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Talk 3: Postnikov towers

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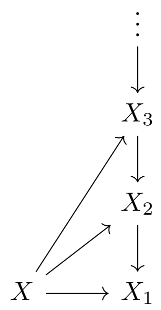
1 Existence and uniqueness

A Postnikov tower of a space is a decomposition into spaces where higher homotopy groups are killed. Intuitively it is some sort of approximation of the space. The main application of Postnikov towers in this seminar is obstruction theory, which will be discussed in talk 4 .

Definition 1.1. A *Postnikov tower* for a path-connected space X is a commutative diagram



such that

(P1) the map $X \to X_n$ induces an isomorphism on π_i for $i \leq n$.

(P2) $\pi_i(X_n) = 0$ for $i > n$.

Example 1.2. Let X be a $K(G,n)$ space (e.g., $X = S^1$). Then a Postnikov tower of X consists of spaces

$$X_i = \begin{cases} K(G,n) & \text{if } i \geq n, \\ * & \text{else.} \end{cases}$$

with maps either constant or the identity.

Proposition 1.3. Every connected CW complex has a Postnikov tower.

The following lemma will be used in the proof of the above proposition.

Lemma 1.4 (Extension lemma)**.** Let (X,A) be a CW pair, $f\colon A \to Y$ a map with Y path-connected. Assume $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n. Then f extends to a map $X \to Y$.

Proof. Assume inductively that f has been extended over the $(n-1)$-skeleton. Then an extension over an n-cell exists if and only if the composition of the cell's attaching map $S^{n-1} \to X$ with $f\colon X^{n-1} \to Y$ is nullhomotopic. □

Proof. (Proposition 1.3). Let $n \geq 1$ and X a connected CW complex. Construct CW complex X_n from X by first attaching cells of dimension $n+2$ to cellular maps $f\colon S^{n+1} \to X$ that generate $\pi_{n+1}(X)$. This makes π_{n+1} trivial. Next attach cells of dimension $n+3$ to cellular generators of π_{n+2} of the newly formed space. Continue this procedure on to kill all higher homotopy groups.

To obtain the maps in the tower, apply the extension lemma on (X_{n+1}, X) and the inclusion $X \to X_n$. Here $X_{n+1} \setminus X$ only has cells of dimension greater than n as X_{n+1} is obtained from X by attaching cells of dimension greater than $n+2$. Moreover, $\pi_i(X_n) = 0$ for all $i > n$, thus the inclusion $X \to X_n$ extends to $X_{n+1} \to X_n$. □

Remark 1.5. For a connected CW complex the Postnikov tower is unique up to homotopy equivalence. As a reference use Hatcher Corollary 4.19.

The next proposition states that every space with a Postnikov tower has an equivalent Postnikov tower where all maps in the tower are fibrations. This is nice as fibrations have for example a long exact sequence in homotopy groups. Furthermore, the fibers of these fibrations turn out the be Eilenberg-MacLane spaces.

Proposition 1.6. For every space X that has a Postnikov tower, there is a Postnikov tower of X satisfying

(P3) the maps $X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$.

Proof. Recall that for any map $f\colon A \to B$ there exists a homotopy equivalence $A \to C$ and a fibration $C \to B$ such that the composition is f. Use this construction to replace $X_2 \to X_1$ by a fibration $X_2' \to X_1$. Next inductively replace the map $X_n \to X_{n-1} \to X_{n-1}'$ by a fibration $X_n' \to X_{n-1}'$.

It remains to show that the fibers are Eilenberg-MacLane spaces. Consider the long exact sequence of the fibration $F_n \to X_n \to X_{n-1}$

$$\cdots \to \pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1}) \to \pi_i(F_n) \to \pi_i(X_n) \to \pi_i(X_{n-1}) \to \cdots$$

Inspecting this sequence shows that $F_n = K(\pi_n(X), n)$. □

To retrieve a space from its Postnikov tower a limit process is needed. Recall that the *inverse limit* of a sequence of spaces $\cdots \to X_2 \to X_1$ is a subspace of $\prod X_n$ consisting of points $x_n \in X_n$ such that x_n is send to x_{n-1} under the map $X_n \to X_{n-1}$. The inverse limit is denoted $\varprojlim X_n$. The same definition holds for groups.

