The Pontrjagin-Thom Construction

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Abstract

We follow [BD70] and [Die08] closely to establish the construction of Pontrjagin-Thom.

1 Thom Spaces

Definition 1.1 (Thom space)**.** The Thom space of a k-dimensional, real vector bundle with a Riemannian metric $B \xrightarrow{\xi} BO(k)$, is defined as $Th(\xi) \coloneqq D(\xi)/S(\xi)$. If B is compact, we have a homeomorphism $Th(\xi) \cong E(\xi) \cup \{\infty\} = E(\xi)^+$, with $E(\xi)$ being the total space of the bundle ξ.

Lemma 1.2. *A bundle map $F : \xi \to \eta$, which is a metric preserving homeomorphism fibre-wise, induces a pointed map $Th(F) : Th(\xi) \to Th(\eta)$.*

Proof. As the bundle map preserve the metric, there is an induced map on $F_D : D(\xi) \to D(\eta)$, and $F_S : S(\xi) \to S(\eta)$. Thus, the map $Th(F) : Th(\xi) \to Th(\eta)$ is well-defined. □

Remark 1.3. We don't need the bundle map F to preserve the metric in each fibre, this is just for convenience.

Lemma 1.4. *Let $B \xrightarrow{\xi} BO(k)$ be a vector bundle, and $\mathbb{R}$ the trivial line-bundle. Then $Th(\xi \oplus \mathbb{R}) \cong \Sigma Th(\xi)$.*

Proof. Let $\phi : D^{k+1} \to D^k \times I$ be an $O(k)$-equivariant homeomorphism. This induces a homeomorphism $D(\xi \oplus \mathbb{R}) \to D(\xi) \times I$. This induces the following.

$$\begin{aligned} Th(\xi \oplus \mathbb{R}) &= D(\xi \oplus \mathbb{R})/S(\xi \oplus \mathbb{R}) \xrightarrow{\cong} (D(\xi) \times I)/(S(\xi) \times I \cup D(\xi) \times \{0, 1\}) \\ &= \Sigma(D(\xi)/S(\xi)) \\ &= \Sigma Th(\xi) \end{aligned}$$

This concludes the proof. □

Remark 1.5. Doing this n-times, hence with $\mathbb{R}^n$, yields a homeomorphism $Th(\xi \oplus \mathbb{R}^n) \cong \Sigma^n Th(\xi)$. More generally, given two vector bundles ξ, η, we have an isomorphism $Th(\xi \times \eta) \cong Th(\xi) \wedge Th(\eta)$. This follows from X, Y locally compact Hausdorff, then $(X \times Y)^+ \cong X^+ \wedge Y^+$. In this way, one obtains Lemma 1.4 by considering $\mathbb{R}^+ \cong S^1$, hence $Th(\xi \oplus \mathbb{R}) \cong Th(\xi) \wedge S^1 \cong \Sigma Th(\xi)$.

2 The Pontrjagin-Thom Construction

We fix some further notation before describing the Pontrjagin-Thom construction. From now on, $B \xrightarrow{\xi} BO(k)$ denotes a smooth, real k-dimensional vector bundle over a closed manifold B. Let $j : M^n \to \mathbb{R}^{n+k}$ be a smooth embedding of a closed, smooth, n-dimensional manifold M^n. We denote by $\nu_j : E(\nu_j) \to M^n$ the normal bundle $E(\nu_j) \coloneqq j^*(T\mathbb{R}^{n+k})/TM^n \subseteq M^n \times \mathbb{R}^{n+k}$. A tubular neighbourhood of M^n in $\mathbb{R}^{n+k}$ is an inclusion $N : E(\nu_j) \to \mathbb{R}^{n+k}$, such that the restriction to the zero section is $j(M^n)$, and

