TALK 16: MODEL CATEGORIES

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Abstract. The goal of this talk is to provide an introduction into model categories and motivate why they are interesting and strong objects to consider. In particular, we will give an overview into how model categories are used to compare homotopy theories.

Preface

This talk is largely based on the lecture course of Najib Idrissi from 2022 [Idr22]. Since the notes are in French, I would like to refer those who cannot speak it to the book of Mark Hovey [Hov99] which is very good and similarly structured. As a matter of fact, it is one of the references of the lecture notes. If you are interested, you can also take a look at the original lecture notes of Daniel G. Quillen [Qui67] who first introduced model categories in '67. While some of the definitions have been formulated slightly differently in the last 60 years, it is still very nice to read.

1. Simplicial Sets

Since simplicial sets were not covered in previous classes but will be used in examples later, we shall introduce them here.

Definition 1.1. The simplex category Δ is comprised of

- objects of the form $[n] = \{0, \ldots n\}$ for all $n \in \mathbb{N}$
- morphisms $\phi : [n] \to [m]$ which are monotonically increasing.

Remark 1.2. Any morphism in Δ is a composition of *cofaces* δ_i and *codegenerecies* σ_i

$$\delta_i : [n] \to [n+1] \qquad\qquad \sigma_i : [n] \to [n-1]$$

$$i \mapsto \begin{cases} j & \forall j \leq i \\ j+1 & \forall j > i \end{cases} \qquad\qquad i \mapsto \begin{cases} j & \forall j \leq i \\ j-1 & \forall j > i \end{cases}$$

Definition 1.3. A simplicial set is a functor $X : \Delta^{op} \to Set$.

Remark 1.4. A simplicial set is completely given by the following data:

- A collection $\{X_n\}_{n\in\mathbb{N}}$ of sets,
- faces $d_i : X_n \to X_{n-1}$ for all $0 \leq i \leq n$,
- degenerecies $d_i : X_n \to X_{n+1}$ for all $0 \leq i \leq n$

such that the following relations hold

- $d_i d_j = d_{j-1} d_i \quad \forall i < j$
- $s_i s_j = s_{j+1} s_i \quad \forall i \leq j$
- $d_i s_j = \begin{cases} s_{j-1} d_i & \forall i < j \\ id & \forall i = j,\, i = j+1 \\ s_j d_{i-1} & \forall i > j+1 \end{cases}$

Definition 1.5. We call an element $x \in X_n$ degenerate if there exists an $y \in X_{n-1}$ and an i s.t. $s_i(y) = x$. Elements which are not degenerate are called non-degenerate.

Definition 1.6. A morphism f between two simplicial sets X and Y is a natural transformation, i.e. $f = (f_n)_{n \in \mathbb{N}}$ where $f_n : X_n \to Y_n$ is a map in (Set) s.t. for all $\phi : [m] \to [n]$ the following diagram commutes.

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle X(\phi)}\downarrow & & \downarrow{\scriptstyle Y(\phi)} \\ X_m & \xrightarrow{f_m} & Y_m \end{array}$$

This forms a category $(sSet)$.

Remark 1.7. It suffices to show that f commutes with the faces and degenerecies.

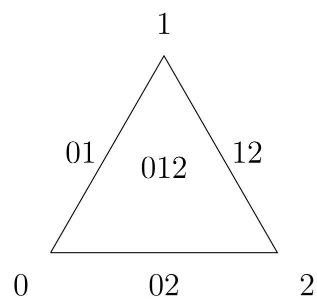
Since this definition seems incredibly abstract, let us look at a very concrete example.

Example 1.8. The *standard n-simplex* Δ^n is defined as follows:

$$\Delta^n_0 = \{0, 1, \dots n\}$$
$$\Delta^n_k = \{i_0 \dots i_k \,|\, i_0 \leq i_1 \leq \dots i_k\}$$

The i-th face will skip the i-th entry, the i-th degenerecy will copy the i-th entry. We can also write it as a functor: $\mathrm{Hom}_\Delta(-, [n])$.

In the case of the standard 2-simplex, we can see that there are only 7 non-degenerate elements: 0, 1, 2, 01, 02, 12, 012. We can vizualise these:



Based on the standard n-simplicies, we can also define a few others.

Example 1.9. We can define the boundary $\partial\Delta^n$ as arrising from Δ^n by removing $01 \dots n$ and all its degenerecies.

We can define the k-th horn of as $\bigwedge^n_k$ as removing the k-th face (i.e. $01 \dots (k-1)(k+1) \dots n$) and its degenerecies from the boundary.

