Talk 4: Obstruction Theory

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1 Extension Problem

Let (W, A) be a CW pair and X a CW complex. Obstruction theory studies the following extension problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ W & & \end{array} \tag{1}$$

So given a map $A \to X$ defined on a subcomplex A of W, we are trying to extend the map to W. The general idea is to split this big extension problem into many smaller extension problems by approximating X with a Postnikov tower and to study if we are able to construct a solution inductively by going up the Postnikov tower. In the previous talk we learned that a connected CW complex X has a Postnikov tower with principal fibrations iff $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n \in \mathbb{N}$. For the rest of this text we will assume that X is connected and $\pi_1(X)$ acts trivially to ensure that X has a Postnikov tower with principal fibrations.

2 Obstruction Theory

First we will justify our approach by showing that our original extension problem is equivalent to the extension problems induced by the Postnikov tower.

Lemma 1. *The extension problem (1) has a solution iff there exists a cone $(W \to X_n)_{n\geq 1}$ that extends the cone $(A \to X_n)_{n\geq 1}$ level-wise.*

Proof. $X \to \lim X_n$ is a weak equivalence. First apply Proposition 12.5 from Topology 2 to find a solution up to homotopy. Then turn it into a solution of (1) by applying the homotopy extension property. □

2.1 Initial Case

Now we can study in which cases it is possible to construct solutions inductively to the extension problems induced by the Postnikov tower. First we have to make sure that the initial case has a solution. For this we extend the Postnikov tower by $X_0 = *$. The map $X_1 \to X_0$ is a principal fibration iff $\pi_1(X)$ is abelian. So the initial case is covered by assuming that $\pi_1(X)$ is abelian.

2.2 Inductive Step

For the inductive step we assume that we already have an extension $W \to X_{n-1}$ of $A \to X_{n-1}$. To extend $W \to X_{n-1}$ to X_n means to find a lift in

$$\begin{array}{ccc} A & \longrightarrow & X_n \\ \big\downarrow & \nearrow & \big\downarrow \\ W & \longrightarrow & X_{n-1} \end{array} \tag{2}$$

Since $X_n \to X_{n-1}$ is a principal fibration, we can replace it up to weak equivalences by a homotopy fiber $F \to E$:

$$\begin{array}{ccccccc} A & \longrightarrow & X_n & \xrightarrow{\sim} & F & \longrightarrow & PK_{n-1} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ W & \longrightarrow & X_{n-1} & \xrightarrow{\sim} & E & \longrightarrow & K_{n-1} \end{array}$$

where the right square is a pullback diagram, PK_{n-1} is the space of paths in K_{n-1} that start in the base point of K_{n-1}, and $K_{n-1} = K(\pi_n(X), n+1)$ is an Eilenberg-Mac Lane space.

Lemma 2. *The lifting problem (2) has a solution iff*

$$\begin{array}{ccccc} A & \longrightarrow & F & \longrightarrow & PK_{n-1} \\ \big\downarrow & \nearrow & \big\downarrow & & \big\downarrow \\ W & \longrightarrow & E & \longrightarrow & K_{n-1} \end{array} \tag{3}$$

has a solution.

Proof. Again use Proposition 12.5 to get a solution of (2) up to homotopies. Then turn it into a solution by applying the homotopy extension property and the relative homotopy lifting property of a fibration. □

Now we have maps into an Eilenberg-Mac Lane space and we want to use this to construct a cohomology class that characterizes if (2) has a solution. Observe that for any space S a map $S \to F$ is equivalent to a map $S \to E$ with a nullhomotopy of the map $S \to E \to K_{n-1}$ which is equivalent to a map $S \to E$ and a compatible map $CS \to K_{n-1}$. Therefore (3) is equivalent to extending the induced map $W \cup CA \to K_{n-1}$ to a map $CW \to K_{n-1}$. We were successful in reformulating the problem in terms of maps into the Eilenberg-Mac Lane space and can define the obstruction class.

Definition 3. *The obstruction class of the inductive step (2) is the cohomology class*

$$\omega_n \in H^{n+1}(W \cup CA; \pi_n(X)) \cong H^{n+1}(W, A; \pi_n(X))$$

induced by the map $W \cup CA \to K_{n-1}$.

Therefore we can prove the following proposition.

Proposition 4. *The inductive lifting problem (2) has a solution iff* $\omega_n = 0$.