$$E(\nu_j) \to j^*(M^n) \to j^*(T\mathbb{R}^{n+k})/TM^n = E(\nu_j)$$

is the identity. We denote the image by $N(M^n)$. A tubular neighbourhood does exist, and it is unique up to isotopy, see [Hir97]. One can think about this as rescaling the fibres to the open disc bundle $\dot{D}_\epsilon(\nu_j)$ until there is no self intersection with $j(M^n)$ given as the zero-section. We now want to assign to each such embedding $j : M^n \to \mathbb{R}^{n+k}$ with a bundle map $F : \nu_j \to \xi$ a pointed homotopy class $\mathcal{P}(j, F) \in \pi_{n+k}(Th(\xi))$. Before we describe the map explicitly, we give an argument on how this is a natural thing to think about. If $f : U \to L$ is an open inclusion between locally compact Hausdorff spaces, then the one-point compactification induces a pointed map $f^* : L^+ \to U^+$. In this sense, the one-point compactification defines a contravariant functor $(-)^+ : \mathbf{LCH}^{\mathrm{op}}_{\hookrightarrow} \to \mathbf{CH}_\bullet$, from the categeory of locally compact Hausdorff spaces with open embeddings into the category of compact Hausdorff spaces with pointed continuous maps.

Definition 2.1 (Pontrjagin-Thom construction)**.** Consider the tubular neighbourhood $N : E(\nu_j) \to \mathbb{R}^{n+k}$. This induces a map $S^{n+k} \cong (\mathbb{R}^{n+k})^+ \to E(\nu_j)^+ \cong Th(\nu_j) \xrightarrow{Th(F)} Th(\xi)$. Explicitly, consider the following diagram.

$$\begin{array}{ccccc}
S^{n+k} & \xrightarrow{\text{Quotient}} & S^{n+k}/(S^{n+k} \setminus N(M^n)) & \xrightarrow{\cong} & D_\epsilon(\nu_j)/(D_\epsilon(\nu_j) \setminus \dot{D}_\epsilon(\nu_j)) \\
{\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \cong} \\
Th(\xi) & & \xleftarrow[Th(F)]{\hspace{6em}} & & Th(\nu_j)
\end{array}$$

We define the class $\mathcal{P}(j, F) \coloneqq [g] \in \pi_{n+k}(Th(\xi))$.

Remark 2.2. The tubular neighbourhood is unique up to isotopy. Two different tubular neighbourhoods give maps in the same homotopy class. This can be seen as bundle maps preserve the structure of the fibres, and there is a linear homotopy in each fibre. Analogously, a different choice of ϵ does not change the homotopy class.

We will now define a certain bordism-relation for such pairs (j, F).

Definition 2.3 (ξ-bordism)**.** Two such pairs (j_0, F_0), and (j_1, F_1), with $j_i : M_i^n \to \mathbb{R}^{n+k}$, and $F_i : \nu_{j_i} \to \xi$, are ξ-bordant, if there exists a smooth submanifold $J : W^{n+1} \to \mathbb{R}^{n+k} \times I$ such that

$$W^{n+1} \cap (\mathbb{R}^{n+k} \times [0, \frac{1}{3})) = M_0^n \times [0, \frac{1}{3}) \text{ and } W^{n+1} \cap (\mathbb{R}^{n+k} \times (\frac{2}{3}, 1]) = M_1^n \times (\frac{2}{3}, 1]$$

and $\partial W^{n+1} = M_0^n \times \{0\} \cup M_1^n \times \{1\}$. Furthermore, we want a bundle map $F : \nu_J \to \xi$ such that the following diagram commutes.

$$\begin{array}{ccccc} & & \overset{F_i}{\frown} & & \\ E(\nu_{j_i}) & \xrightarrow[\subseteq_i]{} & E(\nu_J) & \xrightarrow[F]{} & E(\xi) \\ \downarrow & & \downarrow & & \downarrow \\ M_i^n & \hookrightarrow & W^{n+1} & \longrightarrow & B \end{array}$$

The total space $E(\nu_J)$ of the normal bundle is naturally a subspace of $T(\mathbb{R}^{n+k} \times I)|_{J(W^{n+1})} = W^{n+1} \times \mathbb{R}^{n+k} \times \mathbb{R}$.

Lemma 2.4. *ξ-bordism defines an equivalence relation. We denote by $\mathcal{L}_n(\xi)$ the set of all ξ-bordism classes of such pairs (j, F) with $j : M^n \to \mathbb{R}^{n+k}$, and $F : \nu_j \to \xi$.*

Proof. Trivial, similar to the proof that usual bordism defines an equivalence relation. □

We now want to define the Pontrjagin-Thom map $\mathcal{P} : \mathcal{L}_n(\xi) \to \pi_{n+k}(Th(\xi))$ by mapping a pair (j, F) to the pointed homotopy class $\mathcal{P}(j, F)$ given by the Pontrjagin-Thom construction. A priori, it is not evident that this is well-defined.