One of the big advantages of considering simplicial sets (as opposed to geometric complexes) is that quotients are well defined. As such we can also consider $\Delta^n / \partial\Delta^n$.

Now let us take a look at how simplicial sets relate to topological spaces.

Definition 1.10. The geometrical n-simplex $|\Delta^n|$ is the convex hull of $\langle e_0, \dots e_n \rangle$ in $\mathbb{R}^{n+1}$. We can define continuous morphisms $\delta_i : |\Delta^n| \to |\Delta^{n+1}|$ which is the embedding of $|\Delta^n|$ into the i-th face of $|\Delta^{n+1}|$ and $\sigma_i : |\Delta^n| \to |\Delta^{n-1}|$ where a point in the convex hull $(t_0, \dots, t_n)$ is send to $(t_0, \dots, t_i + t_{i+1}, \dots, t_n)$.

Definition 1.11. The geometric realisation of a simplicial set X is defined as

$$|X| := \coprod_{n\in\mathbb{N}} X_n \times |\Delta^n| \big/ \sim$$

where the equivalence relation $\sim$ is generated by $(x, \delta_i(y)) \sim (d_i(x), y)$ and $(x, \sigma_i(z)) \sim (s_i(x), z)$ where $x \in X_n$, $y \in |\Delta^{n-1}|$, $z \in |\Delta^{n+1}|$ and $0 \leq i \leq n$.

A similar construction turns morphisms of simplicial sets to morphisms of their geometric realisation. Thus we gain a functor

$$|\,.\,| : (sSet) \to (Top).$$

As a matter of fact, the image of $|\,.\,|$ lies completely in the subcategory of compactley generated weak Hausdorff spaces. (These are essentially "nice" spaces, in particular, they include CW complexes). When restricted to this description, it has a right adjoint, the singular functor $Sing = Hom_{Top}(|\Delta^*|, -)$. $Sing$ itself can be fully defined on the whole category (Top) and is used when calculating homology.

In the following, when talking about (Top) we will mean CGWH since we are focusing on the adjunction. We will say a bit more about this at the end.

Definition 1.12. A Kan complex is a simplicial set X s.t. we can fill all horns, i.e. for all $n > 0$ and all $0 \leq k \leq n$:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ \Delta^n & & \end{array}$$

Definition 1.13. A Kan fibration is a simplicial map $f : X \to Y$ s.t. for all $n > 0$ and all $0 \leq k \leq n$:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

Remark 1.14. X is a Kan complex if and only if the simplicial map $X \to *$ is a Kan fibration where $*$ is the terminal object of $(sSet)$.

Definition 1.15. Two simplicial maps $f, g : X \to Y$ are homotopic if there exists a simplicial map $H : X \times \Delta^1 \to Y$ s.t. $H(-, 0) = f$ and $H(-, 1) = g$.

The product here is defined the obvious way.

Warning 1.16. This does not define an equivalence relation since Δ^1 has a direction and as such cannot be simply turned around. Also, it cannot simply be split into two copies of itself. However, if limited to Kan complexes, it does indeed become an equivalence relation (via filling horns).

Definition 1.17. A simplicial map $f : X \to Y$ is a weak homotopy equivalence if its geometric realisation $|f|$ is one in (Top).

Theorem 1.18. *If X and Y are Kan complexes, any map $f : X \to Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

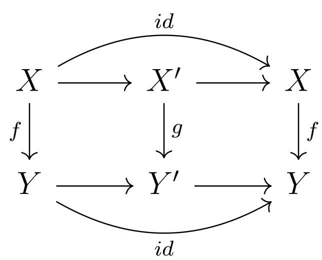
Theorem 1.19. *Every simplicial set is weakly equivalent to a Kan complex.*

2. Model Categories

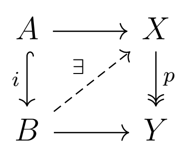
1. **Definition and Examples.**

Definition 2.1. A model category is a category $\mathscr{M}$ equipped with three classes of morphisms $(\mathscr{C}, \mathscr{F}, \mathscr{W})$ where we call $\mathscr{C}$ the cofibrations and denote them with $\hookrightarrow$, $\mathscr{F}$ the fibrations $\twoheadrightarrow$ and $\mathscr{W}$ the weak equivalences $\xrightarrow{\sim}$. We call for the following 5 axioms to be satisfied.