Theorem 1.7. For the Postnikov tower of a connected CW complex X, the natural map (from the universal property) $X \to \varprojlim X_n$ is a weak homotopy equivalence.

The next lemma states when the homotopy group commutes with the inverse limit. This will be used in proving the above theorem. We omitted it during the talk, as a reference see Hatcher Proposition 4.67.

Lemma 1.8. Let $\cdots \to X_2 \to X_1$ be a sequence of fibrations. Then

(i) The natural map (from the universal property) $\lambda\colon \pi_i(\varprojlim_n X_n) \to \varprojlim_n \pi_i(X_n)$ is surjective.

(ii) If $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ is surjective for large enough n, then λ is injective.

Proof. (Theorem 1.7). First of all, for $n \geq i$ there is an isomorphism $\pi_i(X) \to \pi_i(X_n)$ by (P1). Thus $\pi_i(X) \to \varprojlim_n \pi_i(X_n)$ is an isomorphism by the universal property. Secondly, $\pi_i(X_n) \to \pi_i(X_{n-1})$ is an isomorphism for $n > i$ (as they are both isomorphic to $\pi_i(X)$). Thus by Lemma 1.8 the map $\lambda\colon \pi_i(\varprojlim_n X_n) \to \varprojlim_n \pi_i(X_n)$ is an isomorphism. Finally, two out of three maps in the commutative diagram

$$\begin{array}{ccc} \pi_i(X) & \longrightarrow & \pi_i(\varprojlim_n X_n) \\ & \cong\searrow \quad \swarrow\cong & \\ & \varprojlim_n \pi_i(X_n) & \end{array}$$

are isomorphisms, thus it follows that the third map is also an isomorphism. □

2 k-invariant

Given a connected CW complex we saw how to construct it's Postnikov tower. In this section we investigate whether given an infinite sequence of abelian groups $G_1, G_2, \ldots$, can we construct a Postnikov tower such that $K(G_n, n)$ is the homotopy fiber of the map $X_n \to X_{n-1}$. In other words, in which cases does knowing all the homotopy groups determine the space uniquely up to weak homotopy equivalence?

We start with an useful proposition.

Proposition 2.1. Let $p\colon E \to B$ be a fibration, and let $f, g\colon A \to B$ be homotopic. Then the pullbacks $f^*(E)$ and $g^*(E)$ along the fibration are homotopy equivalent.

Proof. Let $H\colon A \times I \to B$ be a homotopy from f to g. As $p\colon E \to B$ is a fibration, it follows that the pullback $H^*(E) \to A \times I$ is also a fibration (see Talk 1 Proposition 2). The fibers of a fibration are all homotopy equivalent (see Talk 1 Proposition 4). Here $f^*(E)$ is the fiber of $A \times \{0\}$ and $g^*(E)$ the fiber of $A \times \{1\}$. □

To answer the above question, the idea is to extend the fibration $K(G_n, n) \to X_n \to X_{n-1}$ in the Postnikov tower to a sequence of fibrations

$$K(G_n, n) \to X_n \to X_{n-1} \to K(G_n, n+1).$$

The motivation for this is that now X_n is the homotopy fiber of $X_{n-1} \to K(G_n, n+1)$. And such homotopy fiber is the same as the pullback of the path fibration $K(G_n, n+1)^I \to K(G_n, n+1)$. Thus by Proposition 2.1 the homotopy type of X_n only depends on the homotopy class of $X_{n-1} \to K(G_n, n+1)$. And $[X_{n-1}, K(G_n, n+1)]$ is isomorphic to $H^{n+1}(X_{n-1}; G_n)$ (see Talk 2 Theorem 4). Thus the homotopy type of X_n only depends on a cohomology class if the fibration can be extended to a sequence of fibrations. In a more general setting we have the following definition.

