Talk 5: Classifying spaces

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The proofs of all results mentioned can be found in chapter 14 of *Algebraic topology* by Tammo tom Dieck.

All throughout this document, G denotes a topological group. Our main goal is to show existence of a classifying space for an arbitrary G. In our specific case, a classifying space BG is the base space of some numerable G-principal bundle $EG \to BG$ that has the following property: any numerable G-principal bundle is isomorphic to a pullback of the bundle $EG \to BG$. The existence of BG as well as this latter property will be proved in the last section, while the first two sections will be devoted to G-principal bundles and a homotopy theorem, respectively.

1 Principal bundles

Definition 1. *Consider a topological group G, a total space E and a base space B, a continuous right action of G on E, denoted as $r : E \times G \to E : (x, g) \mapsto xg$, and a continuous "projection" map $p : E \to B$. The pair (p, r) is a (right) G-**principal bundle** if the following two axioms are satisfied:*

- *For all $x \in E$, $g \in G$, we have $p(xg) = p(x)$.*
- *For all $b \in B$, there exists some open neighbourhood $U \subseteq B$ of b as well as a G-homeomorphism $\varphi : p^{-1}(U) \to U \times G$ which is a trivialization of p over U with typical fibre G:*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi, \cong} & U \times G \\ & \searrow^{p} & \downarrow \mathrm{pr}_1 \\ & & U \end{array}$$

Note that we can similarly define left principal bundles. And also note that a G-principal bundle is a fibre bundle if we forget about the action/the group structure of G.

Remark 2.

- *The axioms imply that we want G to act freely on E due to G-linearity of the trivialization maps.*
- *In contrast to an arbitrary locally trivial map with typical fibre G, the local trivializations in a principal bundle have to be compatible with the group action.*
- *The map p factors through the orbit map $q : E \to E/G$ and induces a continuous bijection $h : E/G \to B$. Since q and p are open maps, hence quotient maps, h must be a homeomorphism:*

$$\begin{array}{ccc} E & \xrightarrow{q} & E/G \\ & \searrow^{p} & \downarrow h, \cong \\ & & B \end{array}$$

Definition 3. *A G-principal bundle with a discrete group G is a G-**principal covering**.*

Example 4. *As an elementary example of a principal bundle, consider the universal cover $p : \mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$, where $G = \mathbb{Z}$ acts on $\mathbb{R}$ by translation.*

Definition 5. *Let $E \times G \to E$ be some free action, and denote by $C(E)$ the set $\{(x, xg) \mid x \in E, g \in G\}$. The map $t = t_E : C(E) \to G : (x, xg) \mapsto g$ is the **translation map** of the action. The action is **weakly proper** if t is continuous.*

Similarly, we call the space E or the induced orbit map weakly proper whenever the corresponding action is.

Lemma 6. *Let $E \times G \to E$ be a free action such that $p : E \to E/G$ is locally trivial. Then the translation map is continuous.*

This lemma implies that a principal bundle is weakly proper.

Proposition 7. *Let the free G-action on E be weakly proper. Then the orbit map $p : E \to E/G = B$ is isomorphic to $\mathrm{pr}_1 : B \times G \to B$ if and only if p has a section. Here an isomorphism between the two maps means a G-homeomorphism $E \to B \times G$, compatible with the projections to B:*

$$\begin{array}{ccc} E & \xrightarrow{\ p\ } & E/G = B \\ \downarrow & & \vdots \\ B \times G & \xrightarrow{\ \mathrm{pr}_1\ } & B \end{array}$$

From this result we conclude that a principal bundle is a trivial bundle if and only if it admits a global section. The same is not true for other fiber bundles. For instance, vector bundles always have a zero section whether they are trivial or not.

