Talk 15: (Co-)homology with local coefficients

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The goal of this talk is to generalize Poincaré duality to non-orientable manifolds. In order to do this we will need to introduce (co-)homology with local coefficients.

1 Local coefficients via bundles of groups

Definition 1.1. Let X be a topological space and G be a discrete group. A bundle of groups over X with fiber G is a continuous map $p : E \to X$ s.t. for every $x \in X$

- $p^{-1}(x)$ has the structure of a group;
- there are $U \subseteq X$ open neighborhood of x and a homeomorphism $h : p^{-1}(U) \to U \times G$ of spaces over U s.t. $h_{|p^{-1}(y)} : p^{-1}(y) \to \{y\} \times G$ is a group homomorphism $\forall y \in U$.

Remark.

- $p^{-1}(x) \cong G$ as groups. This isomorphism is non canonical, in particular every choice of trivialization gives such an isomorphism.
- Since G is discrete E is a covering space.
- For any $U \subseteq X$ open the set of local sections on U

  $$\Gamma(U, E) = \{s : U \to E : \ p \circ s(x) = x \ \forall x \in U\}$$

  is a group with multiplication defined pointwise. More explicitly for $s, t \in \Gamma(U, E)$ $s \cdot t(x) := s(x)t(x)$, where in the right-hand side we are using the group operation of $p^{-1}(x)$.

Example 1.1. The simplest example is the trivial bundle i.e. $X \times G \to G$. By definition every bundle is locally isomorphic to the trivial bundle.

Example 1.2. Let M be a d-manifold, let $o(M) \to M$ be the orientation double cover with non-trivial deck transformation $\tau : o(M) \to o(M)$. Define the orientation bundle of M with fiber $\mathbb{Z}$ as

$$o(M)_{\mathbb{Z}} := o(M) \times_{\mathbb{Z}/2} \mathbb{Z} = o(M) \times \mathbb{Z}/(x, n) \sim (\tau(x), -n).$$

Since $\mathbb{Z}/2$ acts on $\mathbb{Z}$ by group automorphisms it can be checked that $o(M)_{\mathbb{Z}}$ is indeed a bundle of groups.

Alternatively the orientation bundle can be described as follows:

$$o(M)_{\mathbb{Z}} = \bigcup_{x \in M} H_d(M, M \setminus x; \mathbb{Z})$$

with topology induced by the basis consisting of all sets $\{\alpha_{U|x} \ \forall x \in U\}$ where U is any open subset of M, $\alpha_U \in H_d(M, M \setminus U; \mathbb{Z})$ and $\alpha_{U|x}$ denotes the image of α_U under the map in homology induced by the inclusion $(M, M \setminus U) \to (M, M \setminus x)$.

This is, by definition, the étale space associated to the locally constat presheaf of abelian groups

$$U \mapsto H_d(M, M \setminus U; \mathbb{Z}).$$

Note that $o(M) = \{\mu_x \in o(M)_{\mathbb{Z}} : \ \mu_x \text{ generates the fiber over } x\}$. From this description it's easy to see that an orientation on M is just a continuous global section of $o(M) \to M$. From this it follows that M is orientable iff $o(M) = M \coprod M$ iff $o(M)_{\mathbb{Z}} = M \times \mathbb{Z}$.

Definition 1.2. Let X be a topological space and $E \to X$ a bundle of abelian groups on X. We define the chain complex of X with local coefficients in E as

$$\begin{aligned} C_n(X; E) &:= \bigoplus_{\sigma : \Delta^n \to X} \Gamma(\Delta^n, \sigma^* E) \\ &= \left\{ \sum_{i=1}^{m} n_i \sigma_i : \ \sigma_i : \Delta^n \to X \text{ and } n_i : \Delta^n \to E \text{ lifts } \sigma_i \right\} \end{aligned}$$

with boundary given by

$$\partial(n\sigma) := \sum_{i=0}^{n} (-1)^i (n \circ \delta^i)(\sigma \circ \delta^i)$$

where $\delta^i : \Delta^{n-1} \to \Delta^n$ is the inculsion onto the i-th face.

Remark.

