# Spectra and Homology Theories

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As we saw on last week's talk, we can realize any cohomology theory as homotopy classes of maps from a space into an $\Omega$-spectrum.

**Question 0.1.** Can we construct homology theories via homotopy groups/classes?

**Question 0.2.** If true, do they resemble in any way ordinary homology?

**Question 0.3.** If true, does every homology theory come from such a construction?

In this talk we will try to give a positive answer to these questions. For that, we will define the notions of stable homotopy groups and spectra, taking particular interest of those arising from Eilenberg-McLane spaces.

## 1. Preliminaries

In this introductory section we will state some definitions and results that will be used throughout the rest of the talk.

**Fact 1.1.** Let $X$ be a $n$-connected CW-complex. Then the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n+1$. In particular, $SX$ is $(n+1)$-connected.

Hence, for any CW-complex $X$, regardless of its connectivity, the sequence

$$\pi_i(X) \longrightarrow \pi_{i+1}(SX) \longrightarrow \pi_{i+2}(S^2X) \longrightarrow \dots$$

stabilizes.

**Definition 1.2.** Let $X$ be a CW-complex. We define its $i$-th stable homotopy group $\pi_i^s(X)$ to be the resulting group in the above sequence.

**Remark 1.3.** For any CW-complex $X$, its suspension $SX$ and reduced suspension $\Sigma X$ are homotopy equivalent. Thus, we can reformulate the definition of stable homotopy groups in terms of the sequence

$$\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X) \longrightarrow \pi_{i+2}(\Sigma^2 X) \longrightarrow \dots$$

which is equivalent to the original one.

**Definition 1.4.** A *reduced homology theory* on the category $\mathcal{C}$ of basepointed CW-complexes and basepoint-preserving maps is a sequence of functors $h_n : \mathcal{C} \to \mathbf{Ab}$ such that the following properties hold:

(1) (Homotopy invariance) If $f \simeq g : X \to Y$, then $f_* = g_* : h_n(X) \to h_n(Y)$.

(2) (Exactness) For any $A \subset X$, the sequence $h_n(A) \to h_n(X) \to h_n(X/A)$ is exact.

(3) (Wedge sum) For any collection $(X_i)_i$ of objects in $\mathcal{C}$, the product map induces an isomorphism $\bigoplus h_n(X_i) \to h_n(\bigvee_i X_i)$, where $\bigvee_i X_i$ denotes the wedge sum of spaces and these maps are induced by the inclusions $X_i \to \bigvee_i X_i$.

(4) (Suspension isomorphism) For any object $X$ of $\mathcal{C}$, there is a natural isomorphism $h_n(X) \cong h_{n+1}(\Sigma X)$.

Notice that this is equivalent to the definition given for a reduced cohomology theory on the same category. One can check, by the same arguments, that this definition agrees with the usual one of a reduced homology theory.

In the later parts of the talk, we will use the following result that will guarantee us that a certain homology theory that we will introduce is actually ordinary homology.

**Fact 1.5.** Let $h_*$ be an (reduced) homology theory on the category of CW-pairs such that $h_n(S^0) = 0$ for $n \neq 0$. Then, for all CW-pairs $(X, A)$ there is an isomorphism $h_n(X, A) \cong \widetilde{H_n}(X, A, h_0(S^0))$ for all $n \in \mathbb{N}$.

## 2. Stable homotopy theory

We saw in one of the first talks of the seminar that we can get reduced cohomology theory from homotopy classes of spaces into $\Omega$-spectra. In this section, we will prove that we can also obtain an homology theory defined through homotopy groups.

**Theorem 2.1.** *The sequence $h_i(X) = \pi_i^s(X)$ forms a reduced homology theory on the category of basepointed CW-complexes and basepoint-preserving maps.*

Proof. (1) is inmediate.