Lemma 2.5. *Let (j_0, F_0) and (j_1, F_1) be ξ-bordant in $\mathcal{L}_n(\xi)$. Then $\mathcal{P}(j_0, F_0) = \mathcal{P}(j_1, F_1)$.*

Proof. Consider a ξ-bordism (J, F), $J : W^{n+1} \to \mathbb{R}^{n+k} \times I$. One can apply the Pontrjagin-Thom construction to (J, F), and easily obtain

$$\begin{array}{ccccc} S^{n+k} \times I & \longrightarrow & (S^{n+k} \times I)/((S^{n+k} \times I) \setminus N(W^{n+1})) & \longrightarrow & D_\epsilon(\nu_J)/S_\epsilon(\nu_J) \\ \vdots & & & & \downarrow \\ Th(\xi) & & \longleftarrow & & Th(\nu_J) \end{array}$$

defining a homotopy between the maps obtained from (j_0, F_0) and (j_1, F_1) under the Pontrjagin-Thom construction. □

With that, the Pontrjagin-Thom map $\mathcal{P} : \mathcal{L}_n(\xi) \to \pi_{n+k}(Th(\xi))$ is well-defined. We will now state the main theorem of this section.

Theorem 2.6. *Let $B \xrightarrow{\xi} BO(k)$ be a smooth, real, k-dimensional vector bundle over a smooth, closed manifold B. Then the Pontrjagin-Thom map $\mathcal{P} : \mathcal{L}_n(\xi) \to \pi_{n+k}(Th(\xi))$ is a bijection.*

Before we turn to the proof, observe that elements $f : S^{n+k} \to Th(\xi)$ in the obtained class from the Pontrjagin-Thom construction are of a very special type.

(a) Consider $E(\xi) \subseteq Th(\xi) \cong E(\xi)^+$. The induced map $f : f^{-1}(E(\xi)) \to E(\xi)$ is smooth, proper, and $f \pitchfork s(B)$, with $s : B \to E(\xi)$ being the zero section.

(b) The pre-image $f^{-1}(s(B))$ of the zero section has a tubular neighbourhood $U \subseteq S^{n+k}$ such that $f(x) = \infty \in Th(\xi)$ if and only if $x \notin U$. We can find an embedding $j : f^{-1}(s(B)) \to \mathbb{R}^{n+k}$.

(c) If U is the just established neighbourhood, there exists a bundle map defined as follow.

$$E(\nu_j) \to \dot{D}_\epsilon(\nu_j) \to U \xrightarrow{f} E(\xi)$$

Lemma 2.7. *Every map $f : S^{n+k} \to Th(\xi)$ is homotopic to a map f' satisfying* **(a)**-**(c)**.

Proof. The techniques used stem from differential topology, one can use [BD70], or [Die08] as good references. □

Remark 2.8. If f_0 and f_1 are homotopic and satisfy **(a)** to **(c)**, we can find a homotopy $H : S^{n+k} \times I \to Th(\xi)$ satisfying **(a)** to **(c)** as well.

To prove Theorem 2.6, we construct an inverse $\mathcal{Q}$ to the map $\mathcal{P}$.

Proof of Theorem 2.6. As we want to construct an inverse $\mathcal{Q} : \pi_{n+k}(Th(\xi)) \to \mathcal{L}_n(\xi)$, we can assume to work with maps $f : S^{n+k} \to Th(\xi)$, satisfying the properties **(a)** to **(c)**. Let $s : B \to E(\xi)$ be the zero section. Define $M^n := f^{-1}(s(B))$. As f is proper, and B is assumed to be closed, this is a closed, smooth manifold of dimension n, using **(a)**. As f is smooth, this induces a smooth bundle map $F : \nu_j \to \xi$, with an embedding $j : M^n \to \mathbb{R}^{n+k} \subseteq (\mathbb{R}^{n+k})^+ \cong S^{n+k}$, using **(b)** and **(c)**. We define the map $\mathcal{Q}$ to send $[f]$ to the ξ-bordism class $[j, F]$. We need to check that $\mathcal{Q}$ is well-defined. Let $f_0, f_1 : S^{n+k} \to Th(\xi)$ lie in the same homotopy class, satisfying **(a)** to **(c)**. We need to show that $\mathcal{Q}$ sends both to the same ξ-bordism class. We can choose a homotopy H between f_0 and f_1, such that $H|_{[0,\frac{1}{3})} = f_0$, $H|_{(\frac{2}{3},1]} = f_1$. By Remark 2.8, we can assume H to satisfy **(a)** to **(c)**. Now, taking the pre-image $W^{n+1} := H^{-1}(s(B))$ of the zero section, embedding this into $\mathbb{R}^{n+k} \times I$ via some map J, using **(c)** to obtain a bundle map $F : \nu_J \to \xi$, we are left with a ξ-bordism. Hence, $\mathcal{Q}$ is well-defined. By construction, and using **(a)** to **(c)**, $\mathcal{Q}$ is an inverse. Hence, $\mathcal{P}$ is bijective. □

