S-Duality

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The goal of this talk is to discuss abstract duality in the context of spectra. The homology and cohomology theories given by a spectrum then can be related using the dual of a spectrum. And as an application, we get an analogon for Brown's representability theorem for homology theories, at least if we restrict our attention to finite CW-complexes.

To set the stage for this abstract duality, we need to construct a version of the stable homotopy category.

1. A construction for the stable homotopy category

The construction below almost certainly will appear somewhat arbitrary. Still, the construction turns out to satisfy an important universal property.

A morphism of spectra $f : E \to F$ is called a stable equivalence if the induced maps on stable homotopy groups $\pi_k(f) : \pi_k(E) \to \pi_k(F)$ are all isomorphisms.

The stable homotopy category $\mathcal{SH}$ has a natural functor $Q : \mathcal{S}p \to \mathcal{SH}$ which is (in some sense) universal among the functors sending stable equivalences to isomorphisms. So we think of $\mathcal{SH}$ as a category in which we have made the stable equivalences into isomorphisms, *universally.*

Let's start with some preliminaries. Let X be a CW-complex with n-cells α_i^n. Then the n-cells of $S^1 \times X$ are of either of the following forms:

- $D_i^n \xrightarrow{(*,\alpha_i^n)} S^1 \times X$
- $D^n \xrightarrow{\cong} D^1 \times D^{n-1} \xrightarrow{\beta \times \alpha_i^{n-1}} S^1 \times X,$

where β denotes the 1-cell for S^1. Thus the space $* \times X \cup S^1 \times *$ is a subcomplex of $S^1 \times X$. Quotienting this space out yields a CW-structure on the (reduced) suspension ΣX, whose $(n+1)$-cells are precisely the n-cells of X (except for the basepoint) and with precisely one 0-cell.

Definition 1.1. A CW-spectrum is a (sequential) spectrum E such that all E_n carry CW-complex structure and the structure maps

$$\Sigma E_n \to E_{n+1}$$

are subcomplex inclusions.

Now, by the previous discussion, the (non-basepoint) k-cells in E_n give us $(k+1)$-cells in E_{n+1}. Call such a non-basepoint k-cell of E_n a stable $(k-n)$-cell. We consider two stable m-cells to be equal, if their suspensions eventually agree. Notice that we can have negative stable cells. A CW-spectrum will be called finite, if it has a finite number of stable cells.

Morphisms in our stable homotopy category are somewhat more involved. We work towards a definition.

Definition 1.2. Let E, F be CW-spectra.

- A morphism of spectra $\iota : E \to F$ is a CW-subspectrum if ι is levelwise a subcomplex inclusion.
- A CW-subspectrum $E' \subset E$ is called cofinal if for each k-cell e^k of E_n, there is an l such that the corresponding $(k+l)$-cell of E_{l+n} is already in E.

Think of a cofinal $E' \subset E$ as a spectrum containing all the cells of E, but possibly only on higher levels.

Definition 1.3. A eventually-defined map of CW-spectra from E to F is a pair (X, f), where X is a cofinal $X \subset E$ and f is a morphism of spectra $X \to F$.

We consider eventually-defined maps of CW-spectra up to equivalence relation. Two maps (X, f), (Y, g) are equivalent, if there is a cofinal $Z \subset X, Y$ such that $f|_Z = g|_Z$.

Given two eventually-defined maps of CW-spectra $(X, f) : E \to F$, $(Y, g) : F \to G$, we wish to compose them. One can show that the preimage $f^{-1}(Y)$ contains a cofinal spectrum Z (8.13 in [**1**]). This means we can define the composition of the two maps of spectra as $(Z, g \circ f|_Z)$.

Now we are ready to construct the *stable homotopy category.*

Definition 1.4. Let E, F be CW-spectra.

- Let f, g be eventually-defined maps $E \to F$. A homotopy from f to g is an eventually-defined map $H : X \wedge I_+ \to Y$ such that $H \circ \iota_0 = f$ and $H \circ \iota_1 = g$.
- Define $[E, F]$ to be the set of eventually-defined maps $E \to F$ up to eventually-defined homotopy. Define the stable homotopy category $\mathcal{SH}$ to have CW-spectra as objects and homotopy classes of eventually-defined maps as morphisms.

