mathcal{A}^{G_\alpha}_{G_\alpha}(P_s U_\alpha)$ and $\mathcal{A}^{G_\alpha}_{G_\alpha}(P_s V_\alpha)$ using a Mayer-Vietoris sequence and then further decompose these two spaces into disjoint unions of subspaces with lower decomposition complexity. But we only have a version of the disjoint union axiom for "arbitrary far apart" subspaces. For this reason we have to work with a bounded product where we apply FDC with growing parameter $r$. For simplicity in this section we will denote $\mathcal{A}^{G_\alpha}_{G_\alpha}$ only by $\mathcal{A}^{G_\alpha}$. We will prove Theorem 2.3.1 using the following proposition.

**Proposition 2.3.3.** *Let $\{\{X_\alpha, G_\alpha\}_{\alpha\in A_r}\}_{r\in\mathbb{N}}$ be a sequence of equivariant metric families, such that for each $r \in \mathbb{N}$ the family $\{(X_\alpha, G_\alpha)\}_{\alpha\in A_r}$ has FDC and the family $\{X_\alpha\}_{\alpha\in A_r}$ has bounded geometry uniformly. Then*

$$\operatorname*{colim}_{s\in \boldsymbol{Seq}} K_n\left(\operatorname*{colim}_{R\in\mathbb{N}} \prod_{r\geq R}^{bd} \prod_{\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}(P_{s_r} X_\alpha)\right) = 0$$

*for all $n \in \mathbb{Z}$, where **Seq** denotes the partially ordered set consisting of all non-decreasing sequences of positive real numbers, with ordering $(s_1, s_2, \ldots) \leq (s'_1, s'_2, \ldots)$ if $s_i \leq s'_i$ for all $i \in \mathbb{N}$.*

*Proof of Theorem 2.3.1 using Proposition 2.3.3.*
Let $s > 0$ and $x \in K_n\big(\prod^{bd}_{\alpha\in A} \mathcal{A}^{G_\alpha}(P_s X_\alpha)\big)$ be given. Let $\mathbf{s} := \{s\}_{r\in\mathbb{N}} \in \mathbf{Seq}$. For

$\mathbf{s}' = \{s'_r\} \geq \mathbf{s}$ we define the following maps

$$\Delta_*\colon K_n\left(\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_s X_\alpha)\right) \to K_n\left(\prod_{r\in\mathbb{N}}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_s X_\alpha)\right)$$

$$q_{\mathbf{s}'}\colon K_n\left(\prod_{r\in\mathbb{N}}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_r} X_\alpha)\right) \to K_n\left(\operatorname*{colim}_{R\in\mathbb{N}}\prod_{r\geq R}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_r} X_\alpha)\right)$$

$$\mu_{\mathbf{s}'}\colon K_n\left(\prod_{r\in\mathbb{N}}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_s X_\alpha)\right) \to K_n\left(\prod_{r\in\mathbb{N}}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_r} X_\alpha)\right)$$

$$\bar{\mu}_{\mathbf{s}'}\colon K_n\left(\operatorname*{colim}_{R\in\mathbb{N}}\prod_{r\geq R}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_s X_\alpha)\right) \to K_n\left(\operatorname*{colim}_{R\in\mathbb{N}}\prod_{r\geq R}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_r} X_\alpha)\right)$$

The first map is the one induced by the diagonal map, the second one is the map to the colimit, the third and forth map are those induced by the inclusions of the respective Rips complexes. By Proposition 2.3.3 there is $\mathbf{s}' = \{s'_r\} \in \mathbf{Seq}$ with $\mathbf{s}' \geq \mathbf{s}$ such that

$$q_{\mathbf{s}'}\big(\mu_{\mathbf{s}'}(\Delta_*(x))\big) = \bar{\mu}_{\mathbf{s}'}\big(q_{\mathbf{s}}(\Delta_*(x))\big) = 0.$$

For $R \in \mathbb{N}$ let

$$p_R\colon K_n\left(\prod_{r\in\mathbb{N}}^{bd}\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_r} X_\alpha)\right) \to K_n\left(\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_R} X_\alpha)\right)$$

be the map induced by the projection, then in particular there exists $R \in \mathbb{N}$ such that for the above $\mathbf{s}' \in \mathbf{Seq}$

$$0 = p_R\big(\mu_{\mathbf{s}'}(\Delta_*(x))\big) \in K_n\left(\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_R} X_\alpha)\right).$$

Let

$$\mu_{s'_R}\colon K_n\left(\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_s X_\alpha)\right) \to K_n\left(\prod_{\alpha\in A}^{bd} \mathcal{A}^{G_\alpha}(P_{s'_R} X_\alpha)\right)$$

be induced by the inclusion of Rips complexes. Then $\mu_{s'_R}(x) = p_R\big(\mu_{\mathbf{s}'}(\Delta_*(x))\big) = 0$. This proves Theorem 2.3.1. □

In the remainder of this section we will prove Proposition 2.3.3 by induction. The following lemma gives the induction beginning.

**Lemma 2.3.4.** *For each $r \in \mathbb{N}$ let $\{(X_\alpha, G_\alpha)\}_{\alpha\in A_r} \in e\mathfrak{D}_0 = e\mathfrak{B}$ and let the $X_\alpha$ have* *bounded geometry uniformly. Then*

$$\operatorname*{colim}_{s\in \boldsymbol{Seq}} K_n\left(\operatorname*{colim}_{R\in\mathbb{N}} \prod^{bd}_{r\geq R}\prod^{bd}_{\alpha\in A_r} \mathcal{A}^{G_\alpha}(P_{s_r}X_\alpha)\right) = 0$$

*for all $n \in \mathbb{Z}$.*

*Proof.* Since $\{(X_\alpha, G_\alpha)\}_{\alpha\in A_r} \in e\mathfrak{B}$ there exist $t_r > 0$ such that either $d(x,y) < t_r$ or $d(x,y) = \infty$ for all $\alpha \in A_r, x, y \in X_\alpha$. For $x \in X_\alpha$ let $U_x := \{y \in X_\alpha \mid d(x,y) < \infty\}$, $G_x := \{g \in G_\alpha \mid gU_x = U_x\}$, then $\operatorname{diam} U_x < t_r$ and as metric spaces $G_\alpha U_x \cong \coprod_{G_\alpha/G_x} U_x$. Therefore,

$$X_\alpha \cong \coprod_{i\in I_\alpha}\coprod_{G_\alpha/G_i} U_i,$$

for some index set $I_\alpha$, where $G_i \leq G_\alpha$ is a subgroup and $U_i \subseteq X_\alpha$ is a $G_i$-subspace of diameter smaller than $t_r$ for all $i \in I_\alpha$. By Proposition 1.7.14 we get the following equivalence

$$\mathcal{A}^{G_\alpha}(P_{s_r}X_\alpha) \cong \left(\prod^{bd}_{i\in I_\alpha}\prod^{bd}_{G_\alpha/G_i} \mathcal{A}_{G_\alpha}(P_{s_r}U_i)\right)^{G_\alpha} \cong \prod^{bd}_{i\in I_\alpha} \mathcal{A}^{G_i}(P_{s_r}U_i)$$

for all $s_r > 0$. This implies that there are $k_i \in \mathbb{N}$ such that

$$\mathcal{A}^{G_\alpha}(P_{s_r}X_\alpha) \cong \prod^{bd}_{i\in I_\alpha} \mathcal{A}^{G_i}(\Delta^{k_i}) = \prod_{i\in I_\alpha} \mathcal{A}^{G_i}(\Delta^{k_i}),$$

for all $s_r \geq t_r$. Each $\Delta^{k_i}$ is equivariantly metrically homotopy equivalent to its midpoint, thus by Proposition 1.7.5 we get $\mathbb{K}\big(\mathcal{A}^{G_i}(\Delta^{k_i})\big) \cong \mathbb{K}\big(\mathcal{A}^{G_i}(pt)\big)$. The category $\mathcal{A}^{G_i}(pt)$ admits an Eilenberg swindle and therefore it is flasque and has trivial $K$-theory, compare [Bar03b, Remark 3.20].

This implies that for $\mathbf{s} = \{s_r\}_{r\in\mathbb{N}}$ big enough the $K$-theory of

$$\prod^{bd}_{r\geq R}\prod^{bd}_{\alpha\in A_r} \mathcal{A}^{G_\alpha}(P_{s_r}X_\alpha) \cong \prod_{r\geq R}\prod_{\alpha\in A_r}\prod_{i\in I_\alpha} \mathcal{A}^{G_i}(\Delta^{k_i})$$

is trivial as well since $K$-theory commutes with products. □

Let $\gamma$ be a successor ordinal with $\gamma = \beta+1$. For the remainder of this section we assume that we already have proved Proposition 2.3.3 for $\beta$. Let $\mathbf{s} = \{s_r\}_{r\in\mathbb{N}} \in \mathbf{Seq}$ be fixed. For $r \in \mathbb{N}$ let $N_r \geq 2$ be such that for every $\alpha \in A_r, x \in X_\alpha$ the ball $B_{s_r}(x)$ contains at most $N_r$ points. The $N_r$ are finite because the $X_\alpha$ have bounded geometry uniformly. Define $C_r := (2\sqrt{2}+1)^{N_r-2}$. By the definition of FDC there are decompositions

$$X_\alpha = U^{s_r}_\alpha \cup V^{s_r}_\alpha,$$

with finer decompositions

$$U_\alpha^{s_r} = \bigcup_{i\in I(s_r,\alpha)}^{C_r s_r r-disj} U_i \quad \text{and} \quad V_\alpha^{s_r} = \bigcup_{j\in J(s_r,\alpha)}^{C_r s_r r-disj} V_j$$

such that $G_\alpha$ acts on $I(s_r,\alpha)$ and $J(s_r,\alpha)$ and for every $g\in G_\alpha$ we have $gU_i = U_{gi}$ and $gV_j = V_{gj}$. Furthermore, the families

$$\left\{\left(\coprod_{i\in I(s_r,\alpha)} U_i, G_\alpha\right)\right\}_{\alpha\in A} \text{ and } \left\{\left(\coprod_{j\in J(s_r,\alpha)} V_j, G_\alpha\right)\right\}_{\alpha\in A}$$

have to lie in $e\mathfrak{D}_\beta$.

The parameter $C_r s_r r$ above is chosen because we want to use Lemma 2.1.3, i.e. for any subspace $U\subseteq X$ the $r$-neighborhood $(P_{s_r}(U))^r$ of $P_{s_r}(U)$ in $P_{s_r}(X)$ is contained in $P_{s_r}\left(U^{C_r s_r (r+2)}\right)$. And so the subspaces $P_{s_r}(U_i)$ become further apart with increasing $r$.