Proof. If (2) has a solution, the map $W \cup CA \to K_{n-1}$ is nullhomotopic therefore the induced obstruction class is zero. If $\omega_n = 0$ then we can pick a nullhomotopy of $W \cup CA \to K_{n-1}$. Use the homotopy extension property of the CW pair $(CW, W \cup CA)$ to get an extension $CW \to K_{n-1}$ of $W \cup CA \to K_{n-1}$. □

Remark 5. *The obstruction class characterizes the inductive step for a given extension* $W \to X_{n-1}$ *in the sense that it can be extended further to* X_n *iff the obstruction class vanishes. The obstruction classes are unique in the sense that they don't depend on the Postnikov tower nor the replacement* $F \to E$ *of* $X_n \to X_{n-1}$. *But the obstruction class is only defined for a given extension* $W \to X_{n-1}$ *and depends on it. If a given extension* $W \to X_{n-1}$ *has a nonzero obstruction class then it can't be extended to* X_n *but there could exists a different extension* $W \to X_{n-1}$ *that can be extended further.*

The following corollary summarizes the discuss above.

Corollary 6. *If* X *is a connected CW complex with abelian* $\pi_1(X)$ *that acts trivially on all* $\pi_n(X)$ *and* (W, A) *is a CW pair such that*

$$H^{n+1}(W, A, \pi_n(X)) = 0$$

for all $n \in \mathbb{N}$, *then every map* $A \to X$ *can be extended to a map* $W \to X$.

3 Application

We can apply obstruction theory to prove a stronger version of the Whitehead Theorem.

Proposition 7. *If* X, Y *are connected CW complexes with abelian* π_1*'s that act trivially on all* π_n*'s, then a map* $X \to Y$ *that induces isomorphisms on all homology groups is a homotopy equivalence.*

Proof. Use the mapping cylinder to reduce to the case of an inclusion of a subcomplex. Since $H^{n+1}(Y, X, \pi_n(X)) = 0$, by obstruction theory there exists an extension of the identity $X \to X$ to $Y \to X$. Then $\pi_1(X)$ acts trivially on $\pi_n(Y, X)$ and Hurewicz Theorem and Whitehead Theorem imply that $X \to Y$ is a homotopy equivalence. □

4 Lifting Problem

In the same way we used Postnikov towers to study the extension problem we can use Moore-Postnikov towers to study the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ W & \longrightarrow & Y \end{array} \tag{4}$$

where (W, A) is a CW pair and p is a fibration. We know that p has a Moore-Postnikov tower with principal fibrations if X and Y are connected CW complexes and $\pi_1(X)$ acts trivially on all $\pi_n(M_p, X)$ where M_p is the mapping cylinder of p. In this case we have the following diagram

$$\begin{array}{ccccccc} & & & & Z_2 & \longrightarrow & K(\pi_2(F), 3) \\ & & & \nearrow & \big\downarrow & & \\ A & \longrightarrow & X & \longrightarrow & Z_1 & \longrightarrow & K(\pi_1(F), 2) \\ \big\downarrow & & & \searrow^{p} & \big\downarrow & & \\ W & & \longrightarrow & & Y & & \end{array} \tag{5}$$

As in for the extension problem we can try to construct a solution of (4) inductively.

The initial case is asking if we are able to lift $W \to Y$ to Z_1. By taking Z_1 to be the covering space of Y corresponding to the subgroup $p_*(\pi_1(X))$ of $\pi_1(Y)$ the initial case is covered by assuming that A is connected.

The inductive step asks if we are able to lift a given lift $W \to Z_{n-1}$ to Z_n. We can again define the obstruction class

$$\omega_n \in H^n(W \cup CA; \pi_{n-1}(F)) \cong H^n(W, A; \pi_{n-1}(F))$$

to be induced by the map $W \cup CA \to K(\pi_{n-1}(F), n)$ where F is the homotopy fiber of p. By an analogous argument the inductive step is possible iff ω_n vanishes.

5 Special Case

To ask whether two maps are homotopic is a special case of the extension problem. Let $f, g : A \to X$ be two maps. Then f and g are homotopic iff the extension problem

$$\begin{array}{ccc} A \times \{0, 1\} & \xrightarrow{f \coprod g} & X \\ \big\downarrow & \nearrow & \\ A \times I & & \end{array}$$

has a solution. If we already know that f and g are homotopic after composing with a third map $p : X \to Y$ then the there exists a compatible homotopy from f to g iff the lifting problem

$$\begin{array}{ccc} A \times \{0,1\} & \xrightarrow{f \coprod g} & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ A \times I & \longrightarrow & Y \end{array}$$

has a solution.