Vector bundles

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1 Bundle maps as pullbacks of vector bundles

We will focus on the ground field $k = \mathbb{R}$, but most things also apply for other fields, e.g. $k = \mathbb{C}$.

Definition 1.1. Given a map $p : E \to B$ and a n-dim $\mathbb{R}$-vector space structure on each fibre $E_b = p^{-1}(b)$. A bundle chart or trivialization over an open set $U \subseteq B$ is a homeomorphism $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$, which is fibrewise linear and for which the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\varphi\quad} & U \times \mathbb{R} \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle pr_U} & \\ & U & \end{array}$$

A bundle atlas is a collection of bundle charts such that the basic domains cover B. The data $p : E \to B$ together with the n-dim $\mathbb{R}$-vector space structures on the fibres is a vector bundle over the space B, if a bundle atlas exists.

Given two bundle charts $(U, \varphi), (V, \psi)$ of p the transition map is

$$\psi \circ \varphi^{-1} : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n, (b, v) \mapsto (b, g_x(v))$$

where $g_x \in Gl_n(\mathbb{R})$. The assignment $U \cap V \to G, x \mapsto g_x$ is continuous.

Definition 1.2. Let $\xi : E(\xi) \to B$ and $\eta : E(\eta) \to C$ be vector bundles. A bundle morphism over a map $\varphi : B \to C$ is a map $\phi : E(\xi) \to E(\eta)$, which is fibrewise linear, such that the following diagram commutes:

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\quad\phi\quad} & E(\eta) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \eta} \\ B & \xrightarrow{\quad\varphi\quad} & C \end{array}$$

We call ϕ bundle map, if ϕ is fibrewise bijective. Sometimes we write $(\phi, \varphi) : \xi \to \eta$ for the data above.

Remark. One should think of a n-dim vector bundle as a continuous collection of n-dim vector spaces. This idea allows us to think of constructions in linear algebra and apply them to vector bundles. For example, one can define a subbundle in the usual way. The subfibres are subspaces of the original fibres. Furthermore kernels, images and cokernels of bundle morphisms with constant rank are again vector bundles.

Lemma 1.3. *A bundle map ϕ over the identity is a bundle isomorphism.*

Proof. Since ϕ is on each fibre linear and bijective, one can define a linear inverse ϕ^{-1}. We have to check, that ϕ^{-1} is continuous. Using bundle charts, this only has to be checked for maps between trivial bundles $\phi : B \times \mathbb{R}^n \to B \times \mathbb{R}^n, (b, v) \mapsto (b, g_b(v))$. The inverse is given by $(b, v) \mapsto (b, g_b^{-1}(v))$, which is continuous because $b \mapsto g_b$ is continuous. □

Proposition 1.4. *Given a pullback*

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\ \phi\ } & E(\eta) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \eta} \\ B & \xrightarrow{\ \varphi\ } & C \end{array}$$

where $\eta : E(\eta) \to C$ is a vector bundle. Then there exists a unique vector bundle structure on ξ such that the diagram above is a bundle map. We call ξ the induced bundle from η along φ and sometimes write $\xi = \varphi^*\eta$.

Proof. The fibre $\xi^{-1}(b) = \{(b, e) \in E(\xi) \mid \eta(e) = \varphi(b)\}$ is bijective to $\eta^{-1}(\varphi(b))$ via ϕ. We get a unique vector space structure on each of the fibres $\xi^{-1}(b)$, such that ϕ is linear. For a bundle chart $h : \eta^{-1}(U) \to U \times \mathbb{R}^n$ of η we define a bundle chart $h' : \xi^{-1}(\varphi^{-1}(U)) \to \varphi^{-1}(U) \times \mathbb{R}^n, (b, e) \mapsto (b, h_2(e))$ of ξ, which is fibrewise linear, since h is. The inverse is given by $(b, v) \mapsto (b, h^{-1}(b, v))$. □

Remark. Given a bundle map $(\phi, \varphi) : \xi \to \eta$ we get an isomorphism of vector bundles $E(\xi) \to \varphi^* E(\eta), e \mapsto (\xi(e), \phi(e))$. The inverse is given by $(b, e) \mapsto x \in \xi^{-1}(b)$ such that $\phi(x) = e$.

This shows that a bundle map is a pullback of a vector bundle.

2 Classification of vector bundles

In the previous seminar talk we have seen the Classification theorem, which states that there is a bijection between the isomorphism classes of numerable G-principal bundles and the homotopy classes of a classifying map, i.e. a map from B to the classifying space BG:

$$\mathfrak{B}(B, G) \cong [B, BG]$$

We want to use this theorem to classify vector bundles. To do this, we have to connect vector bundles and G-principal bundles.