For $t>0$ let

$$\mathbb{W}_\alpha^{s_r,t} := \left\{(U_i)^{tC_r s_r} \cap (V_j)^{tC_r s_r} \mid i\in I(s_r,\alpha), j\in J(s_r,\alpha)\right\}.$$

**Notation 2.3.5.** Let $\mathbf{s}'\geq \mathbf{s}\in \mathbf{Seq}$ be given. From now on we will use the following abbreviations, where we use the notations from above.

$$P_{\mathbf{s}}\mathcal{X} := \operatorname*{colim}_{R\in\mathbb{N}} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(P_{s_r}(X_\alpha)\right)$$

$$P_{\mathbf{s}}(\mathcal{U}\oplus\mathcal{V}) := \operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left((P_{s_r}U_\alpha^{s_r})^t\right) \oplus \mathcal{A}^{G_\alpha}\left((P_{s_r}V_\alpha^{s_r})^t\right)$$

$$P_{\mathbf{s},\mathbf{s}'}(\mathcal{X},\mathbb{W}_t) := \operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(P_{s_r,s'_r}\left(X_\alpha,\mathbb{W}_\alpha^{s_r,t}\right)\right)$$

$$P_{\mathbf{s}}(\mathcal{U}\cap\mathcal{V}) := \operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left((P_{s_r}U_\alpha^{s_r})^t \cap (P_{s_r}V_\alpha^{s_r})^t\right)$$

$$P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}\cap\mathcal{V},\mathbb{W}_t) := \operatorname*{colim}_{R\in\mathbb{N},t,t'>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(\left(P_{s_r,s'_r}\left(U_\alpha^{s_r},\mathbb{W}_\alpha^{s_r,t}\right)\right)^{t'} \cap \left(P_{s_r,s'_r}\left(V_\alpha^{s_r},\mathbb{W}_\alpha^{s_r,t}\right)\right)^{t'}\right)$$

$$P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}\oplus\mathcal{V},\mathbb{W}_t) := \operatorname*{colim}_{R\in\mathbb{N},t,t'>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(\left(P_{s_r,s'_r}\left(U_\alpha^{s_r},\mathbb{W}_\alpha^{s_r,t}\right)\right)^{t'}\right) \oplus \mathcal{A}^{G_\alpha}\left(\left(P_{s_r,s'_r}\left(V_\alpha^{s_r},\mathbb{W}_\alpha^{s_r,t}\right)\right)^{t'}\right)$$

$$P_{\mathbf{s}''}(\mathcal{U}^t\oplus\mathcal{V}^t) = \operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(P_{s''_r}\left((U_\alpha^{s_r})^{tC_r s_r}\right)\right) \oplus \mathcal{A}^{G_\alpha}\left(P_{s''_r}\left((V_\alpha^{s_r})^{tC_r s_r}\right)\right)$$

Here the neighborhoods of the relative Rips complexes are taken inside $P_{s_r,s'_r}\left(X_\alpha, \mathbb{W}_\alpha^{s_r,t}\right)$.

**Theorem 2.3.6.** *With the above notations we get the following commutative diagram of (Mayer-Vietoris type) Karoubi sequences*

$$\begin{array}{ccc} P_{\boldsymbol{s}}(\mathcal{U}\cap\mathcal{V}) & \longrightarrow & P_{\boldsymbol{s},\boldsymbol{s}'}(\mathcal{U}\cap\mathcal{V},\mathbb{W}_t) \\ \Big\downarrow{\scriptstyle (i_1,i_2)} & & \Big\downarrow{\scriptstyle (i'_1,i'_2)} \\ P_{\boldsymbol{s}}(\mathcal{U}\oplus\mathcal{V}) & \longrightarrow & P_{\boldsymbol{s},\boldsymbol{s}'}(\mathcal{U}\oplus\mathcal{V},\mathbb{W}_t) \\ \Big\downarrow{\scriptstyle j_1-j_2} & & \Big\downarrow{\scriptstyle j'_1-j'_2} \\ P_{\boldsymbol{s}}(\mathcal{X}) & \longrightarrow & P_{\boldsymbol{s},\boldsymbol{s}'}(\mathcal{X},\mathbb{W}_t) \end{array}$$

*where all maps are given by the appropriate inclusions.*

*Proof.* The non-equivariant version of Theorem 2.3.6 is [RTY, Theorem 4.12]. The argument for the equivariant version is the same. □

For the induction step of the proof of Proposition 2.3.3 we need the following two lemmas.

**Lemma 2.3.7.** *For each $x \in K_{n-1}(P_{\boldsymbol{s}}(\mathcal{U}\cap\mathcal{V}))$ there exists $\boldsymbol{s}' \geq \boldsymbol{s}$ such that $\rho_{\boldsymbol{s},\boldsymbol{s}'}(x) = 0$, where*

$$\rho_{\boldsymbol{s},\boldsymbol{s}'} \colon K_{n-1}(P_{\boldsymbol{s}}(\mathcal{U}\cap\mathcal{V})) \to K_{n-1}(P_{\boldsymbol{s},\boldsymbol{s}'}(\mathcal{U}\cap\mathcal{V},\mathbb{W}_t))$$

*is the map induced by the inclusion of the underlying Rips complexes.*

The idea of the proof of Lemma 2.3.7 is that the pieces of the intersection $\mathcal{U}\cap\mathcal{V}$ become further apart with increasing $r$ and have lower decomposition complexity. So by the induction hypothesis we can increase the parameter of the Rips complex to get the vanishing result. We have to work with the relative Rips complex to make sure that after increasing the parameter the pieces of $\mathcal{U}$ and $\mathcal{V}$ are still far apart. The proof of the lemma will be given later.

**Lemma 2.3.8.** *For all $\boldsymbol{s}' \geq \boldsymbol{s}$ and each*

$$x \in K_n(P_{\boldsymbol{s},\boldsymbol{s}'}(\mathcal{U}\oplus\mathcal{V},\mathbb{W}_t)),$$

*there exist $\boldsymbol{s}'' \geq \boldsymbol{s}'$ such that $\mu_{\boldsymbol{s},\boldsymbol{s}',\boldsymbol{s}''}(x) = 0$, where*

$$\mu_{\boldsymbol{s},\boldsymbol{s}',\boldsymbol{s}''} \colon K_n(P_{\boldsymbol{s},\boldsymbol{s}'}(\mathcal{U}\oplus\mathcal{V},\mathbb{W}_t)) \to K_n(P_{\boldsymbol{s}''}(\mathcal{U}^t\oplus\mathcal{V}^t))$$

*is the map induced by the inclusion of the underlying Rips complexes. For this inclusion we use that for metric spaces $U, W \subseteq X$, where $X$ has bounded geometry, the following inclusion holds*

$$(P_{s,s'}(U,W))^t \subseteq P_{s'}\left((U\cup W)^{(t+2)Cs}\right),$$

*where $C = (2\sqrt{2}+1)^{\dim P_s(X)-1}$, see Lemma 2.1.3.*

The families $\mathbb{W}_{\alpha}^{s_r,t}$ are constructed in a way, so that the second parameter of the relative Rips complex does not change the property of the pieces of $\mathcal{U}$ and $\mathcal{V}$ to be far apart. Since they have lower decomposition complexity, we can use the induction hypothesis to prove the lemma. Before giving the proof, we first prove Proposition 2.3.3.

*Proof of Proposition 2.3.3 assuming Lemma 2.3.7 and Lemma 2.3.8.* The induction beginning was proved in Lemma 2.3.4. For the case of a successor ordinal $\gamma = \beta + 1$ consider the following commutative diagram,

$$\begin{array}{ccccc}
 & & K_n(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U} \oplus \mathcal{V}, \mathbb{W}_t)) & \xrightarrow{\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}} & K_n(P_{\mathbf{s}''}(\mathcal{U}^t \oplus \mathcal{V}^t)) \\
 & & \downarrow{\scriptstyle j_1'-j_2'} & & \downarrow{\scriptstyle \iota_1-\iota_2} \\
K_n(P_{\mathbf{s}}(\mathcal{X})) & \xrightarrow{\gamma_{\mathbf{s},\mathbf{s}'}} & K_n(P_{\mathbf{s},\mathbf{s}'}(\mathcal{X}, \mathbb{W}_t)) & \xrightarrow{\gamma_{\mathbf{s},\mathbf{s}',\mathbf{s}''}} & K_n(P_{\mathbf{s}''}(\mathcal{X})) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
K_{n-1}(P_{\mathbf{s}}(\mathcal{U} \cap \mathcal{V})) & \xrightarrow{\rho_{\mathbf{s},\mathbf{s}'}} & K_{n-1}(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U} \cap \mathcal{V}, \mathbb{W}_t)) & &
\end{array}$$

where the first two vertical sequences are the Mayer-Vietoris sequences coming from Theorem 2.3.6. The other maps are induced by inclusions of Rips complexes. By Lemma 2.3.7 for each $x \in K_n(P_{\mathbf{s}}(\mathcal{X}))$ there exists $\mathbf{s}' \geq \mathbf{s}$ such that $\rho_{\mathbf{s},\mathbf{s}'}(\partial(x)) = 0$. Therefore, $\partial(\gamma_{\mathbf{s},\mathbf{s}'}(x)) = 0$ and $\gamma_{\mathbf{s},\mathbf{s}'}(x)$ has a preimage $y \in K_n(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U} \oplus \mathcal{V}, \mathbb{W}_t))$. By Lemma 2.3.8 there exists $\mathbf{s}'' \geq \mathbf{s}'$ such that $\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''}(y) = 0$, thus the image of $x$ in $K_n(P_{\mathbf{s}''} X)$ is zero. Both $\gamma_{\mathbf{s},\mathbf{s}'}$ and $\gamma_{\mathbf{s},\mathbf{s}',\mathbf{s}''}$ are induced by inclusions of Rips complexes, so the composition is again induced by the inclusion. This proves the case for a successor ordinal.