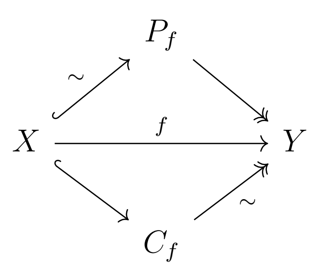
(M1) $\mathscr{M}$ is complete and cocomplete (i.e. all small limits exist).
(M2) $\mathscr{W}$ satisfies "2 out of 3", i.e. if $h = g \circ f$ and two of the morphism are in $\mathscr{W}$ then so is the third.
(M3) If f is a retract of g and g lies in any of the three classes, f lies in the same. A retract is defined via the following commutative diagramme:



(M4) For every commutative square of the following form where $i \in \mathscr{W}$ or $p \in \mathscr{W}$ there exists a lift.



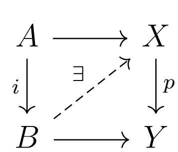
(M5) Every morphism $f : X \to Y$ admits functorial factorisations



We call P_f path space and C_f cylinder.

Definition 2.2. The elements of $\mathscr{C} \cap \mathscr{W}$ are called acyclic cofibrations, those of $\mathscr{F} \cap \mathscr{W}$ acyclic fibrations.

Definition 2.3. Consider two morphisms $i : A \to B$ and $p : X \to Y$. We say that i has the left lifting property w.r.t. p and p has the right lifting property w.r.t. i if for every commutative square of the following form there exists a lift:



We write $i \in {}^{\perp}(p)$ and $p \in (i)^{\perp}$

Definition 2.4. An object X of $\mathscr{M}$ is called

- cofibrant if the unique map from the initial object to X is a cofibration: $\varnothing \hookrightarrow X$.
- fibrant if the unique map from X to the terminal object is a fibration: $X \twoheadrightarrow *$.

Remark 2.5. Due to (M5), every object admits a

- functorial cofibrant replacement: $\varnothing \hookrightarrow Q_X \overset{\sim}{\twoheadrightarrow} X$.
- functorial fibrant replacement: $X \overset{\sim}{\hookrightarrow} R_X \twoheadrightarrow *$.

Example 2.6. Let $\mathscr{M}$ be a complete and cocomplete category. Then there are three trivial model category structures on it:

$\mathscr{W}$	$\mathscr{F}$	$\mathscr{F}$
iso	$\mathscr{M}$	$\mathscr{M}$
$\mathscr{M}$	iso	$\mathscr{M}$
$\mathscr{M}$	$\mathscr{M}$	iso

Here, $\mathscr{M}$ denotes all morphisms of the category. That these are indeed model category structures is easy to see.

Let us move on to more interesting examples. We will not prove that any of the following are indeed model categories, since more often than not, this is the hardest part of working with model categories.

Example 2.7 (Quillen '67). (Top) has a model structure given by

- $\mathscr{W}$: weak homotopy equivalences
- $\mathscr{F}$: Serre fibrations
- $\mathscr{C}$: retracts of inclusions of relative cell complexes

Here, all objects are fibrant, and the retracts of relative cell complexes are cofibrant (in particular CW complexes).

There is more than one non-trivial model structure on (Top).

Example 2.8. (Top) has a model structure given by

- $\mathscr{W}$: homotopy equivalences
- $\mathscr{F}$: Hurewicz fibrations
- $\mathscr{C}$: retracts of Hurewicz cofibrations with closed image

Here, all objects are fibrant and cofibrant.

Later, when talking about (Top), we will always consider the model strcuture of Quillen.

Example 2.9. $(sSet)$ has a model structure given by

- $\mathscr{W}$: weak homotopy equivalences
- $\mathscr{F}$: Kan fibrations
- $\mathscr{C}$: inclusions

Here, all objects are cofibrant and Kan complexes are fibrant.

Example 2.10. $Ch_{\geqslant 0}(\mathbb{Z})$ has a model structure given by

- $\mathscr{W}$: quasi-isomorphisms
- $\mathscr{F}$: surjections in degree $\geqslant 1$
- $\mathscr{C}$: injections with projective cokernel

Here, all objects are fibrant and projective chain complexes are cofibrant. This is called the projective model structure.

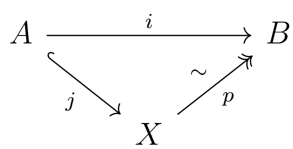
Looking at these examples, two out of the three classes of morphisms seem very controlled while the third does not. That is not a coincidence. Indeed, two classes of morphisms always determine the third. (Whether these three form a model category is a different question.) Let us show this.