The full subcategory of spectra spanned by the CW-spectra $\mathcal{S}p_{CW}$ has a canonical functor $Q : \mathcal{S}p_{CW} \to \mathcal{SH}$. By a version of the Whitehead theorem for spectra ([**1**], Corollary 8.24), it turns out to send stable equivalences to isomorphisms. It even satisfies the universal property informally discussed in the beginning.[1] Every functor $F : \mathcal{S}p_{CW} \to \mathcal{SH}$ sending stable equivalences to isomorphisms factors uniquely through Q, i.e.

$$\begin{array}{ccc} \mathcal{S}p & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle Q}\downarrow & \nearrow_{\exists!} & \\ \mathcal{SH}_{CW} & & \end{array} .$$

We continue by discussing the properties of spectra that we need.

Define the shift functor $sh : \mathcal{S}p \to \mathcal{S}p$ by $(shE)_n := E_{n+1}$. Similarly, sh shifts the structure maps and is defined on morphisms by shifting.

Proposition 1.5. *Let E be a CW-spectrum. The suspension $\Sigma E := E \wedge S^1$ is naturally isomorphic to the shift shE in the stable homotopy category.*

Proof. [**1**], Theorem 8.26. □

[1] In the beginning, we did not restrict to CW-spectra, but here our claim uses $\mathcal{S}p_{CW}$ instead of $\mathcal{S}p$. One can also use $\mathcal{S}p$ if one is willing to approximate by CW-spectra before applying Q, but then one has to state the universal property slightly differently. Essentially the question is whether one wants the localization of a category to be determined *up to isomorphism* or only *up to equivalence.*

Proving this fact is not hard, but surprisingly not obvious. The morphism $E \wedge S^1 \to shE$ defined by twisting and using the structure map is *not* a morphism of spectra, so one has to find a workaround. If instead we were using a more structured version of spectra, like orthogonal or symmetric spectra, we could define such a morphism.

Let E be a spectrum. Define another shift functor $sh^{-1}(E)$ to be $*$ on the 0th level and E_n on the (n+1)st level. The 0th structure map is the unique map and the other structure maps are similarly shifted. Moreover we can make sh^{-1} into a functor. We can use $sh^{-1} : \mathcal{SH} \to \mathcal{SH}$ to define an inverse to $sh : \mathcal{SH} \to \mathcal{SH}$. Thus, the suspension functor Σ has an inverse Σ^{-1}.

Finally we can use this fact to prove that the $[E, F]$ have abelian group structure, for two CW-spectra E,F. By the above remarks,

$$[E, F] \cong [\Sigma^2 E, \Sigma^2 F] \cong [E \wedge S^2, F \wedge S^2].$$

Notice that the pinch map $\rho : S^2 \to S^2 \vee S^2$ gives us a 'diagonal'

$$\Delta : E \wedge S^2 \xrightarrow{E\wedge\rho} E \wedge (S^2 \vee S^2) \cong (E \wedge S^2) \vee (E \vee S^2).$$

giving us with a way to add $f, g : [E \wedge S^2, F \wedge S^2]$ by setting

$$f + g := (f \vee g) \circ \Delta,$$

where $f \vee g \in [(E \wedge S^2) \vee (E \wedge S^2), F \wedge S^2]$ is induced by the universal property of the wedge sum.

One can now show analogously to how it was done for higher homotopy groups, that this yields an abelian group structure on $[E, F]$.

2. E-Homology and E-cohomology

For a CW-spectrum E, we can now define E-homology and E-cohomology.

Definition 2.1. Let X be a CW-spectrum. Define the k-th E-homology of X as the abelian group

$$E_k(X) := [\Sigma^k \mathbb{S}, E \wedge X]$$

and the k-th E-cohomology of X as the abelian group

$$E^k(X) := [\Sigma^{-k}\Sigma^\infty X, E].$$

Also define the suspension isomorphisms in the obvious way.

In can be checked that these actually form (co-)homology theories on $\mathcal{SH}$ (see 8.40. in [**1**]).

These formulas for E-homology and E-cohomology can be rewritten to a more familiar form.