Proof. One implication of the statement is trivial. Conversely, let $s : B \to E$ be a section of p. Then $B \times G \to E : (b, g) \mapsto s(b)g$ and $E \to B \times G : x \mapsto (p(x), t(spx, x))$ are inverse G-homeomorphisms[1], compatible with the projections to B. □

Proposition 8. *Let X and Y be free G-spaces and $\Phi : X \to Y$ a G-map. If $\varphi = \Phi/G$ (the induced map on the orbit spaces) is a homeomorphism and Y weakly proper, then Φ is a homeomorphism:*

$$\begin{array}{ccc} X & \xrightarrow{\ \Phi\ } & Y \\ \downarrow & & \downarrow \\ X/G & \xrightarrow{\varphi = \Phi/G} & Y/G \end{array}$$

Definition 9. *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\ F\ } & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{\ f\ } & B \end{array},$$

with G-principal bundles p and q, and a G-map F. Call F or the pair (F, f) a ***bundle map****. If f is the identity, then F is a* ***bundle isomorphism****.*

Note that, if f is a homeomorphism, then F is a homeomorphism by the last proposition.

We can also construct a principal bundle via pullback, as follows. Given a G-principal bundle $p : X \to B$ and a map $f : C \to B$, we get a pullback

$$\begin{array}{ccc} Y' & \xrightarrow{\ F'\ } & X \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{\ f\ } & B \end{array},$$

where $Y' = \{(c, x) \mid f(c) = p(x)\} \subset C \times X$. Here Y' is equipped with the subspace topology. The maps q' and F' are the restrictions to Y' of the obvious projection maps. To see that q' forms a principal bundle, note that we have a G-action on Y' via $(c, x)g = (c, xg)$, which lies in Y' because $p(x) = p(xg)$. And (as for fibre bundles) one can check that q' is trivial over $f^{-1}(V)$ whenever p is trivial over V. Suitably, we call q' the bundle **induced** from p by f. It is clear that F is a G-map, hence a bundle map. Using the universal property of a pullback, one can check that the bundle map diagram from Definition 9 is a pullback as well; meaning that any bundle map of principal bundles is a pullback.

Up until now we considered G-spaces of which we assumed that the orbit map is a principal bundle. But one can also start with a G-space and ask whether the orbit map is a principal bundle. To make this more concrete, consider the following result:

Proposition 10. *Let U be a right G-space. The following are equivalent:*

[1] Note that t is continuous by assumption of E being weakly proper.

(1) There exists a G-map $f : U \to G$.

(2) U is a free G-space, $p : U \to U/G$ has a section, and t_U is continuous.

Definition 11. *A right G-space U is* ***trivial*** *if it satisfies the first item of the above proposition. A right G-space is* ***locally trivial*** *if it has an open covering by trivial G-subspaces.*

The total space E of a G-principal bundle is thus locally trivial. Conversely, if E is locally trivial, then $E \to E/G$ is a G-principal bundle.

Example 12.

- *Consider the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ as a free S^1-space with action induced by scalar multiplication: $((z_1, \dots, z_n), \lambda) \mapsto (z_1\lambda, \dots, z_n\lambda)$. If we let U_j be the subset of points $z = (z_k)$ in S^{2n-1} with $z_j \neq 0$, then the map $U_j \to S^1 : z \mapsto z_j|z_j|^{-1}$ shows that U_j is a trivial S^1-space by the definition above. The orbit space of this action of S^1 on S^{2n-1} is what we call $\mathbb{C}P^{n-1}$. The orbit map is thus an S^1-principal bundle $p : S^{2n-1} \to \mathbb{C}P^{n-1}$, and we recall from the first talk that this is called a* ***Hopf fibration****.*
- *Similarly, recalling that the $(n-1)$-sphere S^{n-1} is a two-sheeted covering space of the real projective space $\mathbb{R}P^{n-1}$, we have a $\mathbb{Z}_2$-principal bundle $p : S^{n-1} \to \mathbb{R}P^{n-1}$.*

2 The homotopy theorem

An essential tool for stating the classification theorem is the homotopy theorem. We restrict our attention to bundles that are numerable, so let us first define this concept.