Lemma 2.11. *We have the following equivalences.*

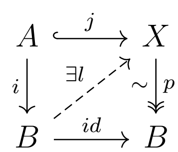
(1) $i \in \mathscr{C} \Leftrightarrow i \in {}^{\perp}(\mathscr{W} \cap \mathscr{F})$
(2) $i \in \mathscr{W} \cap \mathscr{C} \Leftrightarrow i \in {}^{\perp}\mathscr{F}$
(3) $p \in \mathscr{F} \Leftrightarrow i \in (\mathscr{W} \cap \mathscr{C})^{\perp}$
(4) $p \in \mathscr{W} \cap \mathscr{F} \Leftrightarrow p \in \mathscr{F}^{\perp}$
(5) $f \in \mathscr{W} \Leftrightarrow f = p \circ i$ *where* $p \in \mathscr{W} \cap \mathscr{F}$ *and* $i \in \mathscr{W} \cap \mathscr{C}$

Proof. We will only proof the very first part since the next three are very similar and (5) is trivial.

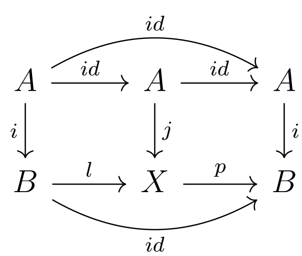
"⇒" Follows directly from (M4).
"⇐" Let $i \in {}^{\perp}(\mathscr{W} \cap \mathscr{F})$. By (M5) there exists a factorization.



By enlarging this triangle with an identity, we can use the left lifting property of i:



Now we can use this to show that i is a retract of j and thus by (M3) a cofibration.



□

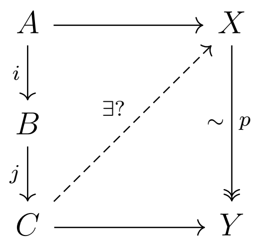
Corollary 2.12. *In a model category, any two classes of morphisms define the third:*

- $\mathscr{F} = (\mathscr{W} \cap \mathscr{C})^{\perp}$
- $\mathscr{C} = {}^{\perp}(\mathscr{W} \cap \mathscr{F})$
- $\mathscr{W} = \mathscr{C}^{\perp} \circ {}^{\perp}\mathscr{F}$

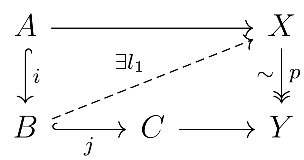
Before we continue on to the homotopy theory of model categories, let us consider another lemma and its proof to see that once we have the data of a model category, it is very easy to prove certain statements.

Lemma 2.13. *The classes $\mathscr{F}$, $\mathscr{C}$ and $\mathscr{W}$ are closed under composition.*

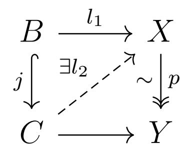
Proof. The case of $\mathscr{W}$ follows directly from (M2). We will only show the case of $\mathscr{C}$ since $\mathscr{F}$ is similar. Consider A $\overset{i}{\hookrightarrow} B \overset{j}{\hookrightarrow} C$. We want to show that this composition has the left lifting property w.r.t. $\mathscr{W} \cap \mathscr{F}$. So we consider



We can rewrite this to recognize a more known diagramme where we can apply (M4).



With this new map l_1, we can now consider another diagramme where we can again use (M4):



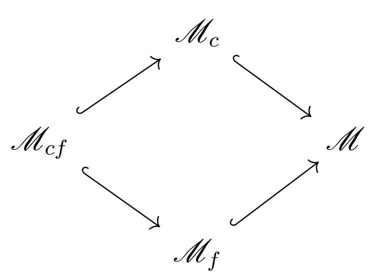
This map l_2 makes the first diagramme commute (which follows from using all informations from the last two diagrammes). □

2. **Homotopy Theory.**

Definition 2.14. The homotopy category $Ho\mathscr{M}$ of a model category $\mathscr{M}$ is defined as the localisation $Ho\mathscr{M} = \mathscr{M}[\mathscr{W}^{-1}]$.

Usually, when talking about localization of categories, we might have size issues. This is not the case when considering weak equivalences of model categories, so this is indeed well defined.