- It can be checked, in the same way one does for singular homology, that $\partial^2 = 0$. Therefore $C_*(X; E)$ is indeed a chain complex. We then define the homology of X with local coefficients in E as the homology of this complex.

- If $E = X \times G$ is the trivial bundle then for any simplex $\sigma : \Delta^n \to X$ $\sigma^* E = \Delta^n \times G$ is again trivial and $\Gamma(\Delta^n, \sigma^* E) \cong G$, where the isomorphism is induced by the projection $\sigma^* E \to G$. So if $C_*^S(X; G)$ denotes the singular chain complex of X with coefficients in G we get $C_*^S(X; G) \cong C_*(X; X \times G)$.

- Since E is a covering space and Δ^n is simply connected then by the lifting property of covering spaces we get $\Gamma(U, E) \cong p^{-1}(\sigma(1, 0, \ldots, 0))$ $n \mapsto n(1, 0, \ldots, 0)$. In other words a lift of σ is determined by its value at one point. Using this identification we could write

  $$C(X; E) = \bigoplus_{\sigma:\Delta^n \to X} p^{-1}(\sigma(1, 0, \ldots, 0))$$

  in this sense the fiber over a point represents the coefficients at that point.

  The issue with this definition is that when defining the boundary we need to restrict a simplex to the 0-th face. Therefore if we used this other definition we would need to "transport" the coefficient in front of the simplex from $p^{-1}(\sigma(1, 0, \ldots, 0))$ to $p^{-1}(\sigma(0, 1, \ldots, 0))$. This can be done using systems of local coefficients.

Definition 1.3. Let X be a topological space and $E \to X$ a bundle of abelian groups on X. We define the cochain complex of X with local coefficients in E as

$$\begin{aligned} C^n(X; E) := & \prod_{\sigma:\Delta^n \to X} \Gamma(\Delta^n, \sigma^* E) \\ & = \{\text{functions } \varphi \text{ assigning to each simplex } \sigma : \Delta^n \to X \text{ a lift } \varphi(\sigma)\} \end{aligned}$$

with coboundary given by $\delta(\varphi) := \varphi \circ \partial$.

Remark.

- Since $\partial^2 = 0$ also $\delta^2 = 0$ therefore $C^*(X; E)$ is a cochain complex. We define the cohomology of X with local coefficients in E to be the cohomology of this complex

- If $E = X \times G$ as before we get $\Gamma(\Delta^n, \sigma^* E) \cong G$ for any $\sigma : \Delta^n \to X$. If $C^*_S(X; G)$ denotes the singular cochain complex of X with coefficients in G we see that

  $$C^n_S(X; G) \cong \mathrm{Hom}_{\mathrm{Sets}}(\{\sigma : \Delta^n \to X\}, G) \cong \prod_{\sigma:\Delta^n \to X} G \cong C^n(X, X \times G).$$

Definition 1.4. Let $p : E \to X$ and $q : F \to Y$ be two bundles of groups. A morphism of bundles $(\tilde{f}, f) : (E, X) \to (F, Y)$ is a commutative square

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

s.t. $\tilde{f}_x : p^{-1}(x) \to q^{-1}(f(x))$ is a group homomorphism for all $x \in X$.

Remark. A morphism of bundles of abelian groups $(\tilde{f}, f) : (E, X) \to (F, Y)$ induces a chain map between the respective complexes by defining

$$C_*(X; E) \to C_*(Y; F) \; n\sigma \mapsto (\tilde{f} \circ n)(f \circ \sigma).$$

Example 1.3. Let X, Y be two spaces and G, H be two discrete abelian groups. Suppose we have a morphism of bundles $(\tilde{f}, f) : (X \times G, X) \to (Y \times H, Y)$ s.t. we can write $\tilde{f} = f \times \varphi$ for some group homomorphism $\varphi : G \to H$.

If we identify local homology with coefficients in the trivial bundle with singular cohomology we see that the map induced by $(\tilde{f}, f)$ is

$$C^S_*(X; G) \cong C_*(X; X \times G) \to C_*(Y; Y \times H) \cong C^S_*(Y; H) \; g\sigma \mapsto \varphi(g)(f \circ \sigma).$$

In other words not only the induced map changes space but also coefficients. Since this is not something one usually does when working with singular homology we will restrict our attention to bundle maps.