Proposition 2.2. *Let E be a CW-spectrum and let X be a CW-complex. We have an isomorphisms*

- $E_k(\Sigma^\infty X) \cong \pi_k(E \wedge X)$
- $E^k(\Sigma^\infty X) \cong [X, E_k]$ *if E is an Ω-spectrum.*

Proof. We prove the first part by showing that for every spectrum F, we have

$$[\mathbb{S}, F] \cong \pi_0(F). \tag{2.1}$$

Notice that the cofinal subspectra of $\mathbb{S}$ are the spectra $sh^{-k}\Sigma^\infty S^k$. Also notice that a map $f : sh^{-k}\Sigma^\infty S^k \to F$ is given uniquely by it's kth level $f_k : S^k \to F_k$, since we have $f_{k+1} = \sigma_F(\Sigma f_k)$.

So define the map

$$[\mathbb{S}, F] \to \pi_0(F)$$

by sending the class of $f : sh^{-k}\Sigma^\infty S^k \to F$ to $[f_k]$.

By the formula $f_{k+1} = \sigma_F(\Sigma f_k)$, we have that two eventually-defined maps are sent to the same class in the stable homotopy group.

To see that the map is also well-defined on homotopy classes, let $H : sh^{-k}\Sigma^\infty S^k \wedge I_+ \to F$ be a homotopy. This now restricts to a (based) homotopy on the kth level, as required.

The map is also a group homomorphism. So let $f, g : sh^{-k}\Sigma^\infty S^k \to F$ represent two classes. Then

$$\begin{aligned}(f+g)_k &= ((f \vee g) \circ \Delta)_k \\ &= (f_k \vee g_k) \circ \rho \\ &= f_k + g_k.\end{aligned}$$

Surjectivity is clear. For injectivity, let $f : sh^{-k}\Sigma^\infty S^k \to F$ represent a class such that $[f_k] = 0$ in the stable homotopy group. This implies that we have a homotopy $H : S^{l+k} \wedge I_+ \to F_{l+k}$ between $\sigma_F^l(\Sigma^l f_k)$ and the constant map. This extends to a homotopy

$$sh^{-(l+k)}\Sigma^\infty S^{l+k} \wedge I_+ \to F,$$

as desired.

Now 2.1 can be used to show the formula for E-homology.

The second part is an exercise for the reader :). It can also be found as theorem 8.42. in [**1**]. □

3. Abstract duality

In order to define duality abstractly in $\mathcal{SH}$, we need to assume the existence of a well-behaved smash product. This means that we require $(\mathcal{SH}, \wedge, \mathbb{S})$ to form a *symmetric monoidal category*, although the reader only needs to know that we require that we have natural isomorphisms

$$\begin{gathered} X \wedge \mathbb{S} \cong X, \\ (X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z) \\ X \wedge Y \cong Y \wedge X \end{gathered}$$

for CW-spectra X, Y and Z.

We will also require the smash product to be compatible with suspension, i.e. for CW-spectra X and Y:

$$(\Sigma X) \wedge Y \cong \Sigma(X \wedge Y) \cong X \wedge \Sigma Y$$

Since Σ is invertible, one also gets

$$(\Sigma^{-1} X) \wedge Y \cong \Sigma^{-1}(X \wedge Y) \cong X \wedge \Sigma^{-1} Y,$$

which one can see by applying Σ to the whole equation. It now follows that

$$(\Sigma^k X) \wedge Y \cong \Sigma^k(X \wedge Y) \cong X \wedge \Sigma^k Y,$$

for $k \in \mathbb{Z}$.

We will also need the symmetric monoidal closed structure on $\mathcal{SH}$. This means that we have a bifunctor F that is contravariant in its first argument and contravariant in its second argument such that we have a natural isomorphism

$$[X \wedge Y, Z] \cong [X, F(Y, Z)]$$

for CW-spectra X,Y and Z. Think of $F(X, Y)$ as the spectrum of maps from X to Y.

The construction of the smash product in Switzer ([**1**] page 254) is quite involved and the construction only yields a symmetric monoidal category after passing to the stable homotopy category. It does not seem to be possible to give a direct construction of a symmetric monoidal smash product on the category of spectra. However, symmetric and orthogonal were designed to fix this deficiency and have a relatively simple construction for the smash product, while still yielding the usual stable homotopy category.