Remark 2.9. If one endows $\mathcal{L}_n(\xi)$ with a suitable group structure, namely disjoint union, this actually becomes a natural isomorphism, as bundle maps induce maps on Thom spaces. This can be seen, as we can choose the tubular neighbourhoods of the disjoint union embedded into $\mathbb{R}^{n+k}$ to be disjoint. Applying the Pontrjagin-Thom construction yields a map $f : S^{n+k} \to S^{n+k} \vee S^{n+k} \to Th(\xi)$.

3 Generalising the Pontrjagin-Thom map

Firstly, we can consider a smooth manifold W with empty boundary. For clarification of the notation, this is not meant to be a ξ-bordism. Instead of considering embeddings $j : M^n \to \mathbb{R}^{n+k}$, one can similarly consider closed submanifolds $i : M \to W$ with an induced smooth bundle map $F : \nu_i \to \xi$. Considering such pairs (i, F) and going through the analogue Pontrjagin-Thom construction

$$\begin{array}{ccccc} W^+ & \longrightarrow & W^+/(W^+ \setminus N(M)) & \longrightarrow & D_\epsilon(\nu_i)/S_\epsilon(\nu_i) \\ {\scriptstyle g}\downarrow & & & & \downarrow \\ Th(\xi) & \longleftarrow & & & Th(\nu_i) \end{array}$$

where the needed induced metric comes from an embedding of W into a large enough euclidean space. Similarly, one can define the ξ-bordism relation, and obtain a bijection $\mathcal{P} : \mathcal{L}(W, \xi) \to [W^+, Th(\xi)]$, given by the Pontrjagin-Thom map. We now want to get rid of the assumption that $B \xrightarrow{\xi} BO(k)$ is a smooth vector bundle over a closed manifold B. We will do this in the following steps.

(a) Consider a bundle $B \xrightarrow{\xi} BO(k)$ of rank k, over a closed manifold B, not necessarily smooth. This is induced by a universal bundle $\gamma_k : E(\gamma_k) \to BO(k)$ and some map $f : B \to BO(k)$, and then taking the pullback.

$$\begin{array}{ccc} E(\xi) & \xrightarrow[F]{} & E(\gamma_k) \\ \downarrow{\scriptstyle \xi} & & {\scriptstyle \gamma_k}\downarrow \\ B & \xrightarrow{f} & BO(k) \end{array}$$

The map f is homotopic to a smooth map g. This result is not trivial, the proof uses the Whitney embedding theorem, see for example [BT82, Proposition 17.8]. The bundle induced by the map g is smooth, and isomorphic to ξ. Therefore, we can drop the assumption that ξ is smooth.

(b) Let $r : X \to B$ be a retraction. Suppose $\mathcal{P}$ is bijective for bundles over X, $B \xrightarrow{\xi} BO(k)$. We can pull back the bundle ξ along r to obtain a bundle $\eta = r^*(\xi)$ over X. As $\mathcal{P}$ is natural and bijective for η, we can conclude that $\mathcal{P}$ is bijective for ξ. In this way, $\mathcal{P}$ is bijective for all bundles over B.

(c) Let B be a compact manifold, with non-empty boundary. Considering $B \cup_{\partial B} B$ yields a closed manifold. There is a retraction $r : B \cup_{\partial B} B \to B$. Hence, $\mathcal{P}$ is a bijection for all compact manifolds.