If $\gamma$ is a limit ordinal the induction step follows directly from the definition of FDC. □

*Proof of Lemma 2.3.7.* By Lemma 2.1.3 we have the following inclusions

$$(P_{s_r} U_i)^t \subseteq P_{s_r}\left((U_i)^{(t+2)C_r s_r}\right) \text{ and } (P_{s_r} V_j)^t \subseteq P_{s_r}\left((V_j)^{(t+2)C_r s_r}\right)$$

for all $i \in I(s_r, \alpha), j \in J(s_r, \alpha)$. Since $U_{\alpha}^{s_r}$ is the $(rC_r s_r)$-disjoint union

$$U_{\alpha}^{s_r} = \bigcup_{i}^{(rC_r s_r)-disj} U_i,$$

this implies

$$(P_{s_r}(U_{\alpha}^{s_r}))^t = \bigcup^{(r-2t-4)-disj} (P_{s_r} U_i)^t$$

and the analogous statement for $V_{\alpha}^{s_r}$. Since the morphisms in $P_{\mathbf{s}}(\mathcal{U} \cap \mathcal{V})$ are uniformly bounded and $\lim_{r\to\infty} r-2t-4=\infty$ for every $t$ we can decompose $P_{\mathbf{s}}(\mathcal{U}\cap\mathcal{V})$ as:

$$P_{\mathbf{s}}(\mathcal{U}\cap\mathcal{V}) \cong \operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\Big(\coprod_{i,j}(P_{s_r}U_i)^t \cap (P_{s_r}V_j)^t\Big)$$

Using again Lemma 2.1.3 we get an inclusion of the right hand side into

$$\operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\Big(\coprod_{i,j} P_{s_r}\left((U_i)^{tC_rs_r}\cap (V_j)^{tC_rs_r}\right)\Big).$$

By [RTY, Lemma 6.15] we have a map from this category, where the Rips complexes are endowed with the metric coming from $P_{s_r}X_\alpha$, to the category

$$\operatorname*{colim}_{\mathbf{s}'\in\mathbf{Seq},R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\Big(\coprod_{i,j} P_{s'_r}\left((U_i)^{tC_rs_r}\cap (V_i)^{tC_rs_r}\right)\Big),$$

where the Rips complexes are endowed with the intrinsic metric. This map is given by the inclusion of the Rips complexes, where we need to take the colimit over **Seq** to make sure that the morphisms still have bounded propagation with the new metric.

By [RTY, Lemma 6.3] taking a finite thickening does not change the decomposition complexity and therefore we have for all $r,t>0$

$$\left\{\Big(\coprod_{i,j}(U_i)^{tC_rs_r}\cap (V_j)^{tC_rs_r}, G_\alpha\Big)\middle|\, \alpha\in A_r\right\}\in e\mathfrak{D}_\beta.$$

This implies that by the induction hypothesis for $\beta$ for every $t>0$

$$\operatorname*{colim}_{\mathbf{s}'\in\mathbf{Seq}} K_n\Bigg(\operatorname*{colim}_{R\in\mathbb{N}} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\Big(\coprod_{i,j} P_{s'_r}\left((U_i)^{tC_rs_r}\cap (V_j)^{tC_rs_r}\right)\Big)\Bigg)=0,$$

for the intrinsic metric. Here we use that $P_s\big(\coprod_{i\in I}X_i\big)=\coprod_{i\in I}P_s(X_i)$ for any family of metric spaces $\{X_i\}_{i\in I}$. Therefore after taking the colimit over $t$ we have,

$$\operatorname*{colim}_{\mathbf{s}'\in\mathbf{Seq}} K_n\Bigg(\operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha}^{bd} \mathcal{A}^{G_\alpha}\Big(\coprod_{i,j} P_{s'_r}\left((U_i)^{tC_rs_r}\cap (V_j)^{tC_rs_r}\right)\Big)\Bigg)=0.$$

The Rips complex $P_{s'_r}\left((U_i)^{tC_rs_r}\cap (V_i)^{tC_rs_r}\right)$ with the intrinsic metric includes into the intersection $P_{s_r,s'_r}\big(U_\alpha^{s_r},\mathbb{W}_\alpha^{s_r,t}\big)\cap P_{s_r,s'_r}\big(V_\alpha^{s_r},\mathbb{W}_\alpha^{s_r,t}\big)$ with the metric coming from the complex $P_{s_r,s'_r}(X_\alpha, \mathbb{W}_\alpha^{s_r,t})$. This induces a map

$$\operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha}^{bd} \mathcal{A}^{G_\alpha}\Big(\coprod_{i,j} P_{s'_r}\left((U_i)^{tC_r s_r} \cap (V_j)^{tC_r s_r}\right)\Big) \to P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}\cap\mathcal{V}, \mathbb{W}_t).$$

The composition of these maps in $K$-theory is $\rho_{\mathbf{s},\mathbf{s}'}$ because all maps are induced by inclusions of Rips complexes. And for $\mathbf{s}' \in \mathbf{Seq}$ large enough the image of $x$ under this map is zero by the above. □

*Proof of Lemma 2.3.8.* Since $(P_{s_r}U_i)^r \subseteq P_{s_r}\big(U_i^{rC_r s_r}\big)$ by Lemma 2.1.3 and we have $d(U_i, U_j) \geq rC_r s_r$ for all $i, j \in I(s_r,\alpha)$, we also get $d(P_{s_r}(U_i), P_{s_r}(U_j)) \geq r$. Since $t$ is independent of $r$, this implies for $r$ large enough that $P_{s_r,s'_r}\big(U, \mathbb{W}_\alpha^{s_r,t}\big)$ is the disjoint union of the pieces $P_{s_r,s'_r}\big(U_i, \{U_i^{tC_r s_r} \cap V_j^{tC_r s_r}\}_{j\in J(s_r,\alpha)}\big)$ with $i \in I(s_r,\alpha)$. For fixed $i_0 \in I(s_r,\alpha)$ define

$$X_1 := U_{i_0}, \quad X_2 := \bigcup_{i\neq i_0} U_i$$

and

$$\mathbb{W}_1 := \big\{(U_{i_0})^{tC_r s_r} \cap (V_j)^{tC_r s_r} \mid j \in J(s_r,\alpha)\big\} \subseteq \mathbb{W}_\alpha^{s_r,t}, \quad \mathbb{W}_2 := \mathbb{W}_\alpha^{s_r,t} \setminus \mathbb{W}_1.$$

By Lemma 2.1.4 we have that

$$d(P_{s,s'}(X_1,\mathbb{W}_1), P_{s,s'}(X_2,\mathbb{W}_2)) \geq \frac{d((U_{i_0})^{tC_r s_r}, (U_r^\alpha \setminus U_{i_0})^{tC_r s_r})}{s_r C_r} - 2 \geq r - 2t - 2.$$

Therefore, the above disjoint union is $r-2t-2$ disjoint, i.e.

$$P_{s_r,s'_r}\big(U, \mathbb{W}_\alpha^{s_r,t}\big) = \bigcup_{i\in I(s_r,\alpha)}^{(r-2t-2)-disj} P_{s_r,s'_r}\big(U_i, \{U_i^{tC_r s_r} \cap V_j^{tC_r s_r}\}_{j\in J(s_r,\alpha)}\big).$$

Since every morphism in $P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}\oplus\mathcal{V}, \mathbb{W}_t)$ is bounded, arguing as in the proof of Lemma 2.3.7 we can decompose $P_{\mathbf{s},\mathbf{s}'}(\mathcal{U}\oplus\mathcal{V}, \mathbb{W}_t)$ as

$$\operatorname*{colim}_{R\in\mathbb{N},t,t'>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \Bigg[\mathcal{A}^{G_\alpha}\Big(\coprod_i \Big(P_{s_r,s'_r}(U_i, \{U_i^{tC_r s_r} \cap V_j^{tC_r s_r}\}_j)\Big)^{t'}\Big) \oplus \mathcal{A}^{G_\alpha}\Big(\coprod_j \Big(P_{s_r,s'_r}(V_j, \{U_i^{tC_r s_r} \cap V_j^{tC_r s_r}\}_i)\Big)^{t'}\Big)\Bigg].$$

We can include $\big(P_{s_r,s'_r}(U_i, \{U_i^{tC_r s_r} \cap V_j^{tC_r s_r}\}_j)\big)^{t'}$ into $\big(P_{s'_r}(U_i^{tC_r s_r})\big)^{t'}$ for every $i \in I(s_r,\alpha)$ and by Lemma 2.1.3 this can further be included into $P_{s'_r}\big(U_i^{(t+t'+2)C_r s_r}\big)$. This induces a map from the above category to

$$\operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(\coprod_i P_{s'_r}(U_i^{tC_r s_r})\right) \oplus \mathcal{A}^{G_\alpha}\left(\coprod_j P_{s'_r}(V_j^{tC_r s_r})\right).$$

As in the proof of Lemma 2.3.7 the induction hypothesis for $\beta$ implies that for each

$$x \in K_n\left(\operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(\coprod_i P_{s'_r}(U_i^{tC_r s_r})\right) \oplus \mathcal{A}^{G_\alpha}\left(\coprod_j P_{s'_r}(V_j^{tC_r s_r})\right)\right)$$

there exists $\mathbf{s}'' \geq \mathbf{s}'$ such that the image of $x$ is zero in

$$K_n\left(\operatorname*{colim}_{R\in\mathbb{N},t>0} \prod_{r\geq R,\alpha\in A_r}^{bd} \mathcal{A}^{G_\alpha}\left(\coprod_i P_{s''_r}(U_i^{tC_r s_r})\right) \oplus \mathcal{A}^{G_\alpha}\left(\coprod_j P_{s''_r}(V_j^{tC_r s_r})\right)\right).$$

Finally, we get a map to $K_n(P_{\mathbf{s}''}(\mathcal{U}^t \oplus \mathcal{V}^t))$, which again is induced by the inclusion of Rips complexes. Since all maps are induced by inclusions, the map

$$\mu_{\mathbf{s},\mathbf{s}',\mathbf{s}''} \colon K_n(P_{\mathbf{s},\mathbf{s}'}(\mathcal{U} \oplus \mathcal{V}, \mathbb{W}_t)) \to K_n(P_{\mathbf{s}''}(\mathcal{U}^t \oplus \mathcal{V}^t))$$

is the composition of the above maps. □

# 3 On the injectivity of the assembly map

In this chapter the main result is proved. This is the following theorem.

**Theorem 3.2.2.** *Let $G$ be a discrete group such that $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC and let $\mathcal{A}$ be a small additive $G$-category. Assume that there is a finite dimensional $G$-CW-model for the classifying space for proper $G$-actions $\underline{E}G$.*

*Then the assembly map in algebraic $K$-theory $H_*^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$ is split injective.*

The outline of the proof is the following:

We will show that the metric equivariant family $\{(G,H)\}_{H\in\mathcal{F}in}$, see Definition 2.2.1, has FDC if the family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC. Then Theorem 2.3.1 implies that

$$\operatorname*{colim}_s K_n\left(\prod_{H\in\mathcal{F}in}^{bd} \mathcal{A}_G^H(P_sG)\right) = 0,$$

for all $n \in \mathbb{N}$.

This information about the fixed point categories for finite subgroups can be used to gain information about the long exact sequence

$$\ldots \to K_n\mathcal{A}_G^G(\underline{\mathrm{E}}G) \to K_n\mathcal{A}_G^G(\underline{\mathrm{E}}G)^\infty \xrightarrow{\partial} K_{n-1}\mathcal{A}_G^G(\underline{\mathrm{E}}G)_0 \to K_{n-1}\mathcal{A}_G^G(\underline{\mathrm{E}}G) \to \ldots$$

where fixed points with respect to the whole group $G$ are taken. Following the strategy of [BR07b] we will use a Descent Principle for this comparison, but the version of [BR07b] can only be applied in the case where $\underline{\mathrm{E}}G$ has a finite model. In the first section we introduce a modified version of homotopy fixed points and prove a Descent Principle which can be used for groups that only admit a finite dimensional model for $\underline{\mathrm{E}}G$ instead of a cocompact one. The proof of Theorem 3.2.2 is given in the second section.