(2) For $A \subset X$, we know that there is an exact sequence $\pi_n(A) \to \pi_n(X) \to \pi_n(X, A)$. By proposition 4.28 of Hatcher, there is an isomorphism $\pi_n(X, A) \to \pi_n(X/A)$ for $n \leq r + s$ when $X$ is $r$-connected and $A$ is $s$-connected. By taking as many suspensions as necessary,

this connectivity assumptions will be reached eventually. Thus, this gives exactness of $\pi_n^s(A) \to \pi_n^s(X) \to \pi_n^s(X/A)$.

(3) We can restrict ourselves to the case in which we have a finite wedge sum. Indeed, maps from compact spaces into CW-complexes factor through finite subcomplexes. Thus, as homotopy groups come from maps from spheres, which are compact, checking the axiom for finite sums suffices.

Moreover, if we suppose to have a finite wedge sum, we can restrict ourselves to the two summands case by induction. In this case, we note that, given two complexes $X$ and $Y$, $\Sigma^i X \vee \Sigma^i Y$ is the $2i-1$ skeleton of $\Sigma^i X \times \Sigma^i Y$ (Think of $X = Y = S^0$ and the resulting torus). Thus, for $n+i < 2i-1$ or $i > n+1$, we have an isomorphism $\pi_{n+i}(\Sigma^i X \vee \Sigma^i Y) \cong \pi_{n+i}(\Sigma^i X \times \Sigma^i Y)$. We know that the latter is isomorphic to $\pi_{n+i}(\Sigma^i X) \oplus \pi_{n+i}(\Sigma^i Y)$. As we take the limit on $i$, we finally obtain the desired result.

(4) Note that the sequences for $\pi_i^s(X)$ and $\pi_{i+1}^s(\Sigma X)$ turn out to be the same after the term $\pi_{i+1}(\Sigma X)$. Thus their stable homotopy groups are the same and we get the desired isomorphism.

The coefficients of this homology theory are given by $h_n(S^0) = \pi_n^s(S^0)$, which do not vanish for $n \neq 0$. For example, $\pi_1^s(S^0) = \mathbb{Z}_2$.

We can actually generalize this homology theory via the smash product. Recall that the *smash product* of two spaces $X$ and $Y$ is defined to be $X \wedge Y := X \times Y / X \vee Y$. During the rest of the talk, we will be using the following properties of the smash product.

**Fact 2.2.** Let $X$ and $K$ be CW-complexes.

i) $\Sigma^k(X) \cong X \wedge S^k$.

ii) $\Sigma(X \wedge K) \cong (\Sigma X) \wedge K \cong S^1 \wedge X \wedge K$.

iii) $(X \wedge K)/(A \wedge K) \cong (X/A) \wedge K$.

iv) For any collection $\{X_i\}$ of CW-complexes, $(\bigvee_i X_i) \wedge K = \bigvee_i (X_i \wedge K)$

**Corollary 2.3.** *Let $K$ be a CW-complex. The sequence $h_i(X) = \pi_i^s(X \wedge K)$ forms a reduced homology theory on the category of basepointed CW-complexes and basepoint-preserving maps.*

Proof. All the axioms follow either trivially or combining the previous theorem with Fact 2.2.

**Remark 2.4.** The coefficients of this homology theory are given by $h_n(S^0) = \pi_n^s(S^0 \wedge K) \cong \pi_n^s(K)$. By Fact 1.5, it suffices that these groups vanish for $n \neq 0$ to see that this homology theory agrees with the ordinary one. So, by giving the complex $K$ a certain structure, one could give a positive answer to Question 0.2. There is where spectra come into play.

## 3. Spectra and Homology theories

**Definition 3.1.** i) A sequence of pointed CW-complexes $K = \{K_n\}_n$ is said to be a *spectrum* if there is a sequence of basepoint-preserving maps $\Sigma K_n \to K_{n+1}$ for all $n$

ii) Given a CW complex $X$, a *suspension spectrum* is a spectrum where each $K_n$ is given by $\Sigma^n X$, and the maps $\Sigma K_n \to K_{n+1}$ by the identity map.