Definition 1.5. Let $p : E \to X$ and $q : F \to Y$ be two bundles of groups. A bundle map $(\tilde{f}, f) : (E, X) \to (F, Y)$ is a morphism of bundles s.t. $\tilde{f}_x$ is a group isomorphism for all $x \in X$.

Remark.

- If $(\tilde{f}, f) : (E, X) \to (F, Y)$ is a bundle map then the induced map $E \to f^*F$ is an isomorphism of bundles over X.
- A bundle map $(\tilde{f}, f) : (E, X) \to (F, Y)$ between bundles of abelian groups induces a map between local cohomology groups in the following way. Define $C^*(Y; F) \to C^*(X; E)$ by $\varphi \mapsto \sigma \times \varphi(f \circ \sigma)$. Here $\sigma \times \varphi(f \circ \sigma)$ denotes the unique simplex $\Delta^n \to E$ s.t. the compostitions $\Delta^n \to E \to X$ and $\Delta^n \to E \to F$ are σ and $\varphi(f \circ \sigma)$ respectively. We know such a simplex exists and is unique by the universal property of the pullback bundle to which E is isomorphic by the previous remark.

Definition 1.6. Let (X, A) be a pair of spaces and $E \to X$ be a bundle of abelian groups. Define the relative chain complex with local coefficients in E as

$$C_*(X, A; E) = \operatorname{coker}(C_*(A; E_{|A}) \to C_*(X; E))$$

and the relative cochain complex with local coefficients in E as

$$C^*(X, A; E) = \ker(C_*(X; E) \to C_*(A; E_{|A}))$$

Theorem 1.1. *Let $\mathcal{C}$ be the category whose objects are triples (X, A, E) where (X, A) is a pair of spaces and $E \to X$ is a bundle of abelian groups. A morphism in $\mathcal{C}$ is a pair $(\tilde{f}, f) : (X, A, E) \to (Y, B, F)$ where $f : (X, A) \to (Y, B)$ is a map of pairs and $(\tilde{f}, f) : (X, E) \to (Y, F)$ is a bundle map.*

Then (co-)homology with local coefficients is a (co-)homology theory on $\mathcal{C}$.

Remark. Given two homotopic maps $f, g : X \to Y$ and a bundle of groups $F \to Y$ it can be shown that $f^*F \cong g^*F$ as bundles over X.

Since we restricted our attention to bundle maps the axiom of homotopy invariance simply reads:

(h.i.) Given two morphisms in $\mathcal{C}$ $(\tilde{f}_i, f_i) : (X, A, E) \to (Y, B, F)$, $i = 1, 2$ s.t. $f_1 \simeq f_2$ then $(\tilde{f}_1, f_2)$ and $(\tilde{f}_2, f_2)$ induce the same map in (co-)homology.

2 Local coefficients via π-modules

We would like now to give a more computable description of local homology. In this section X will be a path-connected, locally path-connected and semi-locally simply-connected space. This assumptions ensure us that X admits a universal cover which we will denote by $\tilde{p} : \tilde{X} \to X$. Moreover we know that there is an isomorphism $\pi := \pi_1(X, x_0) \cong G(\tilde{X})$, where $G(\tilde{X})$ is the group of deck transformations of $\tilde{p}$. In particular π acts on the universal cover via homeomorphisms therefore π acts on $C_*^S(\tilde{X}; \mathbb{Z})$ via chain maps making it into a complex of $\mathbb{Z}[\pi]$-modules.

Remark. To be precise the isomorphism $\pi \cong G(\tilde{X})$ depends on the choice of base-point $\tilde{x}_0 \in \tilde{p}^{-1}(x_0)$. Under our assumptions we know that the universal cover can be described as the set of homotopy classes of paths in X starting from x_0. This means that we have a preferred choice of base-point, namely the identity element of $\pi = \tilde{p}^{-1}(x_0)$. We then chose the isomorphism given by this preferred base-point.

The following proposition tries to explain why there are so many equivalent constructions of local homology.

Proposition 2.1. *The following categories are equivalent*

1. *the functor category* $[\Pi_1(X), \mathrm{Sets}]$*;*
2. *the category of left π-sets;*
3. *the category of covering spaces over X;*
4. *the category of locally constant sheaves on X.*

Proof.