In the last section we also prove an $L$-theoretic version of our main theorem.

## 3.1 The Descent Principle

For the groups we are interested in, in particular linear groups, we can only prove that they have equivariant finite decomposition complexity with respect to finite subgroups, but not equivariant with respect to the whole group itself. The main tool to still get a result for the equivariant assembly map is to compare it with a homotopy fixed point version. We begin with a sketch of the general idea behind this comparison. The proper homotopy fixed points of a $G$-space $X$ are defined as $X^{h_{\mathcal{F}in}G} := \mathrm{Map}_G(\underline{\mathrm{E}}G, X)$. The map $\underline{E}G \to \{pt\}$ induces a map $X^G \to X^{h_{\mathcal{F}in}G}$. The assembly map is equivalent to the boundary map $K_n(\mathcal{A}^G_{G-c}(\underline{E}G)^\infty) \to K_{n-1}(\mathcal{A}^G_{G-c}(\underline{E}G)_0)$ by Proposition 1.7.8. Now the comparison with homotopy fixed points yields the following commutative diagram:

$$\begin{array}{ccc} \pi_n \mathbb{K}(\mathcal{A}^G_{G-c}(\underline{E}G)^\infty) & \longrightarrow & \pi_{n-1}\mathbb{K}(\mathcal{A}^G_{G-c}(\underline{E}G)_0) \\ \downarrow f & & \downarrow \\ \pi_n\left(\mathbb{K}(\mathcal{A}_{G-c}(\underline{E}G)^\infty)^{h_{\mathcal{F}in}G}\right) & \longrightarrow & \pi_{n-1}\left(\mathbb{K}(\mathcal{A}_{G-c}(\underline{E}G)_0)^{h_{\mathcal{F}in}G}\right) \end{array}$$

By [BR07b, Lemma 7.1] the fact that $\mathbb{K}\mathcal{A}^H_{G-c}(\underline{E}G)$ is weakly contractible for every finite subgroup $H \leq G$ implies that the lower map is an isomorphism. If there is a cocompact model for $\underline{E}G$ then the map denoted by $f$ is an isomorphism by [Ros04, Theorem 6.2]. Therefore, if the lower map is (split) injective, so is the upper one. The idea of comparing fixed points and homotopy fixed points to gain injectivity results is called Descent Principle, see for example [BR07b], and it is due to [CP95]. For the case where there only is a finite dimensional model for $\underline{E}G$ we take colimits over the cocompact subsets of $\underline{E}G$. The problem with this is that taking homotopy fixed points does not commute with colimits and for this reason it does no longer suffice to show that $K_n\mathcal{A}^H_{G-c}(\underline{E}G) = 0$ for every finite subgroup $H \leq G$. To get a similar diagram where we can show that the lower map is an isomorphism, we need a more refined version of homotopy fixed points. This will be developed in this section.

Let $Z$ be a simplicial complex and $J$ be the set of simplices of $Z$. A map from $Z$ to a space $X$ is the same as a sequence $(h_\sigma)_{\sigma\in J} \in \prod_{\sigma\in J} \mathrm{Map}(\sigma, X)$ fitting together on their faces. This can be viewed as a sequence $(h_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{Map}(\Delta^n, \prod_{J_n} X)$ with some compatibility conditions, where $J_n$ is the set of $n$-simplices of $Z$ and $\Delta^n$ is the standard $n$-simplex with vertices $\{0, .., n\}$. Let $\mathcal{A}$ be a filtered additive category and $(\mathbb{K}\mathcal{A})_n$ be the $n$-th space of the spectrum $\mathbb{K}\mathcal{A}$. In the case where $X$ is $(\mathbb{K}\mathcal{A})_n$ we can ask that the maps have image in $(\mathbb{K}\prod^{bd}_{J_n}\mathcal{A})_n$ instead of $\prod_{J_n}(\mathbb{K}\mathcal{A})_n$. This will allow us to make an induction over the skeleta of $Z$ instead of the individual cells without losing the control conditions. Now we will make the above precise.

Let $Z$ be a simplicial $G$-CW complex and $\mathcal{A}$ a filtered, additive category with $G$-action. Let $J_k$ be the $G$-set of $k$-simplices in the barycentric subdivision of $Z$. Since the vertices of every simplex in the barycentric subdivision are naturally ordered by the inclusion of the corresponding simplices in $Z$, we get maps

$$s_i \colon J_k \to J_{k-1}, \sigma \mapsto \partial_i\sigma \quad \text{for } 0 \leq i \leq k.$$

Define for each $n \in \mathbb{N}$

$$A^n_k := \mathrm{Map}_G\left(\Delta^k, \left(\mathbb{K}\prod^{bd}_{J_k}\mathcal{A}\right)_n\right) \cong \mathrm{Map}\left(\Delta^k, \left(\mathbb{K}\prod^{bd}_{J_k}\mathcal{A}\right)^G_n\right)$$

and

$$B_k^n := \prod_{i=0}^{k} \operatorname{Map}_G\left(\Delta^{k-1}, \left(\mathbb{K}\prod_{J_k}^{bd}\mathcal{A}\right)_n\right),$$

where $\left(\mathbb{K}\prod_{J_k}^{bd}\mathcal{A}\right)_n$ is the $n$-th space of the spectrum $\mathbb{K}\prod_{J_k}^{bd}\mathcal{A}$. The maps $s_i\colon J_k \to J_{k-1}$ and the inclusions $d_i\colon \Delta^{k-1} \to \Delta^k$ induce maps $f_k^n := (s_i^*)_i\colon A_{k-1}^n \to B_k^n$ respectively $g_k^n := (d_i^*)_i\colon A_k^n \to B_k^n$.

Since the maps $d_i$ are cofibrations the induced maps

$$d_i^*\colon \operatorname{Map}_G\left(\Delta^k, \left(\mathbb{K}\prod_{J_k}^{bd}\mathcal{A}\right)_n\right) \to \operatorname{Map}_G\left(\Delta^{k-1}, \left(\mathbb{K}\prod_{J_k}^{bd}\mathcal{A}\right)_n\right)$$

are fibrations and thus also the maps $g_k^n$ are fibrations.

**Definition 3.1.1.** Let $Z$ be a simplicial $G$-CW complex. The bounded mapping space $\operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})$ is defined as the spectrum whose $n$-th space is the subspace of $\prod_{k\in\mathbb{N}} A_k^n$ consisting of all $(h_k)_k \in \prod_{k\in\mathbb{N}} A_k^n$ with $f_k^n(h_{k-1}) = g_k^n(h_k)$ for all $k \geq 1$. The structure maps are induced by the structure maps of the spectra $\mathbb{K}\left(\prod_{J_k}^{bd}\mathcal{A}\right)$.

We think of the spectrum $\operatorname{Map}_G^{bd}(\underline{E}G, \mathbb{K}\mathcal{A})$ as a bounded version of proper homotopy fixed points. Note that this spectrum depends on the chosen model for $\underline{E}G$ as a simplicial $G$-CW complex.

**Remark 3.1.2.** The inclusion $\prod_{J_k}^{bd}\mathcal{A} \hookrightarrow \prod_{J_k}\mathcal{A}$ together with the natural map $\mathbb{K}\left(\prod_{J_k}\mathcal{A}\right)_n \to \prod_{J_k}(\mathbb{K}\mathcal{A})_n$ induces a map

$$F_k\colon \operatorname{Map}_G\left(\Delta^k, \left(\mathbb{K}\prod_{J_k}^{bd}\mathcal{A}\right)_n\right) \to \operatorname{Map}_G\left(\Delta^k, \prod_{J_k}(\mathbb{K}\mathcal{A})_n\right) \cong \operatorname{Map}_G\left(\coprod_{J_k}\Delta^k, (\mathbb{K}\mathcal{A})_n\right).$$

For $\sigma \in J_k$ let $F_k(h_k)(\sigma)$ denote the restriction of $F_k(h_k)$ to the $\sigma$-component. Since $f_k^n(h_{k-1}) = g_k^n(h_k)$ for every $(h_k)_k \in \left(\operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})\right)_n$, we get

$$F_k(h_k)(\sigma)|_{\partial_i} = d_i^* F_k(h_k)(\sigma) = s_i^* F_{k-1}(h_{k-1})(\sigma) = F_{k-1}(h_{k-1})(\partial_i\sigma).$$

For $\sigma_1, \sigma_2 \in J_k$ with $\partial_i\sigma_1 = \partial_j\sigma_2$ this implies

$$F_k(h_k)(\sigma_1)|_{\partial_i} = F_{k-1}(h_{k-1})(\partial_i\sigma_1) = F_{k-1}(h_{k-1})(\partial_j\sigma_2) = F_k(h_k)(\sigma_2)|_{\partial_j}.$$

This shows that the maps

$$F_k(h_k) \in \operatorname{Map}_G\left(\coprod_{J_k}\Delta^k, (\mathbb{K}\mathcal{A})_n\right)$$

fit together to a map $h \in (\operatorname{Map}_G(Z, \mathbb{K}\mathcal{A}))_n$. Therefore, the inclusions $\prod_{J_k}^{bd}\mathcal{A} \hookrightarrow \prod_{J_k}\mathcal{A}$

induce a map

$$F\colon \operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A}) \to \operatorname{Map}_G(Z, \mathbb{K}\mathcal{A}).$$

Furthermore, the diagonal map $\Delta\colon \mathcal{A} \to \prod_{J_k}^{bd} \mathcal{A}$ induces a map

$$G\colon \mathbb{K}(\mathcal{A}^G) \to \operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})$$

by sending $x \in (\mathbb{K}\mathcal{A}^G)_n$ to $(h_k)_k \in \operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})_n$ with $h_k \equiv \mathbb{K}(\Delta)(x)$ for all $k$. The composition $F \circ G\colon \mathbb{K}(\mathcal{A}^G) \cong \operatorname{Map}_G(*, \mathbb{K}\mathcal{A}) \to \operatorname{Map}_G(Z, \mathbb{K}\mathcal{A})$ is induced by the constant map $Z \to *$.

Next we will show that $\operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})$ can be characterized as a homotopy limit. We will need this later on to see that it commutes with other homotopy limits.

**Proposition 3.1.3.** *Let $Z$ be a simplicial $G$-CW complex and $\mathcal{A}$ a filtered, additive category with $G$-action. Let $(A_k^n, B_k^n, f_k^n, g_k^n)_{k\in\mathbb{N}}$ be as above. Then $\left(\operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})\right)_n$ is a model for the limit as well as the homotopy limit of the diagram $(A_k^n, B_k^n, f_k^n, g_k^n)_{k\in\mathbb{N}}$.*

We will use that pullbacks where one of the two maps is a fibration are homotopy pullbacks and that the limit of a tower of fibrations is a homotopy limit of that tower. These facts are well known and the analogous statements in the category of simplicial sets instead of topological spaces can be found in [BK72, Chapter XI, Examples 4.1(iv)&(v)].