Definition 3.1. A dual of a CW-spectrum X is a CW-spectrum Y equipped with maps $\epsilon : X \wedge Y \to \mathbb{S}$ and $\eta : \mathbb{S} \to Y \wedge X$ such that the following compositions yield the canonical morphisms:

$$\begin{array}{ccc} \mathbb{S} \wedge Y & \xrightarrow{\eta \wedge Y} & Y \wedge X \wedge Y \\ & \searrow & \downarrow{\scriptstyle Y \wedge \epsilon} \\ & & Y \wedge \mathbb{S} \end{array}$$

$$\begin{array}{ccc} X \wedge \mathbb{S} & \xrightarrow{X \wedge \eta} & X \wedge Y \wedge X \\ & \searrow & \downarrow{\scriptstyle \epsilon \wedge X} \\ & & \mathbb{S} \wedge X \end{array}$$

For a dual Y of X, we get an adjunction $\wedge X \vdash \wedge Y$. By uniqueness of adjoints, we get

$$Z \wedge Y \cong F(X, Z)$$

for all CW-spectra Z and in fact such an isomorphism that is natural in Z is equivalent to Y being a dual for X.

In particular, we get $Y \cong F(X, \mathbb{S})$. It follows that the dual of X is unique, so if X has a dual, we write $DX := F(X, \mathbb{S})$ for this unique dual. This formula also lets us make D into a functor by using the contravariant functor structure of F.

Notice that if we have maps $\epsilon : X \wedge DX \to \mathbb{S}$ and $\eta : \mathbb{S} \to DX \wedge X$, we can use the twist to get maps $\epsilon' : DX \wedge X \to \mathbb{S}$ and $\eta' : X \wedge DX \to \mathbb{S}$. The reader can check that these prove that X is a dual of DX. Thus we get the formula $D^2X \cong X$.

Why do we care about duals? It turns out that the homology of the dual is intimately related to the cohomology of the original space.

Theorem 3.2. *Let E be a spectrum. Then we have*

$$E_k(DX) \cong E^{-k}(X)$$

for all CW-spectra X.

Proof.

$$\begin{aligned} E_k(DX) &= [\Sigma^k \mathbb{S}, E \wedge DX] \\ &\cong [\Sigma^k X, E] \\ &= E^{-k}(X) \end{aligned}$$

□

Example 3.3. Define $\mathbb{S}^p := sh^p \mathbb{S}$ to be the p-dimensional sphere, for $p \in \mathbb{Z}$. For $p \geq 0$, we get $\mathbb{S}^p \cong \Sigma^\infty S^p$.

Now notice that for $p \in \mathbb{Z}$, we get

$$\begin{aligned} X \wedge \mathbb{S}^p &\cong X \wedge \Sigma^p \mathbb{S} \\ &\cong \Sigma^p X \wedge \mathbb{S} \\ &\cong \Sigma^p X. \end{aligned}$$

Since Σ^p and Σ^{-p} are inverses to each other, they can be made into an adjunction $\Sigma^p \vdash \Sigma^{-p}$. It follows that $\wedge \mathbb{S}^p \vdash \wedge \mathbb{S}^{-p}$ and thus $D\,\mathbb{S}^p \cong \mathbb{S}^{-p}$.

Duals of suspensions can be easily computed:

PROPOSITION 3.4. *For a dualizable CW-spectrum X, we have a dual $D\Sigma^p X \cong \Sigma^{-p} DX$.*

PROOF. Notice that the adjunctions $\wedge X \vdash \wedge DX$ and $\Sigma^p \vdash \Sigma^{-p}$ compose to one adjunction with left adjoint $\Sigma^p(- \wedge X) \cong \wedge \Sigma^p X$ and right adjoint $(\Sigma^{-p} -) \wedge DX \cong \wedge \Sigma^{-p} DX$. The claim follows. ☐

PROPOSITION 3.5. *For dualizable CW-spectra $X_1, \cdots X_n$, we have a dual $D \bigvee X_i \cong \bigvee DX_i$.*