$1. \simeq 2.$ Since X is path-connected the fully faithful inclusion $\{x_0\} \hookrightarrow \Pi_1(X)$ is an equivalence of categories. By precomposition this equivalence induces $[\Pi_1(X), \mathrm{Sets}] \simeq [\{x_0\}, \mathrm{Sets}]$. Here $\{x_0\}$ is just the group π seen as a one-object category, therefore $[\{x_0\}, \mathrm{Sets}] \simeq \pi$-sets.

$2. \simeq 3.$ This is simply the classification of covering spaces over X. The two functors are given by

$$\mathrm{Cov}_X \to \pi\text{-sets}, \ (C \xrightarrow{p} X) \to p^{-1}(x_0) + \text{left monodromy action of } \pi$$

and

$$\pi\text{-sets} \to \mathrm{Cov}_X,\ M \mapsto \tilde{X} \times_\pi M := \tilde{X} \times M/(x,m) \sim (\gamma\tilde{x}, \gamma m),\ \gamma \in \pi.$$

3. $\simeq$ 4. This is just the restriction of a more general equivalence between étale spaces over X and sheaves on X.

$\square$

Remark. by looking at group objects in each of the above categories we obtain equivalences between the following categories:

1. the category of systems of local coefficients on X i.e. $[\Pi_1(X), \mathrm{Ab}]$;
2. the category of (left) $\mathbb{Z}[\pi]$-modules;
3. the category of bundles of abelian groups on X;
4. the category of locally constant sheaves of abelian groups on X.

Example 2.1. Let M be a connected manifold. Under the above equivalence the orientation bundle $o(M)_\mathbb{Z}$ corresponds to the $\mathbb{Z}[\pi]$-module defined by

$$w : \pi \to \mathrm{Aut}(\mathbb{Z}) \quad w(\gamma) = \begin{cases} \mathrm{id} & \text{if } \gamma \text{ lifts to a loop in } o(M) \\ -\mathrm{id} & \text{otherwise.} \end{cases}$$

If $w(\gamma) = \mathrm{id}$ then we say that γ preserves the orientation.

Definition 2.1. Let M be a $\mathbb{Z}[\pi]$-module. Define the chain complex with local coefficients in M as

$$C_*(X; M) = C_*^S(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} M$$

and the cochain complex with local coefficients in M as

$$C^*(X; M) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*^S(\tilde{X}), M).$$

Define (co-)homology with local coefficients in M as the (co-)homology of the above complex.

Remark.

- Let A, B be two left $\mathbb{Z}[\pi]$-modules. The tensor of A and B over $\mathbb{Z}[\pi]$ can be described as

$$A \otimes_{\mathbb{Z}[\pi]} B = A \otimes_\mathbb{Z} B / a \otimes b \sim \gamma a \otimes \gamma b, \gamma \in \pi.$$

- This contruction is functorial in the following sense. Let $f : X \to Y$ be a map of spaces, let M be a $\mathbb{Z}[\pi_1 X]$-module and N a $\mathbb{Z}[\pi_1 Y]$-module. Note that N can be made into a $\mathbb{Z}[\pi_1 X]$-module by defining for $\gamma \in \pi_1 X$, $n \in N$ $\gamma \cdot n := f_*(\gamma) n$. Just like in the first section we restricted our attention to bundle maps here we consider maps $\varphi : M \to N$ which are isomorphisms of $\mathbb{Z}[\pi_1 X]$-modules. Given such a pair (f, φ) we can define the induced chain maps

$$C_*(X; M) \to C_*(Y; N) \quad \text{and} \quad C^*(Y; N) \to C^*(X; M)$$

in the obvious way.