Definition 13. *A* ***partition of unity*** *over a space X is a locally finite family $T = (t_j : X \to \mathbb{R} \mid j \in J)$ of continuous functions such that the functions t_j take on only non-negative values and for each $x \in X$ we have $\sum_{j\in J} t_j(x) = 1$. A covering[2] $\mathcal{U} = (U_j \mid j \in J)$ of X is* ***numerable*** *if there exists a partition of unity T such that $\operatorname{supp}(t_j) \subset U_j$ holds for each $j \in J$. Finally, a locally trivial map $E \to B$ is* ***numerable*** *if it is trivial over the members of a numerable covering of the base space B.[3] In the next section we will call a G-space* ***numerable*** *if the induced orbit map is a numerable, locally trivial map.*

Theorem 14 (Homotopy theorem)**.** *Let $q : E \to C$ be a numerable G-principal bundle and $h : B \times I \to C$ a homotopy from B to C. Then the bundles p_0 and p_1 induced from q along h_0 and h_1 are isomorphic:*

$$\begin{array}{ccc} * & \longrightarrow & E \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h_0} & C \end{array} \qquad\qquad \begin{array}{ccc} * & \longrightarrow & E \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h_1} & C \end{array}$$

Theorem 15. *Let $q : X \to C$ be a numerable locally trivial map. Then q is a Hurewicz fibration.*

Recall that we saw in the first talk on fibrations that all fibre bundles are Serre fibrations, meaning q has the homotopy lifting property for all CW-complexes. From this theorem we thus see that, if we just add the requirement that q is numerable, the homotopy lifting property then holds for all spaces.

3 Classifying spaces

3.1 Classification theorem

Let us fix an arbitrary topological group G.

Definition 16. *For an arbitrary base space B, denote by $\mathcal{B}(B, G)$ the set of isomorphism classes of numerable G-principal bundles over B.*

Since a continuous map $f : B \to C$ induces via pullback a well-defined map $\mathcal{B}(f) = f^* : \mathcal{B}(C, G) \to \mathcal{B}(B, G)$ as in

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & C \end{array},$$

[2]not necessarily countable

[3]This is for example the case when we have a fibre bundle over a paracompact space.

we obtain a homotopy invariant[4] functor $\mathcal{B}(-, G)$ from the TOP-category to the category of isomorphism classes of numerable G-principal bundles.

Fix a numerable G-principal bundle $q : E \to C$. By the homotopy theorem, homotopic maps induce isomorphic bundles, so we obtain a well-defined map

$$\gamma_B : [B, C] \to \mathcal{B}(B, G) : [f] \mapsto [f^* q].$$

These maps γ_B constitute a natural transformation from $[-, C]$ to $\mathcal{B}(-, G)$.

We now want to construct a base space C such that γ_B is a bijection for all B. This we can only do if C is some particular space that we call the classifying space. We can construct this C by looking at the total space E. We then simply call this space E universal if its orbit space is a classifying space. So let us formalize these concepts.

Definition 17. *The total space EG of a numerable G-principal bundle is **universal** if each numerable free G-space E has up to G-homotopy a unique G-map $E \to EG$, implying EG to be a terminal object in the appropriate homotopy category. The orbit map $p_G : EG \to EG/G$ is a **universal bundle**, the orbit space $BG \coloneqq EG/G$ a **classifying space** of the group G. A map $k : B \to BG$ which induces from $EG \to BG$ a given bundle $q : E \to B$ is called a **classifying map** of the bundle q.*

Remark 18. *By the universal property of EG, one can check that EG is well-defined up to G-homotopy equivalence, and BG is well-defined up to homotopy equivalence.*