PROOF. One can directly check that one has an adjunction $\wedge \bigvee X_i \vdash \wedge \bigvee DX_i$.

$$\begin{aligned}[Y \wedge \bigvee X_i, Z] &\cong [\bigvee (Y \wedge X_i), Z] \\ &\cong \prod [Y \wedge X_i, Z] \\ &\cong \prod [Y, Z \wedge DX_i] \\ &\cong [Y, \bigvee (Z \wedge DX_i)] \\ &\cong [Y, Z \wedge \bigvee DX_i]\end{aligned}$$

Notice that we used that finite products are finite coproducts in $\mathcal{SH}$. One can show that the canonical morphism of spectra from the coproduct to the product is a stable equivalence to show that. Also, the smash product distributes over wedges since it is left-adjoint. ☐

THEOREM 3.6. *Let $f : X \to Y$ be a morphism between dualizable CW-spectra. Then we can give the dual of the cofiber of f by $D(Cf) \cong \Sigma^{-1} C(Df)$. Moreover the dual of a cofiber sequence $X \to Y \to Z$ is a cofiber sequence $DX \leftarrow DY \leftarrow DZ$.*

PROOF. For this proof we use that $\mathcal{SH}$ is a triangulated category. We will also use that the smash product $\wedge$ and the hom-spectrum F preserve exact triangles (i.e. 3.19. in [**2**]).

So notice that we have exact triangles $X \to Y \to Cf$ and $C(Df) \leftarrow DX \leftarrow DY$. By suitably rotating, we get a diagram

$$\begin{array}{ccccccc} F(X,W) & \longleftarrow & F(Y,W) & \longleftarrow & F(Cf,W) & \longleftarrow & \Sigma F(\Sigma X, W) \\ \cong\downarrow & & \cong\downarrow & & \exists\,\vdots & & \downarrow\cong \\ W \wedge DX & \longleftarrow & W \wedge DY & \longleftarrow & W \wedge \Sigma^{-1} CDf & \longleftarrow & \Sigma W \wedge \Sigma^{-1} DX \end{array}$$

which lets us see that we have a morphism $F(Cf, W) \to W \wedge \Sigma^{-1} C(Df)$. By applying $[Z, -]$ to the whole diagram and using the 5-lemma, we see that this map even induces an isomorphism $[Z, F(Cf, W)] \cong [Z, W \wedge \Sigma^{-1} CDf]$. It finally follows that

$$[Z \wedge W, Cf] \cong [Z, W \wedge \Sigma^{-1} CDf],$$

proving the claim[2].

To see that duals preserve cofiber sequences, look at the triangle

$$DX \leftarrow DY \leftarrow \Sigma^{-1} C(Df) \cong DCf.$$

☐

Finally, we get some payoff for our hard labour.

THEOREM 3.7. *Every finite CW-spectrum has a dual.*

[2]Naturality in W follows from the fact that this map commutes with certain other maps when applying $[Z, -]$ to the whole diagram above.

PROOF. Let X be a finite CW-complex. Recall that taking the cofiber of the cell attachments $f : \bigvee S^n \to X^{(n)}$ gives us $X^{(n+1)}$. Thus, since wedge sums and cofibers preserve dualizability, the spectrum $\Sigma^\infty X$ has a dual.

Now notice that every finite CW-spectrum X has a cofinal subspectrum of the form $sh^{-k}\Sigma^\infty X'$ for a finite CW-complex X' for $0 \leq k$. Thus in $\mathcal{SH}$ we get $X \cong \Sigma^{-k}\Sigma^\infty X'$. Thus by our compatibility with suspensions, we get our claim for all finite CW-spectra. □

We wish to prove an analogous result to Brown representability for homology theories. Let $\mathcal{H}$ denote the homotopy category of CW-complexes, i.e. maps are continuous functions up to homotopy. Let $\mathcal{H}_{fin}$ denote the full subcategory given by the finite CW-complexes and let $\mathcal{SH}_{fin}$ be the full subcategory of $\mathcal{SH}$ given by the finite CW-spectra.

THEOREM 3.8. *Let $h_* : \mathcal{H}_{fin} \to Ab$ be a homology theory. Then there exists a CW-spectrum E such that $h_* \cong E_*$.*

PROOF. First, we extend h_* to a homology theory on $\mathcal{SH}$ by setting

$$h_n(E) := colim_k h_{k+n}(E_k)$$

for any CW-spectrum E.