Proposition 2.2. *Let $E \to X$ be a bundle of abelian groups with corresponding $\mathbb{Z}[\pi]$-module M. We have natural isomorphisms*

$$C_*(X; E) \cong C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} M, \qquad C^*(X; E) \cong \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*^S(\tilde{X}; \mathbb{Z}), M).$$

Proof. Recall that $E \cong \tilde{X} \times_\pi M$, in particular we have a covering map $\tilde{X} \times M \to E$ which induces a surjection $C_*^S(\tilde{X} \times M; \mathbb{Z}) \to C_*^S(E; \mathbb{Z})$. We also have a surjection $C_*^S(E; \mathbb{Z}) \to C_*(X; E)$ defined by $n \to n(p \circ n)$, i.e. we are mapping a simplex in E to itself in the group $\Gamma(\Delta^n, (p \circ n)^* E)$. We also have surjections

$$C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[M] \to C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[M] \to C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} M$$

where the second map is induced by the natural surjection $\mathbb{Z}[M] \to M$ $m \mapsto m$. We then get the following commutative diagram

$$\begin{tikzcd}
C_*^S(\tilde{X} \times M; \mathbb{Z}) \arrow[r, two heads] & C_*^S(E; \mathbb{Z}) \arrow[r, two heads] & C_*(X; E) \\
\bigoplus_M C_*^S(\tilde{X}; \mathbb{Z}) \arrow[u, "\cong"] \arrow[d, "\cong"'] & & \\
C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[M] \arrow[r, two heads] & C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[M] \arrow[r, two heads] \arrow[uu, dashed] & C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} M \arrow[uu, dashed]
\end{tikzcd}$$

It can be easily checked that the kernels of the two compositions

$$\bigoplus_M C_*^S(\tilde{X}; \mathbb{Z}) \to C_*^S(\tilde{X} \times M; \mathbb{Z}) \to C_*^S(E; \mathbb{Z})$$

and

$$\bigoplus_M C_*^S(\tilde{X}; \mathbb{Z}) \to C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[M] \to C_*^S(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[M]$$

coincide. Therefore the leftmost dotted arrow exists and is an isomorphism. Similarly for the other dotted arrow.

Since $E \cong \tilde{X} \times_\pi M$ we have $\Gamma(\Delta^n, \sigma^* E) \leftrightarrow \pi \backslash \Gamma(\Delta^n, \sigma^*(\tilde{X} \times G))$ for any $\sigma : \Delta^n \to X$. Moreover the map

$$\mathrm{Hom}_\pi(\Gamma(\Delta^n, \sigma^* \tilde{X}), M) \to \pi \backslash \Gamma(\Delta^n, \sigma^*(\tilde{X} \times G)) \; \varphi \mapsto [\tilde{\sigma} \times \{\varphi(\tilde{\sigma})\}]$$

doesn't depend on the choice of $\tilde{\sigma} \in \Gamma(\Delta^n, \sigma^* \tilde{X})$ and is bijective with inverse

$$\pi \backslash \Gamma(\Delta^n, \sigma^*(\tilde{X} \times G)) \to \mathrm{Hom}_\pi(\Gamma(\Delta^n, \sigma^* \tilde{X}), M) \; [\tilde{\sigma} \times \{g\}] \mapsto (\gamma \tilde{\sigma} \mapsto \gamma g).$$

Here we are using the fact that, since the universal cover is normal, π acts freely and transitively on the set $\Gamma(\Delta^n, \sigma^*\tilde{X})$ of lifts of σ. Therefore

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}[\pi]}(C^S_*(\tilde{X};\mathbb{Z}), M) &\cong \mathrm{Hom}_\pi(\{\Delta^n \to \tilde{X}\}, M) \\
&\cong \prod_{\sigma:\Delta^n \to X} \mathrm{Hom}_\pi(\Gamma(\Delta^n, \sigma^*\tilde{X}), M) \\
&\cong \prod_{\sigma:\Delta^n \to X} \pi\backslash\Gamma(\Delta^n, \sigma^*(\tilde{X} \times G)) \\
&\cong \prod_{\sigma:\Delta^n \to X} \Gamma(\Delta^n, \sigma^* E).
\end{aligned}$$

□

Lemma 2.3. *Let S be a left π-set and C free abelian over S. Let $\pi' \subset \pi$ be a subgroup. Then $C \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi/\pi']$ is free abelian over $\pi'\backslash S$.*