*Proof.* Let $M_m \subseteq \prod_{k\le m} A_k^n$ denote the subspace with $f_k^n(h_{k-1}) = g_k^n(h_k)$. $M_m$ is a limit of the diagram $(A_k^n, B_k^n, f_k^n, g_k^n)_{k\le m}$:

$$\begin{array}{ccccccc}
\dots & \longrightarrow & M_2 & \twoheadrightarrow & M_1 & \twoheadrightarrow & A_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_1} \\
\dots & \twoheadrightarrow & (g_2)^*A_1 & \twoheadrightarrow & A_1 & \xrightarrow{g_1} & B_1 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f_2} & & \\
\dots & \longrightarrow & A_2 & \xrightarrow{g_2} & B_1 & &
\end{array}$$

The limit arises from taking finitely many pullbacks. Since the maps $g_k^n$ are fibrations, the space $M_m$ is a also a homotopy limit of this diagram and the induced maps $M_m \to M_{m-1}$ are fibrations as well. $\left(\operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})\right)_n$ is a limit of the tower

$$\ldots \to M_m \to M_{m-1} \to \ldots \to M_1 \to M_0 = A_0,$$

and since all these arrows are fibrations, it is also a homotopy limit of the tower. Therefore, $\left(\operatorname{Map}_G^{bd}(Z, \mathbb{K}\mathcal{A})\right)_n$ is a model for the limit and the homotopy limit of the diagram $(A_k^n, B_k^n, f_k^n, g_k^n)_{k\in\mathbb{N}}$. □

Proposition 1.7.13 and Proposition 3.1.3 imply that we get the following homotopy fibration sequence:

$$\operatorname{Map}_G^{bd}(Z, \mathbb{K}(\mathcal{A}_{G-c}(X)_0)) \to \operatorname{Map}_G^{bd}(Z, \mathbb{K}(\mathcal{A}_{G-c}(X))) \to \operatorname{Map}_G^{bd}(Z, \mathbb{K}(\mathcal{A}_{G-c}(X)^{\infty}))$$

Now for a simplicial model for $\underline{E}G$ the diagram for the Descent Principle with bounded homotopy fixed points is the following:

$$\begin{array}{ccc}
K_n(\mathcal{A}^G_{G-c}(\underline{E}G)^\infty) & \xrightarrow{\partial} & K_{n-1}(\mathcal{A}^G_{G-c}(\underline{E}G)_0) \\
\downarrow & & \downarrow \\
\operatorname*{colim}_{K\subseteq \underline{E}G \text{ cp.}} \pi_n \operatorname{Map}^{bd}_G(\underline{E}G, \mathbb{K}\mathcal{A}_G(GK)^\infty) & \xrightarrow{\partial} & \operatorname*{colim}_{K\subseteq \underline{E}G \text{ cp.}} \pi_{n-1} \operatorname{Map}^{bd}_G(\underline{E}G, \mathbb{K}\mathcal{A}_G(GK)_0)
\end{array}$$

The next proposition gives a condition under which the lower map in the diagram for the Descent Principle with bounded homotopy fixed points is an isomorphism. Using bounded maps allows us to do an induction over the dimension of $\underline{E}G$ instead of the individual cells.

**Proposition 3.1.4.** *Let $Y$ be a finite dimensional, simplicial $G$-CW complex with finite stabilizers and let $\{X_\alpha\}_{\alpha\in A}$ be a directed system of metric $G$-spaces and equivariant metrically coarse maps such that for every $G$-set $J$ with finite stabilizers*

$$\operatorname*{colim}_{\alpha\in A} K_n\left(\prod_J^{bd} \mathcal{A}_G(X_\alpha)\right)^G = 0, \quad \forall n\in\mathbb{N}.$$

*Then*

$$\operatorname*{colim}_{\alpha\in A} \pi_n\big(\operatorname{Map}^{bd}_G(Y, \mathbb{K}\mathcal{A}_G(X_\alpha))\big) = 0, \quad \forall n\in\mathbb{N}.$$

*Proof.* Let $x_0\in S^n$ be the base point. If no extra base point is added we will always consider $0\in\Delta^k$ as the base point. As above let $J_k$ be the set of $k$-simplices in the barycentric subdivision of $Y$ and let $s_i\colon J_k\to J_{k-1}$ be defined by $\sigma\mapsto\partial_i\sigma$. Let $Z^\alpha_k := \mathbb{K}\big(\prod^{bd}_{J_k}\mathcal{A}_G(X_\alpha)\big)^G$.

An element in $\pi_n(\operatorname{Map}^{bd}_G(Y,\mathbb{K}\mathcal{A}_G(X_\alpha)))$ is represented by a system of maps

$$h_k\in\operatorname{Map}_*(S^n,\operatorname{Map}(\Delta^k,Z^\alpha_k))\cong\operatorname{Map}_*(S^n\wedge\Delta^k_+,Z^\alpha_k)$$

such that

- $h_k|_{S^n\wedge(\partial_i\Delta^k)_+} = (s_i)^*\circ h_{k-1}$.

And a null homotopy of $\{h_k\}$ is a system of maps

$$H_k\in\operatorname{Map}_*(S^n\wedge I,\operatorname{Map}(\Delta^k,Z^\beta_k))\cong\operatorname{Map}_*(S^n\wedge I\wedge\Delta^k_+,Z^\beta_k)\cong\operatorname{Map}_*(S^n\wedge\Delta^{k+1},Z^\beta_k)$$

for some $\beta\ge\alpha$, such that

- $H_k|_{S^n\wedge\partial_i\Delta^{k+1}} = (s_i)^*\circ H_{k-1},\ 1\le i\le k+1$ and
- $H_k|_{S^n\wedge(\partial_0\Delta^{k+1}\cup\{0\})} = h_k$.

We will construct such a null homotopy $\{H_k\}$ by induction on $k$. Since $h_0$ represents an element in $\pi_n Z_0^\alpha$, there exists $\beta \geq \alpha$ such that $h_0$ is null homotopic in $Z_0^\beta$ by assumption. Every such null homotopy is a map

$$H_0 \in \mathrm{Map}_*(S^n \wedge \Delta^1, Z_0^\beta)$$

with

- $H_0|_{S^n \wedge (\partial_0 \Delta^1 \cup \{0\})} = h_0$.

Assume we have already constructed maps

$$H_j \in \mathrm{Map}_*(S^n \wedge \Delta^{j+1}, Z_j^\beta)$$

for $j < k$ and some $\beta \in A$ such that

- $H_j|_{S^n \wedge \partial_i \Delta^{j+1}} = (s_i)^* \circ H_{j-1},\ 1 \leq i \leq j+1$ and
- $H_j|_{S^n \wedge (\partial_0 \Delta^{j+1} \cup \{0\})} = h_j$.

These maps can be glued together to a map

$$\tilde{H}_k \in \mathrm{Map}_*(S^n \wedge \partial\Delta^{k+1}, Z_j^\beta)$$

such that

- $\tilde{H}_k|_{S^n \wedge \partial_i \Delta^{k+1}} = (s_i)^* \circ H_{k-1},\ 1 \leq i \leq k+1$ and
- $\tilde{H}_k|_{S^n \wedge (\partial_0 \Delta^{k+1} \cup \{0\})} = h_k$.

Since $S^n \wedge \partial\Delta^{k+1} \cong S^{n+k}$ the element $\tilde{H}_k$ gives an element in $\mathrm{Map}_*(S^{n+k}, Z_k^\beta)$. By assumption there exists $\beta' \geq \beta$ such that $\tilde{H}_k$ is null homotopic in $\mathrm{Map}_*(S^{n+k}, Z_k^{\beta'})^G)$. Any such null homotopy can be used to extend $\tilde{H}_k$ to a map

$$H_k \in \mathrm{Map}_*(S^n \wedge \Delta^{k+1}, Z_k^{\beta'})$$

with

- $H_k|_{S^n \wedge \partial_i \Delta^{k+1}} = (s_i)^* \circ H_{k-1},\ 1 \leq i \leq k+1$ and
- $H_k|_{S^n \wedge (\partial_0 \Delta^{k+1} \cup \{0\})} = h_k$.

Since $Y$ was assumed to be finite dimensional, after finitely many steps we have constructed the required null homotopy $\{H_k\}$ of $\{h_k\}$. □

This can be used to prove a version of the Descent Principle:

**Theorem 3.1.5** (Descent Principle)**.** *Let $G$ be a discrete group admitting a finite dimensional model for $\underline{E}G$ and let $X$ be a simplicial $G$-CW complex. Assume that for every $G$-set $J$ with finite stabilizers*

$$\operatorname*{colim}_{K\subseteq X \text{ compact}} K_n\Big(\prod_{j\in J}^{bd}\mathcal{A}_G(GK)\Big)^G = 0,$$

*then the assembly map*

$$H_*^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$$

*is a split injection.*

*Proof.* Let $Y$ be a finite dimensional simplicial $G$-CW-model for $\underline{E}G$. For each finite subcomplex $K\subseteq X$ consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{K}(\mathcal{A}_G^G(GK)_0) & \longrightarrow & \mathbb{K}(\mathcal{A}_G^G(GK)) & \longrightarrow & \mathbb{K}(\mathcal{A}_G^G(GK)^\infty) \\
\downarrow & & \downarrow & & \downarrow f \\
\operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(GK)_0)) & \longrightarrow & \operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(GK))) & \longrightarrow & \operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(GK)^\infty)) \\
\downarrow & & \downarrow & & \downarrow g \\
\operatorname{Map}_G(Y,\mathbb{K}(\mathcal{A}_G(GK)_0)) & \longrightarrow & \operatorname{Map}_G(Y,\mathbb{K}(\mathcal{A}_G(GK))) & \longrightarrow & \operatorname{Map}_G(Y,\mathbb{K}(\mathcal{A}_G(GK)^\infty))
\end{array}$$

All three rows in this diagram are induced by Karoubi filtrations and are, therefore, homotopy fibrations. The vertical maps are those from Remark 3.1.2. The composition $g\circ f$ is a weak homotopy equivalence by [Ros04, Theorem 6.2]. Therefore, $f$ induces a split injection on homotopy groups.

(Since K-theory commutes with products, $g$ is a weak homotopy equivalence but we do not need this fact here.)