With some more technical arguments, one can show that B being any space suffices. One shows the assertion for B being a CW-complex, and the uses CW-approximation. This is explained in [Die08].

4 Bordism and Thom Spectra

Definition 4.1 (Thom Spectrum)**.** Let $\gamma_n : E(\gamma_n) \to BO(n)$ be a universal, n-dimensional, real vector bundle. We denote its Thom space by $MO(n) := Th(\gamma_n)$. The Thom spectrum $MO = (MO(n), e_n)$ is the spectrum with the following structure maps $e_n : \Sigma MO(n) \cong Th(\gamma_n \oplus \mathbb{R}) \to MO(n+1)$ induced by classifying maps $\gamma_n \oplus \mathbb{R} \to \gamma_{n+1}$. We set $MO(n) = *$ for $n \leq 0$. For a space X, we denote the associated homology groups to be $MO_n(X) = \pi_n(X \wedge MO)$.

We will now state the main theorem.

Theorem 4.2. *There is a natural isomorphism* $T(X) : \mathcal{N}_n(X) \cong MO_n(X^\bullet)$*. Here,* $X^\bullet$ *is* X *with an added point.*

We will prove this later. For a space X, consider the product bundle $\xi_k(X) = \mathrm{id}_X \times \gamma_k$. We will now define the following map.

$$\Pi_k(X) : \mathcal{L}_n(\xi_k(X)) \to \mathcal{N}_n(X)$$

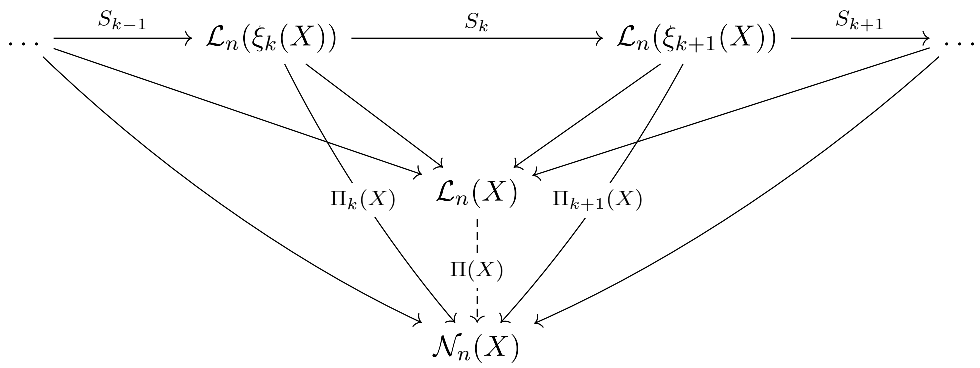
Let $[j, F] \in \mathcal{L}_n(\xi_k(X))$, an n-dimensional $\xi_k(X)$-submanifold, meaning $j : M^n \to \mathbb{R}^{n+k}$ with a given bundle map $F : E(\nu_j) \to E(\xi_k(X))$. Restricting to the first component F_1 of the map $F = (F_1, F_2) : E(\nu) \to E(\xi_k(X)) = X \times E(\gamma_k)$, given by the universal property of the product, yields a map $F_1 : E(\nu_j) \to X$. Via pre-composition with the zero-section, we get $f = F_1 \circ s : M^n \to X$. We define $\Pi_k(X)[j, F] = [M^n, f] \in \mathcal{N}_n(X)$. This is well-defined, as dropping the second component of the bundle map leaves us with the usual bordism-relation. Hence, under the map Π_k, $\xi_k(X)$-bordism classes get mapped to bordism-classes in $\mathcal{N}_n(X)$. Furthermore, there is a suspension map.