Proof. The maps

$$C \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi/\pi'] \to \mathbb{Z}[\pi'\backslash S] \quad s \otimes \pi' \mapsto \pi' s \quad \text{for } s \in S$$

and

$$\mathbb{Z}[\pi'\backslash S] \to C \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi/\pi'] \quad \pi' s \to s \otimes \pi'$$

are inverses of each other. □

Example 2.2. Let $p' : X' \to X$ be a connected covering space with corresponding subgroup $\pi' := p'_*(\pi_1 X') \subseteq \pi$. Recall that $X' \cong \pi'\backslash\tilde{X}$ therefore $\pi'\backslash\{\Delta^n \to \tilde{X}\} \leftrightarrow \{\Delta^n \to X'\}$. By the previous lemma

$$C^S_*(\tilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi/\pi'] \cong C^S_*(X';\mathbb{Z}).$$

Now let G be any abelian group and set $G[\pi/\pi'] := \mathbb{Z}[\pi/\pi'] \otimes_{\mathbb{Z}} G$. Then $G[\pi/\pi']$ is a $\mathbb{Z}[\pi]$-module since $\mathbb{Z}[\pi/\pi']$ is and

$$C(X; G[\pi/\pi']) \cong C^S_*(\tilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi/\pi'] \otimes_{\mathbb{Z}} G \cong C^S_*(X'; G).$$

In this sense local homology already computes the homology of all connected covering spaces.

The same is not true for local cohomology but we get the following result.

Proposition 2.4. *Let X be a finite CW-complex with universal cover $\tilde{X}$ and fundamental group π. Then $H^n(X;\mathbb{Z}[\pi]) \cong H^n_{cs}(\tilde{X},\mathbb{Z})$ for every n.*

Now let X be a connected CW-complex . The universal cover admits an induced CW-structure such that every deck transformation is cellular. Therefore π acts on $\tilde{X}$ cellularly making $C^{cell}_*(\tilde{X};\mathbb{Z})$ into a complex of $\mathbb{Z}[\pi]$-modules.

Proposition 2.5. *Let X be as above and let M be a $\mathbb{Z}[\pi]$-module. Then $H^n(X; M) \cong H^n(C^{cell}_*(\tilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} M)$.*

Example 2.3. Let $X = \mathbb{RP}^n$ for $n \geq 2$ and let $\rho : \pi \cong \mathrm{Aut}(\mathbb{Z})$. Then $\mathbb{Z}^\rho$ is the $\mathbb{Z}[\pi]$-module with underlying abelian group $\mathbb{Z}$ on which the generator $g \in \pi$ acts as $-\mathrm{id}$. Note that if n is even then $\rho = w$ as defined above.

The universal cover of $\mathbb{RP}^n$ is S^n and the group of deck transformations is $\{\pm \mathrm{id}\}$. On $\mathbb{RP}^n$ we pick the standard CW-structure with one cell in every dimension, this lifts to the CW-structure on S^n with two cells in every dimesnion i.e.

$$\begin{tikzcd}
S^0 \arrow[d] & \subseteq & S^1 \arrow[d] & \subseteq & S^2 \arrow[d] & \subseteq & \dots & \subseteq & S^n \arrow[d] \\
* & \subseteq & \mathbb{RP}^1 & \subseteq & \mathbb{RP}^2 & \subseteq & \dots & \subseteq & \mathbb{RP}^n
\end{tikzcd}$$

where $S^i = e^i_+ \cup e^i_-$.

Let $f^i_\pm : (D^i, S^{i-1}) \to (S^i, S^{i-1})$ be the inclusion in the northern (respectively southern) emisphere. Let $[D^i, S^{i-1}] \in H_i(D^i, S^{i-1}; \mathbb{Z})$ be such that $(f^i_\pm)_*([D^i, S^{i-1}]) = e^i_\pm \in H_i(S^i, S^{i-1})$. Note that $(-\mathrm{id}) \circ f^i_+ = f^i_- \circ (-\mathrm{id})$ therefore

$$g \cdot e^i_+ = (-\mathrm{id})_*(e^i_+) = (f^i_-)_*(-\mathrm{id})_*([D^i, S^{i-1}]) = (-1)^i e^i_-,$$

in particular $C^{cell}_i(S^n; \mathbb{Z})$ is a free $\mathbb{Z}[\pi]$-module generated by e^i_+.