Clearly the colimit of the suspension isomorphisms gives a valid suspension isomorphism again. To see that h_n actually defines a functor on $\mathcal{SH}$, we need to show it sends stable equivalences to isomorphisms. By the Freundenthal suspension theorem, a stable equivalence between CW-complexes $f : \Sigma^\infty X \to \Sigma^\infty Y$ is a weak equivalence for $k > n$ for some n. Thus, we get that on the colimits, f also induces an isomorphism $h_n(\Sigma^\infty X) \to h_n(\Sigma^\infty Y)$. The suspension isomorphism gives us that h_n sends *all* stable equivalences to isomorphisms. This proves that h_n actually factors through $\mathcal{SH}$.

Since coproducts commute with colimits, the wedge axiom holds.

For exactness, let $f : X \to Y$ be a map and Cf its cofiber. Levelwise, we get exact sequences

$$h_{k+n}(X_k) \to h_{k+n}(Y_k) \to h_{k+n}(Cf_k).$$

Since filtered colimits preserve exact sequences, the exactness axiom also holds.

Now we define a cohomology theory on $\mathcal{SH}_{fin}$ we wish to represent:

$$h^n(X) := h_{-n}(DX)$$

We check the axioms for generalized cohomology. First, the suspension isomorphism:

$$\begin{aligned} h^{n+1}(\Sigma X) &\cong h_{-(n+1)}(D\Sigma X) \\ &\cong h_{-(n+1)}(\Sigma^{-1}DX) \\ &\cong h_{-n}(DX) \\ &\cong h^n(X) \end{aligned}$$

Now the wedge axiom holds since all interesting wedge sums are finite and finite wedge sums commute with duals. The dual of a cofiber sequence is a cofiber sequence, which yields the correct exact sequence in homology.

Thus, Brown's representability theorem yields a spectrum E such that $h^n(\Sigma^\infty X) \cong E^n(\Sigma^\infty X)$ for all finite CW-complexes X.[3]

Since every for CW-spectrum X we have a CW-complex X' such that $X \cong \Sigma^k\Sigma^\infty X'$, the cohomology of X is uniquely determined by the cohomology of X'.

[3] This really only should work if the domain of the cohomology theory are all CW-complexes, but the way [**3**] proves this by first constructing a π^* universal CW-complex, makes it obvious that the statement we need also holds.

Therefore two cohomology theories that are the same on $\mathcal{S}_{fin}$ are also the same on $\mathcal{SH}_{fin}$.

Using this, we get that $h^n(X) \cong E^n(X)$ for all CW-spectra X. Finally, we can compute

$$\begin{aligned} h_n(X) &\cong h_n(D^2X) \\ &\cong h^{-n}(DX) \\ &\cong E^{-n}(DX) \\ &\cong E_n(X), \end{aligned}$$

as desired. □

EXAMPLE 3.9. We can embed every smooth n-manifold X into an $\mathbb{R}^m$. Let $N_X\mathbb{R}^m$ denote the normal bundle of X in R^m. One can show (using Alexander duality, see [**1**] 14.43.) that we have a dual $DX_+ \cong \Sigma^{-m}Th(N_X\mathbb{R}^m)$. For a CW-spectrum E this implies that

$$E_k(X_+) \cong E^{-k}(\Sigma^{-m}Th(N_X\mathbb{R}^m)) \cong E^{m-k}(Th(N_X\mathbb{R}^n)).$$

In particular, if X is orientable, we can use the Thom isomorphism to get

$$\begin{aligned} H_k(X_+) &\cong E^{m-k}(Th(N_X\mathbb{R}^n)) \\ &\cong E^{m-k-(m-n)}(X_+) \\ &\cong E^{n-k}(X_+) \end{aligned}$$

in ordinary (co-)homology. This could be used as a proof for Poincare-duality.

References

[1] Robert M Switzer. *Algebraic topology - homotopy and homology.* Springer, 2002.
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[3] Allen Hatcher. *Algebraic topology.* Cambridge University Press, 2002.