$$S_k : \mathcal{L}_n(\xi_k(X)) \to \mathcal{L}_n(\xi_{k+1}(X))$$

Consider $j' : M^n \xrightarrow{j} \mathbb{R}^{n+k} \xrightarrow{\mathrm{id}_{\mathbb{R}^{n+k}} \times \{0\}} \mathbb{R}^{n+k+1}$. The normal bundle of M^n in $\mathbb{R}^{n+k+1}$ is $E(\nu_j) \oplus \mathbb{R}$. The bundle map F induces a mapping $F_2' : E(\nu_j) \oplus \mathbb{R} \to E(\gamma_k) \oplus \mathbb{R} \to E(\gamma_{k+1})$ with the latter map being induced by the classifying map. Therefore, we obtain a new bundle map $F' = (F_1, F_2') : E(\nu_j) \to E(\xi_{k+1}(X)) = X \times E(\gamma_{k+1})$. We now set $S_k[j, F] = [j', F']$. Obviously, we have $\Pi_{k+1} \circ S_k = \Pi_k$. Therefore, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{L}_n(\xi_k(X)) & \xrightarrow{\quad S_k \quad} & \mathcal{L}_n(\xi_{k+1}(X)) \\ & {\scriptstyle \Pi_k} \searrow \quad \swarrow {\scriptstyle \Pi_{k+1}} & \\ & \mathcal{N}_n(X) & \end{array}$$

We denote the colimit over the maps S_k by $\mathcal{L}_n(X)$, defining a map $\Pi(X) : \mathcal{L}_n(X) \to \mathcal{N}_n(X)$.



Theorem 4.3. *The map* $\Pi(X) : \mathcal{L}_n(X) \to \mathcal{N}_n(X)$ *is bijective.*

Proof. We first prove surjectivity. Let $[M, f] \in \mathcal{N}_n(X)$ be given. By the Whitney embedding theorem, there exists $k \in \mathbb{N}$ such that we can embed M into $\mathbb{R}^{n+k}$ via some map j. As the normal bundle $\nu_j : E(\nu_j) \to M$ is the pullback of some classifying bundle $\gamma_k : E(\gamma_k) \to BO(k)$ under a map $M \to BO(k)$, there is a bundle map $\kappa_{j,k} : \nu_j \to \gamma_k$. This defines a $\xi_k(X)$-structure given by (j, F), with $F := (f \circ \nu_j, \kappa_{j,k})$. By construction, we have $\Pi_k(X)[j, F] = [M, f]$.
Let us turn our attention to injectivity. Let us assume that $[j_0, F_0]$ and $[j_1, F_1]$ in $\mathcal{L}_n(X)$ have the same image under $\Pi(X)$. By the Whitney embedding theorem, we can assume that $j_i : M_i^n \to \mathbb{R}^{n+k}$ for a suitable k. There exists a bordism B with $\partial B \cong M_0^n \amalg M_1^n$, with an extension $f : B \to X$ of $f_i : M_i^n \to X$, the first component of F_i. For t large enough, there exists an embedding $B \to \mathbb{R}^{n+k+t} \times I$ such that the following holds.

$$B \cap (\mathbb{R}^{n+k} \times \mathbb{R}^t \times [0, \frac{1}{3})) = M_0^n \times \{0\} \times [0, \frac{1}{3}) \text{ and } B \cap (\mathbb{R}^{n+k} \times \mathbb{R}^t \times (\frac{2}{3}, 1]) = M_1^n \times \{0\} \times (\frac{2}{3}, 1]$$

Extending the collars, we can find a bordism B such that $\phi : C := M_0^n \times [0, \frac{1}{2}) \amalg M_1^n \times (\frac{1}{2}, 1] \to B$ is an embedding, and $\partial B = M_0^n \times \{0\} \amalg M_1^n \times \{1\}$. We can embed $C \to \mathbb{R}^{n+k} \times \mathbb{R}^t \times I$ by the mapping $(p, q) \mapsto (p, 0, q)$. By the Tietze extension theorem, there exists an extension $\Phi : B \to \mathbb{R}^{n+k+t} \times I$ such that Φ extends ϕ on $D := M_0^n \times [0, c) \amalg M_1^n \times (c, 1]$ for some $\frac{1}{3} < c < \frac{1}{2}$, and furthermore $\Phi(B \setminus D) \subseteq \mathbb{R}^{n+k+t} \times (\frac{1}{3}, \frac{2}{3})$. If we assume $k + t > n + 1$, we can assume that Φ is an embedding, let us set $J = \Phi$. Now, the bundle maps $F_i : \nu_{j_i} \to \gamma_k$ yield bundle maps $\nu_{j_i} \oplus \mathbb{R}^t \to \gamma_{k+t}$. The classifying maps are unique up to homotopy, and as the inclusion $\partial B \to B$ is a cofibration, we can extend these maps to a bundle map $\nu_J \to \gamma_{k+t}$. Therefore, $[j_i, F_i] \in \mathcal{L}_{n+t}(\xi_k(X))$ are $\xi_{k+t}(X)$-bordant in $\mathcal{L}_n(\xi_{k+t}(X))$. Hence, they lie in the same equivalence class in $\mathcal{L}_n(X)$. □