Assuming that EG exists, we now show that there is an inverse map to γ_B for any base space B. Let $q : E \to B$ be some numerable G-principal bundle. Then there exists a G-map $\Phi : E \to EG$ by definition and an induced map $\overline{\Phi} : B \to BG$ on the orbit spaces (after taking quotients)

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & EG \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p_G} \\ B & \xrightarrow{\overline{\Phi}} & BG \end{array};$$

and G-homotopic maps Φ induce homotopic maps between the base spaces. We assign to $[q]$ the class $[\overline{\Phi}] \in [B, BG]$. Isomorphic bundles yield the same homotopy class, so we obtain a well-defined map $\kappa_B : \mathcal{B}(B, G) \to [B, BG]$, and the κ_B's constitute a natural transformation between functors. One checks immediately that the compositions $\gamma_B \kappa_B$ and $\kappa_B \gamma_B$ are the identity for each base space B.

Supposing that a universal bundle exists, we have thus actually shown the following:

Theorem 19 (Classification theorem)**.** *Fix a topological group G and a space B. We assign to each isomorphism class of numerable G-principal bundles the homotopy class of a classifying map and obtain a well-defined bijection $\mathcal{B}(B, G) \cong [B, BG]$. The inverse assigns to $k : B \to BG$ the bundle induced by k from the universal bundle.*

Example 20. *Recall from Topology 2 that we have principal bundles*

$$O(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n),$$

$$U(k) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n),$$

where the Stiefel manifold $V_k(\mathbb{F}^n)$ is the set of all orthonormal k-frames in $\mathbb{F}^n$, the Grassmannian $G_k(\mathbb{F}^n)$ the set of all k-dimensional linear subspaces of $\mathbb{F}^n$, and the projection maps send a k-frame to the subspace spanned by that frame. In the next talk, we will see that when passing to the limit $n \to \infty$, these bundles become universal bundles.

3.2 Existence of EG

To construct a space EG, we use the notion of a join of a family of spaces, which allows for a functorial construction.

Definition 21. *Let $(X_j \mid j \in J)$ be a family of spaces, for some index set J. The elements of the **join** $X = \bigstar_{j \in J} X_j$ are represented by families*

$$(t_j x_j \mid j \in J), \quad t_j \in [0,1], \quad x_j \in X_j, \quad \sum_{j \in J} t_j = 1,$$

[4]This means that the functor takes the same value on morphisms that are related by a homotopy.

where only a finite number of t_j are different from zero. The families $(t_j x_j)$ and $(u_j y_j)$ are said to represent the same element of X if and only if

(1) $t_j = u_j$ for each $j \in J$,

(2) $x_j = y_j$ whenever $t_j \neq 0$.

For each $j \in J$, we have coordinate maps

$$t_j : X \to [0,1] : (t_i x_i) \mapsto t_j, \quad p_j : t_j^{-1}]0,1] \to X_j : (t_i x_i) \mapsto x_j.$$

*Define the **Milnor topology** on X to be the coarsest topology for which all t_j and p_j are continuous. This means that the topology is characterized by the following property: A map $f : Y \to X$ from any space Y is continuous if and only if the maps $t_j f : Y \to [0,1]$ and $p_j f : f^{-1} t_j^{-1}]0,1] \to X_j$ are continuous. If the spaces X_j are right G-spaces, we can define a continuous action of G on X via $((t_j x_j), g) \mapsto (t_j x_j g)$.*

Definition 22. *The **Milnor space** is*

$$EG = G \star G \star G \star \cdots,$$

which is the join of a countably infinite number of copies of G. Write $BG \coloneqq EG/G$ for the orbit space and $p_G : EG \to BG$ for the orbit map.

We now claim that $p_G : EG \to BG$ is a numerable G-principal bundle. To see this, observe that the coordinate maps t_j are G-invariant and hence induce maps $\tau_j : BG \to [0,1]$, which form a point-finite partition of unity subordinate to the open covering by the V_j/G, where $V_j \coloneqq t_j^{-1}]0,1]$. Now p_G is trivial over the sets V_j/G, since we have G-maps $p_j : V_j \to G$ by construction.[5]

Now that we constructed some candidate for EG, we have to show that it satisfies the necessary properties.