Proposition 2.15. *Denote the subcategory of fibrant objects as $\mathscr{M}_c$, respective $\mathscr{M}_f$ and $\mathscr{M}_{cf}$ for the subcategories of fibrant and cofibrant-fibrant objects. The natural inclusions*



induce equivalences on the homotopy categories.

$$\begin{array}{ccccc} & & Ho\mathscr{M}_c & & \\ & \overset{\sim}{\nearrow} & & \overset{\sim}{\searrow} & \\ Ho\mathscr{M}_{cf} & & & & Ho\mathscr{M} \\ & \underset{\sim}{\searrow} & & \underset{\sim}{\nearrow} & \\ & & Ho\mathscr{M}_f & & \end{array}$$

In particular, when considering the homotopy category, we can w.l.o.g. restrict to cofibrant-fibrant objects.

Now, we want to transform this definition into something we can work with. We want to use equivalence relations. For this, let $f, g : A \to X$ be two morphisms.

Definition 2.16. A left homotpy between f and g is the data of a commutative diagramme

$$\begin{array}{ccccc} A \sqcup A & \xhookrightarrow{(i_0,i_1)} & C & \xrightarrow[p]{\sim} & A \\ & \searrow^{(f,g)} & \downarrow H & & \\ & & X & & \end{array}$$

where $p \circ (i_0, i_1) = (id, id) : A \sqcup A \to A$, i.e. C is a cylinder (not necessarily the one from (M5)). We write $f \simeq_l g$.

Definition 2.17. A right homotpy between f and g is the data of a commutative diagramme

$$\begin{array}{ccccc} & & A & & \\ & & H\downarrow & \searrow^{(f,g)} & \\ X & \xrightarrow[j]{\sim} & P & \underset{(q_0,q_1)}{\twoheadrightarrow} & X \times X \end{array}$$

where $(q_0, q_1) \circ j = (id, id) : X \to X \times X$, i.e. P is a path space (not necessarily the one from (M5)). We write $f \simeq_r g$.

We can see that $\simeq_l$ behaves nicely for fibrant objects, while $\simeq_r$ is well-behaved on cofibrant objects. Putting this together, we obtain the following.

Proposition 2.18. *On cofibrant-fibrant objects $\simeq_l = \simeq_r =: \simeq$ and defines an equivalence relation. Moreover*

$$Ho\mathscr{M} = {}^{\mathscr{M}_{cf}}\!/_{\simeq}$$

3. **Quillen Equivalence.** Now let us consider two model categories $\mathscr{M}$ and $\mathscr{N}$. We want to find a sufficient condition sucht that $Ho\mathscr{M}$ is equivalent to $\mathscr{N}$. We start with an adjunction

$$F : \mathscr{M} \rightleftharpoons \mathscr{N} : G.$$

Definition 2.19. An adjunction $F \dashv G$ is a Quillen adjunction if F preserves $\mathscr{C}$ and $\mathscr{C} \cap \mathscr{W}$.

Remark 2.20. This condition is equivalent to any of the three following.

- G preserves $\mathscr{F}$ and $\mathscr{F} \cap \mathscr{W}$
- F preserves $\mathscr{C}$ and G preserves $\mathscr{F}$
- F preserves $\mathscr{C} \cap \mathscr{W}$ and G preserves $\mathscr{F} \cap \mathscr{W}$

Example 2.21. The adjunction $|\,.\,| \dashv Sing$ is indeed a Quillen adjunction. We can see this by considering the second condition: If f is a Serre fibration then $Singf$ is a Kan fibration. Furthermore, $Sing$ respects weak equivalence by definition.

The following theorem needs far more theory, namely right and left derived functors, then we want to consider here. For that reason, it is not stated in a very precise manner.

Theorem 2.22. *Let $F \dashv G$ be a Quillen adjunction. TFAE*

(1) $F \dashv G$ induces an equivalence between $Ho\mathscr{M}$ and $Ho\mathscr{N}$
(2) For every cofibrant object A in $\mathscr{M}$ and every fibrant object X in $\mathscr{N}$, a morphism $f : FA \to X$ is a weak equivalence if and only if its adjoint $\tilde{f} : A \to GX$ is one.

In this case, we say that the adjunction is a Quillen equivalence. The "induced equivalence" is actually completely determined by F and G.

Example 2.23. The adjunction $|\,.\,| \dashv Sing$ is a Quillen equivalence and as such the homotopy theory of (Top) is equivalent to that of $(sSet)$.

At this point, we should note that while we restricted ourselves to CGWH spaces, there is also a Quillen equivalence between the complete (Top) and $(CGWH)$. Thus, on the level of homotopy, we do not need this restriction.

References

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[Qui67] D. G. Quillen. *Homotopical Algebra*, Lecture Notes in Mathematics, Springer Berlin, Heidelberg, 1967.