To get to Theorem 4.2, we want to consider $\Pi(X)^{-1}$ and then apply the Pontrjagin-Thom map, as in Theorem 2.6. This was only defined on $\mathcal{L}_n(\xi)$ for some vector bundle $B \xrightarrow{\xi} BO(k)$. We will again consider $\xi_k(X) = \mathrm{id}_X \times \gamma_k$, and argue by taking the colimit. Using Remark 1.5, we notice that $Th(\xi_k(X)) = Th(\mathrm{id}_X \times \gamma_k) \cong Th(\mathrm{id}_X) \wedge Th(\gamma_k) \cong X^\bullet \wedge MO(k)$. The Pontrjagin-Thom maps

$\mathcal{P} : \mathcal{L}_n(\xi_k(X)) \to \pi_{n+k}(Th(\xi_k(X))) \cong \pi_{n+k}(X^\bullet \wedge MO(k))$ behave well with the maps S_k. The following diagram commutes.

$$\begin{array}{ccc} \mathcal{L}_n(\xi_k(X)) & \xrightarrow{\quad\mathcal{P}\quad} & \pi_{n+k}(X^\bullet \wedge MO(k)) \\ \Big\downarrow{\scriptstyle S_k} & & \Big\downarrow{\scriptstyle \Sigma_k} \\ \mathcal{L}_n(\xi_{k+1}(X)) & \xrightarrow{\quad\mathcal{P}\quad} & \pi_{n+k+1}(X^\bullet \wedge MO(k+1)) \end{array}$$

Here, the map Σ_k is similarly to S_k defined to be the following map.

$$\begin{array}{ccccc} \pi_{n+k}(X^\bullet \wedge MO(k)) & \xrightarrow{\quad\cong\quad} & \pi_{n+k}(Th(\xi_k(X))) & \xrightarrow{\quad\Sigma\quad} & \pi_{n+k+1}(\Sigma Th(\xi_k(X))) \\ \vdots{\scriptstyle \Sigma_k} & & & & \Big\downarrow{\scriptstyle \cong} \\ \pi_{n+k+1}(X^\bullet \wedge MO(k+1)) & \xleftarrow[\quad\cong\quad]{} & \pi_{n+k+1}(Th(\xi_{k+1}(X))) & \longleftarrow & \pi_{n+k+1}(Th(\xi_k(X) \oplus \mathbb{R})) \end{array}$$

The map Σ becomes an isomorphism for large enough k. Thus, we obtain a natural bijection $\mathcal{P}(X) : \mathcal{L}_n(X) \to MO_n(X^\bullet)$ by taking the colimit.

Proof of Theorem 4.2. We define $T(X) \coloneqq \mathcal{P}(X) \circ \Pi(X)^{-1} : \mathcal{N}_n(X) \to MO_n(X^\bullet)$. This is a natural bijection. It is left to show that $T(X)$ preserves the group structure. This is a similar argument as we have seen in Remark 2.9. We are slightly more precise. Given $[M_i, f_i] \in \mathcal{N}_n(X)$, we can smoothly embed $M_0 \to \{x \in \mathbb{R}^{n+k} : x_{n+k} < 0\}$, and $M_1 \to \{x \in \mathbb{R}^{n+k} : x_{n+k} > 0\}$. Furthermore, choose tubular neighbourhoods $N(M_0) \subseteq \{x \in \mathbb{R}^{n+k} : x_{n+k} < 0\}$, and $N(M_1) \subseteq \{x \in \mathbb{R}^{n+k} : x_{n+k} > 0\}$. This is disjoint, hence it represents $M_0 \amalg M_1$. Again, applying the Pontrjagin-Thom construction yields a map which factors over $S^{n+k} \to S^{n+k} \vee S^{n+k}$. Restricting to each component represents $\mathcal{P}[j_i, F_i]$. Here, we used the bijective relation $[M_i, f_i] \sim [j_i, F_i]$. Note, the group structure on $\mathcal{N}_n(X)$ and $\mathcal{L}_n(X)$ is the same. □

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