Since $\operatorname{colim}_{K\subseteq X \text{ finite}} K_n(\mathcal{A}_G(GK))) \cong K_n(\mathcal{A}_{G-c}(X))$, after taking homotopy groups and colimits over $K\subseteq X$ finite we get the following commutative diagram:

$$\begin{array}{ccc}
K_{n+1}(\mathcal{A}_{G-c}^G(X)^\infty) & \xrightarrow{\partial} & K_n(\mathcal{A}_{G-c}^G(X)_0) \\
\downarrow f_* & & \downarrow \\
\operatorname*{colim}_{K\subseteq X \text{ finite}} \pi_{n+1}(\operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(GK)^\infty))) & \xrightarrow{\partial} & \operatorname*{colim}_{K\subseteq X \text{ finite}} \pi_n(\operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(GK)_0)))
\end{array}$$

The lower horizontal map is an isomorphism by Proposition 3.1.4 and $f_*$ is split injective as stated above. Thus the upper horizontal map is split injective and this map is equivalent to the assembly map by Proposition 1.7.8. □

The next theorem is a special case of Theorem 3.1.5 using the Rips complex.

**Theorem 3.1.6** (Decent Principle)**.** *Let $G$ be a discrete group admitting a finite dimensional model for $\underline{E}G$. Assume that for every $G$-set $J$ with finite stabilizers we have*

$$\operatorname*{colim}_s K_n\Big(\prod_{j\in J}^{bd} \mathcal{A}_G(P_sG)\Big)^G = 0,$$

*then the assembly map*

$$H_*^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$$

*is a split injection.*

*Proof.* Consider the diagram

$$\begin{array}{ccccc}
\mathbb{K}(\mathcal{A}_G^G(P_sG)_0) & \longrightarrow & \mathbb{K}(\mathcal{A}_G^G(P_sG)) & \longrightarrow & \mathbb{K}(\mathcal{A}_G^G(P_sG)^\infty) \\
\downarrow & & \downarrow & & \downarrow f \\
\operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(P_sG)_0)) & \longrightarrow & \operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(P_sG))) & \longrightarrow & \operatorname{Map}_G^{bd}(Y,\mathbb{K}(\mathcal{A}_G(P_sG)^\infty)) \\
\downarrow & & \downarrow & & \downarrow g \\
\operatorname{Map}_G(Y,\mathbb{K}(\mathcal{A}_G(P_sG)_0)) & \longrightarrow & \operatorname{Map}_G(Y,\mathbb{K}(\mathcal{A}_G(P_sG))) & \longrightarrow & \operatorname{Map}_G(Y,\mathbb{K}(\mathcal{A}_G(P_sG)^\infty))
\end{array}$$

which is the analog of the diagram in the proof of Theorem 3.1.5. Taking homotopy groups and colimits over $s > 0$ we can argue as in Theorem 3.1.5 to see that the map

$$\operatorname*{colim}_s K_n(\mathcal{A}_G^G(P_sG)^\infty) \to \operatorname*{colim}_s K_{n-1}(\mathcal{A}_G^G(P_sG)_0)$$

is split injective. Let $P_\infty G$ denote the full simplicial complex with vertices $G$. Since $\operatorname{colim}_s \mathcal{A}_G^G(P_sG)$ is equivalent to $\mathcal{A}_{G-c}^G(P_\infty G)$ by definition, the above map is equivalent to the map

$$K_n(\mathcal{A}_{G-c}^G(P_\infty G)^\infty) \to K_{n-1}(\mathcal{A}_{G-c}^G(P_\infty G)_0).$$

The $G$-space $P_\infty G$ is a model for the classifying space $\underline{E}G$ and thus this map is equivalent to the assembly map by Proposition 1.7.8. □

## 3.2 The main theorem

For any subgroup $H \leq G$ and any finite proper left invariant metric $d$ on $G$ the function

$$d_{H\backslash G}(Hg, Hg') := \inf_{h\in H} d(hg, g')$$

defines a finite proper metric on the quotient $H\backslash G$.

**Proposition 3.2.1.** *Let $G$ be a group such that the metric family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC, then the equivariant metric family $\{(G,H)\}_{H\in\mathcal{F}in}$ has FDC as well.*

*Proof.* Let $\{(X_i, G_i)\}_{i\in I}$ be an equivariant metric family with $G_i \in \mathcal{F}$ and $X_i \subseteq \coprod_{A_i} G$ be a $G_i$-invariant subspace, where $A_i$ is a $G_i$-set. We prove by induction on the decomposition complexity that $\{(X_i, G_i)\}_{i\in I} \in e\mathfrak{D}_{\gamma+1}$ if $\{G_i\backslash X_i\}_{i\in I} \in \mathfrak{D}_\gamma$. For the induction beginning let $\{G_i\backslash X_i\}_{i\in I}$ be in $\mathfrak{D}_0 = \mathfrak{B}$. Then there exist $R > 0$ and $Y_i \subseteq G$ with $\operatorname{diam} Y_i < R$ for all $i \in I$ such that $X_i = G_i Y_i$. This is equivalent to the existence of $G'_i \leq G_i$ and

$$X_i \cong \coprod_{[g]\in G_i/G'_i} gG'_iY_i$$

with $G'_iY_i \subseteq G$.

Let $r > 0$ be given and define $\mathcal{S}_r := \{H \in \mathcal{F}in \mid H = \langle S\rangle, S \subseteq B_{2R+r}(e)\}$ and $k := \max_{H\in\mathcal{S}} |H|$. Let $g_i \in Y_i$ be a fixed base point. Let $H_i \leq G'_i$ be the subgroup generated by those $g \in G'_i$ with $d(Y_i, gY_i) < r$. For these $g$ we have $d(e, g_i^{-1}gg_i) < 2R+r$. Therefore, $g_i^{-1}H_ig_i \in \mathcal{S}$ and $|H_i| \leq k$. We have the decomposition

$$X_i = \bigcup_{[g]\in G_i/H_i} gH_iY_i.$$

This decomposition is $r$-disjoint, since $d(ghy, g'h', y') < r$ with $g, g' \in G_i, h, h' \in H_i$ and $y, y' \in Y_i$ implies that $d(Y_i, h^{-1}g^{-1}g'h'Y_i) < r$ and so by definition $h^{-1}g^{-1}g'h' \in H_i$ which is equivalent to $gH_i = g'H_i$. By definition of $H_i$ we get $\operatorname{diam} gH_iY_i < k(2R+r)$. Thus, $\{(X_i, G_i)\}_{i\in I}$ is $r$-decomposable over $e\mathfrak{D}_0 = e\mathfrak{B}$ for every $r > 0$ and lies in $e\mathfrak{D}_1$.

If $\{G_i\backslash X_i\}_{i\in I}$ lies in $\mathfrak{D}_{\gamma+1}$, then it decomposes over $\mathfrak{D}_\gamma$ and the preimages under the projection $X_i \to G_i\backslash X_i$ give a decomposition of $\{(X_i, G_i)\}$ over $e\mathfrak{D}_{\gamma+1}$ by the induction hypothesis. The induction step for limit ordinals is trivial. □

We can now prove our main theorem:

**Theorem 3.2.2.** *Let $G$ be a discrete group such that $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC and let $\mathcal{A}$ be a small additive $G$-category. Assume that there is a finite dimensional $G$-CW-model for the classifying space for proper $G$-actions $\underline{E}G$.*

*Then the assembly map in algebraic $K$-theory $H_*^G(\underline{E}G; \mathbb{K}_\mathcal{A}) \to K_*(\mathcal{A}[G])$ is a split injection.*

*Proof.* By the Descent Principle 3.1.6 it suffices to show that for every integer $n$ and every $G$-set $J$ with finite stabilizers the following holds

$$\operatorname*{colim}_s K_n\Big(\prod_{j\in J}^{bd} \mathcal{A}_G(P_sG)\Big)^G = 0.$$

Since $\big(\prod_{j\in J}^{bd} \mathcal{A}_G(P_sG)\big)^G$ is equivalent to $\prod_{Gj\in G\backslash J}^{bd} \mathcal{A}_G^{G_j}(P_sG)$, where $G_j$ is the stabilizer of $j \in J$, this is equivalent to showing that for every family of finite subgroups $\{G_i\}_{i\in I}$

over some index set $I$ the following holds

$$\operatorname*{colim}_s K_n\left(\prod_{i\in I}^{bd}\mathcal{A}_G^{G_i}(P_sG)\right)=0.$$

Since $\{(G,H)\}_{H\in\mathcal{F}in}$ has FDC by Proposition 3.2.1 and the category $\mathcal{A}_G^{G_i}(P_sG)$ is equivalent to $\mathcal{A}_{G_i}^{G_i}(P_sG)$ this follows from Theorem 2.3.1. □

## 3.3 L-theory

As already mentioned in the case $L$-theory only minor changes are required and we obtain the following $L$-theoretic version of Theorem 3.2.2.

**Theorem 3.3.1.** *Let $G$ be a discrete group such that $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC. Let $\mathcal{A}$ be a small additive $G$-category with involution. Assume that there is a finite dimensional $G$-CW-model for the classifying space for proper $G$-actions $\underline{E}G$. Assume further that for every finite subgroup $H\leq G$ there is an $i_0\in\mathbb{N}$ such that for $i\geq i_0$, $K_{-i}(\mathcal{A}[H])=0$, where $\mathcal{A}$ is considered only as an additive category.*

*Then the assembly map in $L$-theory $H_*^G(\underline{E}G;\mathbb{L}_\mathcal{A})\to L_*(\mathcal{A}[G])$ is a split injection.*

*Proof.* For the proof of Theorem 3.2.2 we have only used the properties of $K$-theory stated in Theorem 1.6.9, which hold for $L$-theory as well, except that we needed that $K$-theory commutes with products for the proof of Lemma 2.3.4 and the proof of the Descent Principle 3.1.6. We need that $L$-theory also commutes with products in these cases. For the proof of Lemma 2.3.4 this is true without further assumptions because we only need to commute with a product of categories with trivial $K$-theory. In the proof of Descent Principle 3.1.6 the additional assumption about the vanishing of $K_{-i}(\mathcal{A}[H])$ for large $i$ is needed because only then the $L$-theoretic analogue of [Ros04, Theorem 6.2] holds, see Theorem 1.6.10. □

# 4 Examples

Finite decomposition complexity is a coarse geometry property fulfilling strong inheritance properties, see Section 1.5. This allows to show that many classes of groups have FDC, for example all elementary amenable and all (countable) linear groups, see [GTY13, GTY12]. For our main theorem we need the stronger hypothesis that the family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC. This property does not have as good inheritance properties as FDC itself, but in this chapter we will prove the following theorem.

**Theorem 4.1.13.** *Let $R$ be a commutative ring with unit and let $G$ be a finitely generated subgroup of $GL_n(R)$, then $\{H\backslash G\}_{H\leq G,|H|\leq n}$ has FDC for every $n\in\mathbb{N}$.*

This directly implies the following corollary of Theorem 3.2.2.