The boundary for $C^{cell}_*(S^n; \mathbb{Z})$ is given by the composition

$$H_i(S^i, S^{i-1}; \mathbb{Z}) \xrightarrow{\partial} H_{i-1}(S^{i-1}, *; \mathbb{Z}) \xrightarrow{j_*} H_{i-1}(S^{i-1}, S^{i-2}; \mathbb{Z}).$$

By naturality of the long exact sequence of pairs for the map $(f^i_\pm)_*$ we get

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H_i(D^i, S^{i-1}; \mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "(f^i_\pm)_*"] & H_{i-1}(S^{i-1}, *; \mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_i(S^i, *; \mathbb{Z}) \arrow[r, "j_*"] & H_i(S^i, S^{i-1}; \mathbb{Z}) \arrow[r, "\partial"] & H_{i-1}(S^{i-1}, *; \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Therefore $\partial(e^i_\pm) = [S^{i-1}] := \partial[D^i, S^{i-1}]$ and by exactness of the second row $j_*[S^i] = e^i_+ - e^i_-$. Finally $\partial e^i_+ = e^{i-1}_+ - e^i_- = (1 + (-1)^i g) e^i_+$.

This means the cellular chain complex of S^n is given by

$$0 \longrightarrow \mathbb{Z}[\pi] \xrightarrow{1+(-1)^n g} \dots \xrightarrow{1-g} \mathbb{Z}[\pi] \xrightarrow{1+g} \mathbb{Z}[\pi] \xrightarrow{1-g} \mathbb{Z}[\pi] \longrightarrow 0$$

and tensoring it with $\mathbb{Z}^\rho$ gives

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1+(-1)^{n+1}} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$$

Note that if $\mathbb{Z}^\tau$ denotes the $\mathbb{Z}[\pi]$-module with underlying abelian group $\mathbb{Z}$ on which g acts as the identity then $C^{cell}_*(S^n) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^\tau = C^{cell}_*(\mathbb{RP}^n; \mathbb{Z})$.

Taking homology

$$H^i(\mathbb{RP}^n; \mathbb{Z}^\rho) = \begin{cases} \mathbb{Z}/2 \text{ if } 0 \leq i < n \text{ even} \\ \mathbb{Z} \text{ if } i = n \text{ even} \\ 0 \text{ otherwise} \end{cases}$$

By direct comparison we see that if n is even $H_*(\mathbb{RP}^n;\mathbb{Z}^w) \cong H^{n-*}(\mathbb{RP}^n;\mathbb{Z})$.

Example 2.4. Let $X = K$ be the Klein bottle. Its universal cover is $\mathbb{R}^2$ with group of deck transformations given by $\pi = \langle a, b | abab^{-1}\rangle$ where a and b act on $\mathbb{R}^2$ as

$$a \cdot (x, y) = (x + 1, y), \quad b \cdot (x, y) = (-x, y + 1).$$

$w : \pi \to \mathrm{Aut}(\mathbb{Z})$ is given by $w(a) = \mathrm{id}$, $w(b) = -\mathrm{id}$.

On the Klein bottle we pick the standard CW-structure with one 0-cell, two 1-cells and one 2-cell glued along the path $abab^{-1}$ (here we think of a and b as the inclusions of the two 1-cells in the 1-skeleton of K). This lifts to the following CW-structure on $\mathbb{R}^2$

$$\begin{array}{ccccc} \mathbb{Z}\times\mathbb{Z} & \subseteq & (\mathbb{R}\times\mathbb{Z})\cup(\mathbb{Z}\times\mathbb{R}) & \subseteq & \mathbb{R}^2 \\ \downarrow & & \downarrow & & \downarrow \\ * & \subseteq & S^1 \vee S^1 & \subseteq & K \end{array}$$

Since the universal cover is normal π acts freely and transitively on the set of lifts of any given cell of K so that $C_n^{cell}(\mathbb{R}^2)$ is a free $\mathbb{Z}[\pi]$-module whose rank is the number of n-cells of K(this is a general fact). Therefore the cellular chain complex of $\mathbb{R}^2$ is

$$C_*^{cell}(\mathbb{R}^2) \quad = \quad (0 \longrightarrow \mathbb{Z}[\pi] \xrightarrow{f} \mathbb{Z}[\pi]\oplus\mathbb{Z}[\pi] \xrightarrow{g} \mathbb{Z}[\pi] \longrightarrow 0)$$

where $f(1) = (1 + ba^{-1}, a - 1)$, $g(1, 0) = a - 1$ and $g(0, 1) = b - 1$. Tensoring with $\mathbb{Z}^w$ gives

$$C_*^{cell}(\mathbb{R}^2) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w \quad = \quad (0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{(0,2)} \mathbb{Z} \longrightarrow 0)$$