Construction 2.1 (Associated fibre bundles)**.** Let $q : E \to B$ be a G-princial bundle and V a n-dim representation of G. The projection $E \times V \to E \to B$ induces a map $p : E \times_G V \to B$ via passage to orbit spaces. This is a well-defined map: $(e, v) \cdot g = (eg, g^{-1}v) \mapsto eg \mapsto q(eg) = q(e)$. A bundle chart $\varphi : q^{-1}(U) \to U \times G$ of q induces a bundle chart

$$p^{-1}(U) = q^{-1}(U) \times_G V \to (U \times G) \times_G V \cong U \times V$$

of p. The vector space structure on the fibres $p^{-1}(b)$ is uniquely determined, since we want p to be fibrewise linear. This data makes $p : E \times_G V \to B$ a n-dim vector bundle. p is called the associated fibre bundle with typical fibre V.

Example 2.2 (Tautological bundles)**.** Let V be a n-dim vector space and $Gr_k(V)$ the Grassmannian. Define $E_k(V) = \{(x,v) \in Gr_k(V) \times V \mid v \in x\}$. We get the projection $p : E_k(V) \to Gr_k(V), (x,v) \mapsto x$. The fibre of the element $x \in Gr_k(V)$ is the subspace $x \subseteq V$. We want to realize this as an associated fibre bundle. Consider the $O(k)$-principal bundle $V_k(V) \to Gr_k(V)$, where $V_k(V)$ is the Stiefel manifold. The map

$$V_k(V) \times_{O(k)} \mathbb{R}^k \to E_k(V), ((v_1,...,v_k),(\lambda_1,...,\lambda_k)) \mapsto (\langle v_1,...,v_k\rangle, \sum_{i=1}^{k} \lambda_i v_i)$$

is a fibrewise linear homeomorphism.

Construction 2.3 (Frame bundle)**.** Let $p : X \to B$ be a n-dim vector bundle. Set $E_b = Iso(\mathbb{R}^n, X_b)$, the space of linear isomorphism. $G = Gl_n(\mathbb{R})$ acts freely and transitively on E_b by precomposition. Define $E = \bigsqcup_{b\in B} E_b$ and consider the map of sets $q : E \to B, \alpha \in E_b \mapsto b$ with fibrewise $Gl_n(\mathbb{R})$-action. Given a bundle chart

$$\varphi : p^{-1}(U) \to U \times \mathbb{R}^n, \varphi_b : p^{-1}(b) \to \{b\} \times \mathbb{R}^n$$

for p, we define a bundle chart

$$\tilde{\varphi} : q^{-1}(U) = \bigsqcup_{b\in U} E_b \to U \times Gl_n(\mathbb{R}), \alpha \in E_b \mapsto (b, \varphi_b \circ \alpha)$$

for q, which is $Gl_n(\mathbb{R})$-equivariant. For two bundle charts $(U,\varphi),(V,\psi)$ the transition function is

$$U \cap V \times Gl_n(\mathbb{R}) \to U \cap V \times Gl_n(\mathbb{R}), (b,\gamma) \mapsto (b, \psi_b \circ \varphi_b^{-1} \circ \gamma),$$

which is continuous, since $b \mapsto \psi_b \circ \varphi_b^{-1}$ is. Thus there exists a unique topology on E such that the sets $q^{-1}(U)$ are open and the bundle charts $\tilde{\varphi}$ are homeomophisms. Now the $Gl_n(\mathbb{R})$ action becomes continuous. Therefore $q : E \to B$ is a $Gl_n(\mathbb{R})$-principal bundle.

Remark. If the vector bundle has an Euclidean metric, one can do a similar construction for $G = O(n)$.
One can also take n-frames of X_b instead of linear isomorpisms $\mathbb{R}^n \to X_b$. We will use this idea in the following example.

Example 2.4. Consider $TS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid \langle x,v\rangle = 0\}$ the tangent bundle of the sphere with the projection $p : TS^n \to S^n$ onto the first factor. Then $p^{-1}(x)$ is the orhtogonal complement of x in $\mathbb{R}^{n+1}$.
The principal $O(n)$-bundle associated to TS^n is the space of pairs $(x, \underline{v})$ with $x \in S^n$ and $\underline{v} = (v_1,...,v_n)$ an orthonormal n-frame of $x^\perp$. This is the space of all orthonormal $(n+1)$-frames in $\mathbb{R}^{n+1}$, i.e. the Stiefel manifold $V_{n+1}(\mathbb{R}^{n+1}) \cong O(n+1)$. Hence the principal $O(n)$-bundle can be associated to the standard quotient map $O(n+1) \to S^n$, where we associate $S^n \cong O(n+1)/O(n)$.

Theorem 2.5. *There is an equivalence of categories*

$$\left\{ \begin{matrix} Gl_n(\mathbb{R})\text{-principal bundles} \\ \text{bundle maps} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} n\text{-dim real vector bundles} \\ \text{bundle maps} \end{matrix} \right\}$$
$$[q : E \to B] \longmapsto [p : E \times_{Gl_n(\mathbb{R})} \mathbb{R}^n \to B]$$