**Corollary 4.0.1.** *Let $R$ be a commutative ring with unit and let $G$ be a finitely generated subgroup of $GL_n(R)$ with an upper bound on the size of its finite subgroups and a finite dimensional model for $\underline{E}G$. Then the assembly map*

$$H_*^G(\underline{E}G;\mathbb{K}_{\mathcal{A}})\to K_*(\mathcal{A}[G])$$

*is split injective for every additive $G$-category $\mathcal{A}$.*

By Selberg's Lemma [Sel60] the group $GL_n(F)$ is virtually torsion-free for every field $F$ of characteristic zero. In particular, every finitely generated linear group over a field of characteristic zero has a global upper bound on the size of its finite subgroups. By a result of Alperin-Shalen [AS81] a finitely generated subgroup of a linear group over a field of characteristic zero has a finite dimensional model for $\underline{E}G$ if and only if there is a global upper bound on the rank of its abelian subgroups. Therefore, Theorem 4.1.13 implies the following corollary of Theorem 3.2.2.

**Corollary 4.0.2.** *Let $F$ be a field of characteristic zero, $G$ a finitely generated subgroup of $GL_n(F)$ with a global upper bound on the rank of its abelian subgroups. Then the assembly map*

$$H_*^G(\underline{E}G;\mathbb{K}_{\mathcal{A}})\to K_*(\mathcal{A}[G])$$

*is split injective for every additive $G$-category $\mathcal{A}$.*

For a finitely generated linear group $G$ over a field of positive characteristic $\underline{E}G$ admits a finite dimensional model by [DP, Corollary 5]. So Theorem 4.1.13 and Theorem 3.2.2 imply the following.

**Corollary 4.0.3.** *Let $F$ be a field of positive characteristic, $G$ a finitely generated subgroup of $GL_n(F)$. Suppose $G$ has an upper bound on the order of its finite subgroups. Then the assembly map*

$$H_*^G(\underline{E}G; \mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$$

*is split injective for every additive $G$-category $\mathcal{A}$.*

## 4.1 Groups with finite quotient FDC

To show that a group $G$ satisfies the property that $\{H\backslash G\}_{H\in\mathcal{F}}$ has FDC of our main theorem we start with groups having finite asymptotic dimension and then use inheritance properties. If a group $G$ has finite asymptotic dimension we only know that the family $\{H\backslash G\}_{H\leq G,|H|\leq n}$ has FDC for every $n \in \mathbb{N}$, but not if the family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC. This property does not have good inheritance properties. Therefore, we make the two following technical definitions.

**Definition 4.1.1.** A group $G$ has finite quotient FDC (fqFDC) if for every $n \in \mathbb{N}$ the family

$$\{H\backslash G\}_{H\leq G,|H|\leq n}$$

has FDC.

A group $G$ has strong finite quotient FDC (sfqFDC) if for every $n \in \mathbb{N}$ and every extension $\Gamma$ of $G$ (i.e. $G$ is a normal subgroup of $\Gamma$) the family

$$\{H\backslash HG\}_{H\leq\Gamma,|H|\leq n}$$

has FDC.

**Proposition 4.1.2** ([BR07b, Lemma 2.2])**.** *For a metric $G$-space $X$ with finite asymptotic dimension and every $n \in \mathbb{N}$ the family $\{H\backslash X\}_{H\leq G,|H|\leq n}$ has finite asymptotic dimension uniformly. In particular it has FDC by Remark 1.4.8.*

**Corollary 4.1.3.** *If a group $G$ has finite asymptotic dimension, then it has fqFDC.*

**Lemma 4.1.4.** *If $G$ has fqFDC, then any subgroup $H \leq G$ has fqFDC.*

*Proof.* Let $n \in \mathbb{N}$ be given. Considering the coarse embedding

$$\{H'\backslash H\}_{H'\leq H,|H'|\leq n} \to \{H'\backslash G\}_{H'\leq G,|H'|\leq n}$$

embedding $H'\backslash H$ into $H'\backslash G$. The lemma follows from coarse invariance (Lemma 1.5.6). □

For any subgroup $H \leq G$ and any finite proper left invariant metric $d$ on $G$ we defined a finite proper metric on $H\backslash G$ by

$$d_{H\backslash G}(Hg, Hg') := \inf_{h\in H} d(hg, g').$$

If we have a finite proper left invariant metric on $G$ and a normal subgroup $K \trianglelefteq G$, we will always consider the metric

$$d_{G/K}(gK, g'K) := \inf_{k \in K} d(kg, g')$$

on $G/K = K\backslash G$. This metric is again left invariant. Recall that a subgroup $K$ of $G$ is called characteristic if for every automorphism $\varphi$ of $G$ we have $\varphi(K) = K$. The first inheritance property of sfqFDC is the following.

**Lemma 4.1.5.** *sfqFDC is closed under extensions $K \to G \to Q$ where $K$ is a characteristic subgroup of $G$, i.e. if $K, Q$ have sfqFDC, then so does $G$.*

*Proof.* Let $K \trianglelefteq G$ be characteristic, $Q := G/K$ and $G \trianglelefteq \Gamma$. Assume $K$ and $Q$ have sfqFDC.

Since $K$ is a characteristic subgroup of $G$, the group $K$ is normal in $\Gamma$ and $Q$ is normal in $\Gamma/K$. We get a uniformly expansive map

$$\{H\backslash HG\}_{H \leq \Gamma, |H| \leq k} \longrightarrow \{H\backslash HQ\}_{H \leq \Gamma/K, |H| \leq k}$$

and $\{H\backslash HQ\}_{H \leq \Gamma/K, |H| \leq k}$ has FDC because $Q$ has sfqFDC. So by Theorem 1.5.7 it suffices to show that for all $r > 0$ the family $\{H\backslash HqB_r(e)K\}_{H \leq \Gamma, |H| \leq k, q \in G}$ has FDC, where $B_r(e)K = \{gk \mid g \in B_r(e), k \in K\}$. We have a coarse embedding

$$\{H\backslash HqB_r(e)K\}_{H \leq \Gamma, |H| \leq k, q \in G} \to \left\{ \bigcup_{g \in B_r(e)} H\backslash HqgK \right\}_{H \leq \Gamma, |H| \leq k, q \in G}.$$

Since $H\backslash HqgK$ is isometric to $H^{qg}\backslash H^{qg}K$, where $H^{qg} = (qg)^{-1}Hqg$, the lemma follows by Theorem 1.5.9 and the assumption that $K$ has sfqFDC. □

**Lemma 4.1.6.** *Let $G$ be the direct union of groups $G_\alpha$ having finite asymptotic dimension. Then $G$ has sfqFDC.*

*Proof.* Let $k \in \mathbb{N}$ and an extension $G \trianglelefteq \Gamma$ be given. For any $r > 0$ and $H \leq \Gamma$ with $|H| \leq k$ define

$$Z_H := \langle B_r(e) \cap HG \rangle.$$

Since

$$HG = \bigcup_{gZ_H \in HG/Z_H}^{r-disj} gZ_H,$$

in particular

$$HG = \bigcup_{HgZ_H \in H\backslash HG/Z_H}^{r-disj} HgZ_H.$$

Therefore, $\{H\backslash HG\}_{|H| \leq k}$ $r$-decomposes over $\{H\backslash HgZ_H\}_{g \in G, |H| \leq k}$ and it suffices to show that $\{H\backslash HgZ_H\}_{g \in G, |H| \leq k}$ has FDC for any $r > 0$.

Let $\alpha$ be such that $B_{2(k+1)r}(e) \cap G \subseteq G_\alpha$.

**Claim:** $Z_H \cap G \leq G_\alpha$ for all $|H| \leq k$.

Using this we conclude that $\{Z_H \cap G\}_{|H|\leq k}$ has finite asymptotic dimension uniformly. Furthermore, $Z_H/Z_H \cap G \leq HG/G \cong H/H \cap G$ has less or equal to $k$ elements. For every $H$ choose $h_i^H \in Z_H$ with $Z_H = \bigcup_{i=1}^k h_i^H(Z_H \cap G)$. Since $h_i^H(Z_H \cap G)$ is isometric to $Z_H \cap G$ for every $1 \leq i \leq k$ the family $\{h_i^H(Z_H \cap G)\}_{|H|\leq k}$ has finite asymptotic dimension uniformly and, therefore, also

$$\{Z_H\}_{|H|\leq k} = \left\{\bigcup_{i=1}^k h_i^H(Z_H \cap G)\right\}_{|H|\leq k}$$

has finite asymptotic dimension uniformly.

Numerating the elements of each $H$ with $|H| \leq k$ we conclude in the same way that

$$\{HgZ_H\}_{g\in G,|H|\leq k} = \left\{\bigcup_{i=1}^k h_i^H gZ_H\right\}_{g\in G,|H|\leq k}$$

has finite asymptotic dimension uniformly.

If $\{HgZ_H\}_{g\in G,|H|\leq k}$ has finite asymptotic dimension, then $\{H\backslash HgZ_H\}_{g\in G,|H|\leq k}$ has finite asymptotic dimension as well by Proposition 4.1.2.

Therefore, $\{H\backslash HgZ_H\}_{g\in G,|H|\leq k}$ has FDC.

It remains to prove the above claim:

Let $z \in Z_H$ and let $m$ be minimal with

$$z = h_1 g_1 \ldots h_m g_m g$$

for some $h_j \in H, g_j \in G, g \in G_\alpha$ such that $h_j g_j \in B_r(e)$ for all $j$.

Assume $m > k \geq |H|$, then there exist $m - |H| \leq n_1 < n_2 \leq m$ with

$$h_{n_1} \ldots h_m = h_{n_2} \ldots h_m.$$

Therefore,

$$(h_{n_1} g_{n_1} \ldots h_m g_m)^{-1} h_{n_2} g_{n_2} \ldots h_m g_m \in G \cap B_{2(k+1)r}(e) \subseteq G_i$$

and there exists $g' \in G_i$ with

$$h_{n_1} g_{n_1} \cdot \ldots \cdot h_m g_m = h_{n_2} g_{n_2} \cdot \ldots \cdot h_m g_m g'.$$

Thus $m$ is not minimal, a contradiction.

Let $z \in Z_H \cap G$ be represented as

$$z = h_1 g_1 \dots h_m g_m g$$

for some $h_j \in H, g_j \in G, g \in G_\alpha$ with $h_j g_j \in B_r(e)$ and $m \leq k$.

Then $h_1 g_1 \dots h_m g_m \in G \cap B_{kr}(e) \subseteq G_\alpha$ and therefore also $z = h_1 g_1 \dots h_m g_m g \in G_\alpha$. This proves the claim. □

By the classification of finitely generated abelian groups we immediately get the following:

**Corollary 4.1.7.** *Abelian groups have sfqFDC.* □

Combining Lemma 4.1.5 and Corollary 4.1.7 yields:

**Corollary 4.1.8.** *Solvable groups have sfqFDC.* □

To show that finitely generated linear groups have fqFDC we need the following extension property.