Proposition 23. *Let E be a G-space. Any two G-maps $f, g : E \to EG$ are G-homotopic.*

Proof. We start by writing f and g in their coordinate forms:

$$f(x) = (\lambda_1(x) f_1(x), \lambda_2(x) f_2(x), \dots), \quad g(x) = (\mu_1(x) g_1(x), \mu_2(x) g_2(x), \dots).$$

The trick is to show that f and g are G-homotopic to maps with respective coordinate forms

$$\hat{f}(x) \coloneqq (\lambda_1(x) f_1(x), 0, \lambda_2(x) f_2(x), 0, \dots), \quad \hat{g}(x) \coloneqq (0, \mu_1(x) g_1(x), 0, \mu_2(x) g_2(x), \dots).$$

Here 0 denotes an element of the form $0 \cdot y$. We illustrate the construction of this homotopy for f, in an infinite number of steps. In the first step, we consider the homotopy

$$(\lambda_1 f_1, t\lambda_2 f_2, (1-t)\lambda_2 f_2, t\lambda_3 f_3, (1-t)\lambda_3 f_3, \dots),$$

where t is the homotopy parameter. This homotopy eliminates the first zero in $\hat{f}$. We repeat this process iteratively for all ensuing coordinates. The desired final homotopy is then obtained by applying the first homotopy on the time interval $\left[0, \frac{1}{2}\right]$, the second on $\left[\frac{1}{2}, \frac{3}{4}\right]$, and so on. The resulting homotopy is continuous, since in each coordinate only a finite number of homotopies are applied. We repeat this process for g, in order to arrive at the final forms $\hat{f}$ and $\hat{g}$. These can then be connected by the homotopy $((1-t)\lambda_1 f_1, t\mu_1 g_1, (1-t)\lambda_2 f_2, t\mu_2 g_2, \dots) = (1-t)\hat{f} + t\hat{g}$ in the parameter t. □

Proposition 24. *Let E be a G-space. Let $\mathcal{U} \coloneqq (U_n \mid n \in \mathbb{N})$ be an open covering by G-trivial sets. Suppose there exists a point-finite partition of unity $(v_n \mid n \in \mathbb{N})$ of E by G-invariant functions subordinate to the covering $\mathcal{U}$. Then there exists a G-map $\varphi : E \to EG$. In particular, a numerable free G-space E admits a G-map $E \to EG$.*

Proof. Recalling the definition of a trivial G-space, there exist G-maps $\varphi_n : U_n \to G$. The map that we are looking for is then given by $\varphi(z) = (v_1(z)\varphi_1(z), v_2(z)\varphi_2(z), \dots)$. (If $z \notin U_i$, then $v_i(z) = 0$ anyway.) It can be checked that this map is continuous by the universal property of the Milnor topology.
The case of a numerable free G-space satisfies the assumptions of the proposition, except for the fact that $\mathcal{U}$ may not be countable. Luckily there is a fix for this technical matter in tom Dieck's book, chapter 13, where he proposes a method to reduce an arbitrary partition of unity to a countable one. □

By slightly modifying the proofs of the two previous results, one sees that G can be replaced by any X that is a free numerable G-space. Namely, $E = X \star X \star \cdots$ is a universal G-space. For example, if we take H to be a subgroup of G such that G is numerable as an H-space, then EG is, considered as H-space, universal.

[5] A small side note should be made here. We consider a numerable covering to be associated to a locally finite partition of unity. Here we merely showed existence of an associated point-finite partition of unity. To fix this "shortcoming", we can make use of Lemma 13.1.7 in tom Dieck's book, which guarantees a refined covering of BG that does guarantee an associated locally finite partition of unity (though it is possible the functions in this family turn out to be completely different from the ones we started with in the point-finite case). Hence $EG \to BG$ is in fact numerable.