Again by direct comparison we see that $H_*(K;\mathbb{Z}^w) \cong H^{2-*}(K;\mathbb{Z})$

3 Twisted Poincaré duality

Let M be a d-manifold and $E \to M$ a bundle of abelian groups. We can define an E-orientation on M in the same way we do for singular homology. Note that the bundle $o(M)_E := E \otimes_{\mathbb{Z}} o(M)_{\mathbb{Z}}$ can be described as

$$o(M)_E = \bigcup_{x\in M} H_d(M, M \setminus x; E)$$

with topology induced by the basis consisting of all sets $\{\alpha_{U|x}\ \forall x \in M\}$ where U is any open subset of M, $\alpha_U \in H_d(M, M \setminus U; E)$ and $\alpha_{U|x}$ denotes the image of α_U under the map in homology induced by the inclusion $(M, M \setminus U) \to (M, M \setminus x)$. From this description it's easy to see that an E-orientation on M is just a continuous global section of $o(M)_E \to M$ such that the image of every point is a generator of the fiber.

Whenever $(U,h),(V,k)$ are charts trivializing $o(M)_{\mathbb{Z}}$ the transition map

$$(U\cap V)\times\mathbb{Z}\to(U\cap V)\times\mathbb{Z}$$

is either the identity or $\mathrm{id}\times(-\mathrm{id})$. Therefore the transition maps for the bundle $o(M)\mathbb{Z}\otimes_{\mathbb{Z}}o(M)_{\mathbb{Z}}$ are always the identity. This means that $o(M)\mathbb{Z}\otimes_{\mathbb{Z}}o(M)_{\mathbb{Z}}$ is the trivial bundle and therefore every manifold admits a $o(M)_{\mathbb{Z}}$-orientation. This fact is the key ingredient for the proof of twisted Poincaré duality.

For simplicity we define the cap product in local homology only for "nice" spaces.

Definition 3.1. Let X be a path-connected, locally path-connected and semi-locally simply-connected space with fundamental group π. Let M and N be $\mathbb{Z}[\pi]$-modules. The cap product in local homology is defined as

$$\varphi\cap(\tilde{\sigma}\otimes n):=\tilde{\sigma}_{|[n,\ldots,n+k]}\otimes\varphi(\tilde{\sigma}_{|[0,\ldots,n]})\otimes n\in C_k(X;M\otimes_{\mathbb{Z}}N),$$

where $\varphi\in C^n(X;M)$ and $\tilde{\sigma}\otimes n\in C_{n+k}(X;M\otimes_{\mathbb{Z}}N)$.

Remark. In the same way it is done for singular homology it can be proven that the cap product factors through homology giving a map

$$H^n(X;M)\otimes_{\mathbb{Z}}H_{n+k}(X;N)\to H_k(X;M\otimes_{\mathbb{Z}}N).$$

Theorem 3.1. *Let M be a d-manifold without boundary and let $E\to M$ be a bundle of abelian groups. Then there exists a fundamental class $[M]\in H_d(M;o(M)_{\mathbb{Z}})$ and the map*

$$-\cap[M]:H^n_{cs}(M;E)\to H_{d-n}(M;E\otimes_{\mathbb{Z}}o(M)_{\mathbb{Z}})$$

is an isomorphism for every n.

Local cohomology with compact support is defined the same way as for singular homology.

The proof is the same as for orientable manifolds and ordinary coefficients. Indeed for the base case $M=\mathbb{R}^n$ we get back the ordinary statement of Poincaré duality since every bundle over $\mathbb{R}^n$ is trivial. Also the other steps of the proof work in the same way since we know that (co-)homology with local coefficients satisfies properies analogous to the ones of singular homology. In particular we get excision isomorphisms and Mayer-Vietoris sequences.