**Proposition 4.1.9.** *Let $K \to G \to Q$ be an extension and let $K$ have sfqFDC and $Q$ fqFDC. Then $G$ has fqFDC.*

*Proof.* Let $n \in \mathbb{N}$ be given. The map

$$\{H \backslash G\}_{H \leq G, |H| \leq n} \to \{(H \cap K) \backslash Q\}_{H \leq G, |H| \leq n}$$

is uniformly expansive and the family $\{(H \cap K) \backslash Q\}_{H \leq G, |H| \leq n}$ has FDC by assumption. By Theorem 1.5.7 it suffices to show that for all $r > 0$ the family

$$\{H \backslash H g B_r(e) K\}_{g \in G, |H| \leq n} = \left\{ \bigcup_{\gamma \in B_r(e)} H \backslash H g \gamma K \right\}_{g \in G, |H| \leq n}$$

has FDC. This follows from Theorem 1.5.9, the fact that $H \backslash H g \gamma K$ is isometric to $H^{g\gamma} \backslash H^{g\gamma} K$ and the assumption that $K$ has sfqFDC. □

Now we prove the desired statement for linear groups over fields and then use our previous results to generalize this to finitely generated linear groups over general commutative rings.

**Theorem 4.1.10.** *Let $G$ be a finitely generated subgroup of $GL_n(F)$ where $F$ is a field. Then $G$ has fqFDC.*

We only have to prove the following fqFDC version of [GTY12, Lemma 3.9]. It has the same assumptions as the original lemma and it is proved in [GTY12] that these are fulfilled for finitely generated linear groups over fields.

**Lemma 4.1.11.** *Let $G$ be a countable discrete group. Suppose there exists a left invariant, finite pseudo metric $d'$ on $G$ with the following properties:*

*(a) G has finite asymptotic dimension with respect to $d'$.*

*(b) For all $r > 0$ there exists a left invariant, finite pseudo metric $d_r$ on $G$ such that*

*(i) $G$ has finite asymptotic dimension with respect to $d_r$,*

*(ii) $d_r$ is proper when restricted to $B_{r,d'}(e)$, the ball of radius $r$ around $e$ with respect to the metric $d'$.*

*Then $G$ has fqFDC.*

Condition (ii) in (b) in the lemma means precisely that $B_{s,d_r}(e) \cap B_{r,d'}(e)$ is finite for every $s > 0$.

*Proof.* Fix a proper, finite, left invariant metric $d$ on G. By Theorem 1.5.7, applied to the identity map

$$\{H\backslash(G,d)\}_{H \leq G, |H| \leq n} \to \{H\backslash(G,d')\}_{H \leq G, |H| \leq n},$$

it suffices to show that for every $r > 0$ the family $\{H\backslash HB_{r,d'}(g)\}_{g \in G, |H| \leq n}$ has FDC. By Corollary 4.1.3 this is implied if we show that the family $\{B_{r,d'}(g)\}_{g \in G} = \{gB_{r,d'}(e)\}_{g \in G}$ has finite asymptotic dimension uniformly when equipped with the metric $d$. Since all spaces $gB_{r,d'}(e)$ are isometric to $B_{r,d'}(e)$ we only have to show that $B_{r,d'}(e)$ has finite asymptotic dimension.

Let $r > 0$. Pick $d_{2r}$ as in the assumptions. The ball $B_{r,d'}(e) \subseteq G$ has finite asymptotic dimension with respect to the metric $d_{2r}$.

Thus, it remains to show that the metrics $d$ and $d_{2r}$ on $B_{r,d'}(e)$ are coarsely equivalent.

Since balls in $G$ with respect to $d$ are finite, we easily see that for every $s$ there exists $s'$ such that if $d(g,h) \leq s$, then $d_{2r}(g,h) \leq s'$; this holds for every $g$ and $h$ in $G$. Conversely, for every $s$ the set $B_{2r,d'}(e) \cap B_{s,d_{2r}}(e)$ is finite by assumption, and we obtain $s'$ such that for every $g$ in this set $d(g,e) \leq s'$. If $g,h \in B_{r,d'}(e)$ are such that $d_{2r}(g,h) \leq s$, then $g^{-1}h \in B_{s,d_{2r}}(e)$ and $d(g,h) \leq s'$. □

To generalize this to arbitrary commutative rings we need Lemma 5.2.3 from [GTY13]:

**Lemma 4.1.12.** *Let $R$ be a finitely generated commutative ring with unit and let $\mathfrak{n}$ be the nilradical of $R$,*

$$\mathfrak{n} = \{r \in R \mid \exists n : r^n = 0\}.$$

*The quotient ring $S = R/\mathfrak{n}$ contains a finite number of prime ideals $\mathfrak{p}_1, .., \mathfrak{p}_n$ such that the diagonal map*

$$S \to S/\mathfrak{p}_1 \oplus \ldots \oplus S/\mathfrak{p}_n$$

*embeds $S$ into a finite direct sum of domains.*

The next theorem is the fqFDC version of [GTY13, Theorem 5.2.2]. We need to assume that $G$ is finitely generated because we do not know if fqFDC is closed under unions.

**Theorem 4.1.13.** *Let $R$ be a commutative ring with unit and let $G$ be a finitely generated subgroup of $GL_n(R)$, then $G$ has fqFDC.*

*Proof.* Because $G$ is finitely generated we can assume that $R$ is finitely generated as well. With $\mathfrak{n}$ and $S$ as in the previous lemma, we have an extension

$$1 \to I + M_n(\mathfrak{n}) \to GL_n(R) \to GL_n(S) \to 1 \qquad (4.1.14)$$

in which $I + M_n(\mathfrak{n})$ is nilpotent and therefore has sfqFDC by Corollary 4.1.8. In the notation of the previous lemma we have embeddings

$$GL_n(S) \hookrightarrow GL_n(S/\mathfrak{p}_1) \times \ldots \times GL_n(S/\mathfrak{p}_n) \hookrightarrow GL_n(Q(S/\mathfrak{p}_1)) \times \ldots \times GL_n(Q(S/\mathfrak{p}_n))$$

where $Q(S/\mathfrak{p}_i)$ is the quotient field of $S/\mathfrak{p}_i$.

Thus the quotient in the extension (4.1.14) has fqFDC by Theorem 4.1.10. Now, the theorem follows from Proposition 4.1.9. □

## 4.2 Finite wreath products

A group $G$ is said to satisfy the Farrell-Jones Conjecture with finite wreath products if for every finite group $F$ the wreath product $G \wr F$ satisfies the Farrell-Jones Conjecture. The reason to consider this version is that the class $FJCw$ of groups satisfying the Farrell-Jones Conjecture with finite wreath products is closed under finite extensions, i.e. for every extension $H \to G \to F$ such that $F$ is finite and $H \in FJCw$ also $G \in FJCw$.

In this section we want to show that Corollary 4.0.1 also holds for finite wreath products, i.e. we want to prove the following.

**Corollary 4.2.1.** *Let $R$ be a commutative ring with unit and let $G$ be a finitely generated subgroup of $GL_n(R)$ with an upper bound on the size of its finite subgroups and a finite dimensional model for $\underline{E}G$. Let $F$ be a finite group. Then the assembly map*

$$H_*^{G \wr F}(\underline{E}(G \wr F); \mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G \wr F])$$

*is a split injection for every additive $G \wr F$-category $\mathcal{A}$.*

**Definition 4.2.2.** Let $H \leq G$ be a subgroup of finite index, $R$ a ring. A homomorphism $\pi \colon H \to GL_n(R)$ is the same as viewing $R^n$ as an $R[H]$-module. The module $R[G] \otimes_{R[H]} R^n$ is a free $R$-module of rank $[G : H]n$ and for every choice of representatives $x_i$ for $G/H$ the set $\{(x_i, e_j)\}$ is a basis for $R[G] \otimes_{R[H]} R^n$. This induces a map $\pi' \colon G \to GL_{[G:H]n}(R)$. If $\pi$ is injective also $\pi'$ is injective. The module $R[G] \otimes_{R[H]} R^n$ is called the induced representation.

Corollary 4.2.1 follows from the following three observations.

**Lemma 4.2.3.** *If $G$ is a finitely generated subgroup of $GL_n(R)$, $F$ a finite group, then $G \wr F$ is a finitely generated subgroup of $GL_{|F|^2 n}(R)$.*

*Proof.* $G \leq G \wr F$ is a subgroup of index $|F|^2$. The induced representation of $G \wr F$ gives an embedding $G \wr F \to GL_{|F|^2 n}(R)$. □

**Lemma 4.2.4.** *If $G$ has an upper bound on the size of its finite subgroups and $F$ is a finite group, then $G \wr F$ has an upper bound on the size of its finite subgroups.*

*Proof.* Assume every finite subgroup of $G$ has order as most k, then every finite subgroup of $\bigoplus_F G$ has order at most $k^{|F|}$ and considering the extension $\bigoplus_F G \to G \wr F \to F$ we see that every finite subgroup of $G \wr F$ has order at most $|F|k^{|F|}$. □

**Lemma 4.2.5.** *If $G$ has a finite dimensional model for $\underline{E}G$ and $F$ is a finite group, then there is a finite dimensional model for $\underline{E}(G \wr F)$.*

*Proof.* Let $X$ be a finite dimensional model for $\underline{E}G$. Let $\bigoplus_F G$ act on $\prod_F X$ via $(g_f)_{f\in F}.(x_f)_{f\in F} = (g_f x_f)_{f\in F}$. Let $H \leq \bigoplus_F G$ be a subgroup. Then $(x_f)_{f\in F} \in (\prod_F X)^H$ if and only if $x_f \in X^{\mathrm{pr}_f(H)}$, where $\mathrm{pr}_f$ is the projection onto the $f$-factor in $\bigoplus_F G$. This implies that the stabilizers of $\prod_F X$ are finite and the fixed point sets for any finite group are contractible, i.e. $\prod_F X$ is a finite dimensional model for $\underline{E} \bigoplus_F G$. The group $G \wr F$ acts on $\prod_F X$ via $(g, f')(x_f)_{f\in F} = (g_f x_{(f')^{-1}f})_{f\in F}$. A subgroup $H \leq G \wr F$ is finite if and only if $H \cap \bigoplus_F G$ is finite. Therefore, all stabilizers of $\prod_F X$ with respect to the $G \wr F$ action are finite. The stabilizers for a finite subgroup $H$ are given by

$$\left(\prod_F X\right)^H = \left\{ (x_f) \in \left(\prod_F X\right)^{H\cap\bigoplus_F G} \;\middle|\; \forall (g,f) \in G \wr F : x_f = g_f x_{(f')^{-1}f} \right\}.$$

Let $p : G \wr F \to F$ be the projection. We have the following homeomorphism

$$\left(\prod_F X\right)^H \cong \prod_{f\in F/p(H)} X^{pr_f(H\cap\oplus_F G)}$$

and $\left(\prod_F X\right)^H$ is contractible. Thus, $\prod_F X$ is a model for $\underline{E}(G \wr F)$. □

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