**Remark 3.2.** i) Note this is an immediate generalization of a $\Omega$-spectrum, where the maps $\Sigma K_n \to K_{n+1}$ come from homotopy equivalences $K_n \to \Omega K_{n+1}$.

ii) If $K$ is a spectrum such that each $K_n = K(G, n)$ for a group $G$, $K$ is called the *Eilenberg-MacLane spectrum.*

Recall the following: Let $\{G_i\}_i$ be a directed system of groups such that for every $i \leq j$ there is a homomorphism $f_{ij} : G_i \to G_j$ that respects compositions, with $f_{ii} = id$ for every $i$. The *direct limit* $\varinjlim(G_i)$ is defined to be the quotient of $\bigoplus_i G_i$ by the subgroup generated by all elements of the form $a - f_{ij}(a)$, for $a \in G_i$, $j \geq i$. This object can be defined in a larger generality via an universal property over a directed systems of objects and morphisms in any category.

**Fact 3.3.** Direct limits preserve exact sequences and isomorphisms.

**Definition 3.4.** Let $K$ be a spectrum. We define its $i$-th homotopy group $\pi_i(K) = \varinjlim \pi_{i+n}(K_n)$, where the direct limit is given via the following maps

$$\pi_{i+n}(K_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma K_n) \longrightarrow \pi_{i+n+1}(K_{n+1})$$

where the last map is induced by the map $\Sigma K_n \to K_{n+1}$ that the spectrum structure gives by definition. Note that we use the same notation for homotopy groups of spectra and spaces. It should still be clear from context to what homotopy we are referring to.

**Remark 3.5.** i) Suppose $K$ is the suspension spectrum of a CW-complex $X$. Then, by definition, $\pi_i(K) = \pi_i^s(X)$.

ii) Let $K$ be an arbitrary spectrum. Consider the composition $\pi_{i+n}(K_n) \to \pi_{i+n+j}(K_{n+j})$. This factors through $\pi_{i+n+j}(\Sigma^j K_n)$, as we will see in a picture during the talk. Thus, we have that $\pi_i(K) = \varinjlim \pi_{i+n}^s(K_n)$.

This view of spectra homotopy groups as stable homotopy groups will allow us to use our previous results to prove our final theorem. Let us first define how to construct a new spectrum from a given one and another CW-complex via the smash product.

**Remark 3.6.** Given a spectrum $K$ and a CW-complex $X$, we can construct a new spectrum $X \wedge K$ by putting $(X \wedge K)_n = X \wedge K_n$, with the maps given by $\Sigma(X \wedge K_n) = X \wedge \Sigma K_n \to X \wedge K_{n+1}$, with this last map given by the one the spectra $K$ gives by definition.

**Theorem 3.7.** *i) Let $K$ be a spectrum. The sequence $h_i(X) = \pi_i(X \wedge K)$ forms a reduced homology theory on the category of pointed CW-complexes with basepoint-preserving maps.*

*ii) If $K$ is the Eilenberg-MacLane spectrum, this homology theory agrees with ordinary homology.*

Proof. *i)* follows from the fact that direct limits preserve exact sequences and isomorphisms. Hence, the arguments of Corollary 2.3 still work in this setting.

For *ii)*, let's look at the coefficients of this homology theory. $h_i(S^0) = \varinjlim \pi^s_{i+n}(K(G,n))$, which by definition turn out to be 0 unless $i = 0$. By Fact 1.5, we obtain the result.

As we saw for cohomology in the last talk, it is natural to ask if every homology theory comes from a spectrum. This happens to be true if we replace the wedge sum axiom by the *Direct limit axiom*, that states that for any CW-complex $X$, there is an isomorphism $h_i(X) \cong \varinjlim h_i(X_j)$, where $\{X_j\}_j$ is the set of finite subcomplexes of $X$.