Proof. First we show that

$$X \to E = \bigsqcup_{b\in B} Iso(\mathbb{R}^n, X_b \times_{Gl_n(\mathbb{R})} \mathbb{R}^n), x \mapsto [x \mapsto [(x,v)]]$$

is a principal $Gl_n(\mathbb{R})$-bundle morphism. Note that $xg \mapsto [v \mapsto [(xg,v)] = [(x,gv)]] = [v \mapsto [x,v]] \cdot g$. Hence this is a G-map. Every bundle map over the identity is a bundle isomorphism.
The evaluation map

$$E_b \times \mathbb{R}^n \to X_b, (\alpha, v) \mapsto \alpha(v)$$

induces a continuous map $E \times_{Gl_n(\mathbb{R})} \mathbb{R}^n \to X$, which is well-defined since $(\alpha, v)\cdot g = (\alpha g, g^{-1}v) \mapsto \alpha g(g^{-1}v) = \alpha(v)$. This is a bundle map over the identity, hence a bundle isomorphism. □

Now applying the classification theorem yields a 1:1 correspondence between the isomorphism classes of n-dim real vector bundles over B and the homotopy classes of classifying maps $[B, BGl_n(\mathbb{R})]$. But with the theory, which we have developed thus far, we don't know what space $BGl_n(\mathbb{R})$ is. Figuring this out is our next objective.

Let $i : H \hookrightarrow G$ be the inclusion of a subgroup. Restricting the G-action to H we obtain a free, contractible H-space $res_H EG$. If $G \to G/H$ is a numerable H-principal bundle, then $res_H EG$ is a numerable H-space. Thus we have in this case $res_H EG \to (res_H EG)/H$ as a model for $EH \to BH$. Since $EG \times_G G/H \cong EG/H$, we get via the associated fibre bundle a map

$$Bi : BH = BG/H \to EG/G = BG$$

with typical fibre G/H. If G/H is contractible, then Bi is a numerable fibration with contractible fibre, hence a homotopy equivalence. In case of CW-complexes one proves this using the LES of fibrations and the Whitehead theorem.

Proposition 2.6. *The inclusions of subgroups induce homotopy equivalences*

$$BO(n) \to BGl_n(\mathbb{R}) \text{ and } BU(n) \to BGl_n(\mathbb{C})$$

Proof. Since the QR-decomposition is unique for invertible matrices, $Gl_n(\mathbb{R})/O(n) \cong$ space of all upper triangular matrices with positive entries on the diagonal. This space is contractible via the homotopy $A \mapsto tA + (1-t)E_n$. □

Thus we have

$$[B, BO(n)] \cong [B, BGl_n(\mathbb{R})].$$

The classifying space $BO(n)$ is the infinite Grassmanian $Gr_n(\mathbb{R}^\infty)$ of n-planes in $\mathbb{R}^\infty$. The total space is $V_n(\mathbb{R}^\infty)$ the Stiefel manifold of n-dim orthonormal frames in $\mathbb{R}^\infty$.

Definition 2.7. Let $\xi : E(\xi) \to B$ and $\eta : E(\eta) \to C$ be vector bundles. The product $\xi \times \eta : E(\xi) \times E(\eta) \to B \times C$ is again a vector bundle. In the case $B = C$, we define the Whitney sum $\xi \oplus \eta = d^*(\xi \times \eta)$ as the pullback of the diagram

$$\begin{array}{ccc} d^*(\xi \times \eta) & \longrightarrow & E(\xi) \times E(\eta) \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \times B \end{array}$$

where $d : B \to B \times B, b \mapsto (b, b)$ is the diagonal map.

Example 2.8. Inclusion of subgroups induces the fibre bundle

$$S^{n-1} \cong O(n)/O(n-1) \to BO(n-1) \to BO(n),$$

which is numerable and thus a fibration. One should think of a point in $BO(n-1)$ as a point $V \in BO(n)$ plus a unit vector $v \in V$. The LES of homotopy groups yields that $Bi : BO(n-1) \to BO(n)$ is $(n-1)$-connected. Thus $(Bi)_* : [X, BO(n-1)] \to [X, BO(n)]$ is bijective (surjective) for any CW-complex X with $dimX < n-1$ (resp. $dimX \leq n-1$). If $dimX < n-1$,$k \geq n$, then a k-dim vector bundle ξ over X is isomorphic to $\eta \oplus (k-n)\epsilon$ for an n-dim bundle η, which isomorphism class is unique. Here, $(k-n)\epsilon$ is the $(k-n)$-dim trivial bundle.

Definition 2.9. One can define the tensor product $\xi \otimes \eta$ of two vector bundles over B. The underlying set is

$$\bigsqcup_{b \in B} (E(\xi)_b \otimes E(\eta)_b) = E(\xi \otimes \eta).$$

For bundle charts $\varphi : \xi^{-1}(U) \to U \times \mathbb{R}^m, \psi : \eta^{-1}(U) \to U \times \mathbb{R}^n$ define

$$\gamma : \bigsqcup_{b \in U} (E(\xi)_b \otimes E(\eta)_b) \to U \times (\mathbb{R}^m \otimes \mathbb{R}^n), e \otimes e' \mapsto (b, \varphi(e) \otimes \psi(e')).$$

The transition maps of such charts are homeomorphisms, thus there is a unique topology on $E(\xi \otimes \eta)$ such that the sets $\gamma^{-1}(U)$ are open and the γ are homeomorphisms.