3.3 Properties of EG

Proposition 25. *The space EG is contractible.*

Proof. From the proof of Proposition 23, we know that there is a homotopy that goes from the identity to the map $(t_j x_j) \mapsto (t_1 x_1, 0, t_2 x_2, t_3 x_3, \dots)$. This latter map is one end of the nullhomotopy $((1-t)t_1 x_1, te, (1-t)t_2 x_2, \dots)$ with homotopy parameter t, where e denotes the identity element of G. □

The following theorem characterizes universal bundles, so that we need not rely on a special construction.

Theorem 26. *A numerable G-principal bundle $q : E \to B$ is universal if and only if E is contractible.*

Proof. We already know by Proposition 25 that Milnor's space EG is contractible. If q is universal, then the G-space E is G-homotopy equivalent to EG and hence contractible. The proof of the converse direction involves some additional theory on shrinkability, so it is left for personal reading. □

Example 27. *By Theorem 15 and Proposition 25, we know that the bundle $p : EG \to BG$ is a fibration with contractible total space. By the long exact sequence for fibrations, we have an isomorphism of homotopy groups $\partial : \pi_n(BG) \cong \pi_{n-1}(G)$.*

- *For a discrete group, BG must be an Eilenberg-MacLane space of type $K(G, 1)$. We recall that G-principal bundles for a discrete group G are covering spaces; this holds in particular for the universal bundle $EG \to BG$. Since EG is simply connected, it is also the universal covering of BG. Thus the two notions of "universal" meet. Recall from the second talk that Eilenberg-MacLane spaces of the same type $K(G, n)$ are unique up to homotopy equivalence.*
  - *If $B = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{R}P^\infty$ is a model for BG, p_G is obtained by looking at the fibre bundles $\mathbb{Z}/2\mathbb{Z} \to S^n \twoheadrightarrow \mathbb{R}P^n$ and letting $n \to \infty$.*
  - *For $G = \mathbb{Z}$, we just recover our example from the start of the universal cover $p_G : \mathbb{R} \to S^1$.*
  - *More generally, for $G = \mathbb{Z}^n$, we have the universal cover $p_G : \mathbb{R}^n \to \mathbb{T}^n$, with p_G the orbit map of the action given by integral translations along the n coordinate axes of $\mathbb{R}^n$.*
- *As an example where G is not discrete, take $G = S^1$. Then BG is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2)$, so $\mathbb{C}P^\infty$ is a model for BG. Here, p_G is obtained by looking at the fibre bundles $S^1 \hookrightarrow S^{2n-1} \twoheadrightarrow \mathbb{C}P^n$ and letting $n \to \infty$.*

A continuous homomorphism of topological groups $\alpha : K \to L$ induces the map

$$E(\alpha) : EK \to EL : (t_i k_i) \mapsto (t_i \alpha(k_i)),$$

which is compatible with the projections to the classifying spaces. We then obtain an induced map $B(\alpha) : BK \to BL$. In this way, B becomes a functor from the category of topological groups into TOP up to homotopy equivalence.

Proposition 28. *An inner automorphism $\alpha : K \to K : k \mapsto uku^{-1}$, $u \in K$, induces a map $B(\alpha)$ which is homotopic to the identity.*

Proof. The map $EK \to EK : (t_i k_i) \mapsto (t_i u k_i)$ is a K-map and therefore K-homotopic to the identity by Proposition 23. And the assignment $(t_i k_i) \mapsto (t_i u k_i u^{-1})$ induces the same map between the orbit spaces as the previous one:

$$BK \to BK : \overline{(t_i k_i)} = \overline{(t_i k_i u)} \mapsto \overline{(t_i u k_i u u^{-1})} = \overline{(t_i u k_i)}.$$

□

We conclude that B does not detect inner automorphisms.