Definition 2.2. A fibration $F \to E \to B$ is *principal* if there exists a commutative diagram

$$\begin{array}{ccccccc}
 & & F & \longrightarrow & E & \longrightarrow & B \\
 & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\
\Omega B' & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & B'
\end{array}$$

such that the second row is a sequence of fibrations and the vertical maps are weak homotopy equivalences.

In the above setting principal means there is a commutative diagram

$$\begin{array}{ccccccc}
K(G_n, n) & \longrightarrow & X_n & \longrightarrow & X_{n-1} & & \\
\downarrow\simeq & & \| & & \| & & \\
\Omega K(G_n, n+1) & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & K(G_n, n+1)
\end{array}$$

Recall that indeed $\Omega K(G_n, n+1) \simeq K(G_n, n)$. Thus if all fibrations in a Postnikov tower are principal, then

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(G_3, 3) & \longrightarrow & X_3 & \xrightarrow{k_3} & K(G_4, 5) \\
 & & \downarrow & & \\
K(G_2, 2) & \longrightarrow & X_2 & \xrightarrow{k_2} & K(G_3, 4) \\
 & & \downarrow & & \\
K(G_1, 1) & = & X_1 & \xrightarrow{k_1} & K(G_2, 3)
\end{array}$$

So inductively, X_{n+1} can be constructed as the homotopy fiber of $k_n \colon X_n \to K(G_{n+1}, n+2)$. Now notice that the groups $G_1, G_2, \ldots$ together with the maps $k_1, k_2, \ldots$ unique determine a space X up to weak homotopy equivalence by inductively constructing all X_n and taking the inverse limit. Therefore the k_n's are important.

Definition 2.3. The n^{th} *k-invariant* of X is the class in $H^{n+2}(X_n; \pi_{n+1}(X))$ that is equivalent to the map k_n.

These k-invariants show how to construct X from the $K(\pi_n(X), n)$'s.

Example 2.4. Consider the space $S^1 \times \mathbb{CP}^\infty$ (notice this is $K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 2)$). The Postnikov tower is

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
* = K(0, 3) & \longrightarrow & S^1 \times \mathbb{CP}^\infty & \xrightarrow{k_3} & K(0, 5) = * \\
 & & \downarrow & & \\
\mathbb{CP}^\infty = K(\mathbb{Z}, 2) & \longrightarrow & S^1 \times \mathbb{CP}^\infty & \xrightarrow{k_2} & K(0, 4) = * \\
 & & \downarrow & & \\
K(\mathbb{Z}, 1) & = & S^1 & \xrightarrow{k_1} & K(\mathbb{Z}, 3)
\end{array}$$

Now $[S^1, K(\mathbb{Z}, 3)] \cong H^3(S^1; \mathbb{Z}) = 0$, thus k_1 is nullhomotopic. Furthermore, clearly all k_i are nullhomotopic for $i \geq 2$. In general, if X is the product of Eilenberg-MacLane spaces, then all k-invariants are zero. The converse is also true.

The next result states tells us when a Postnikov tower is principal, that is, when we can build a space from its k-invariants.

Theorem 2.5. A connected CW complex X has a Postnikov tower of principal fibrations $\iff$ the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geq 2$ (i.e., the space X is *simple*).

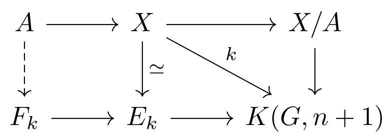
Lemma 2.6. Let (X, A) be a CW pair with X and A connected such that the homotopy fiber of the inclusion is $K(G, n)$. Then the following are equivalent

(i) there exists a fibration $F \to E \to B$ and a map $(X, A) \to (E, F)$ inducing weak homotopy equivalences $X \to E$ and $A \to F$.

(ii) the action of $\pi_1(A)$ on $\pi_{n+1}(X, A)$ is trivial.

Proof. $(i) \implies (ii)$: the action of $\pi_1(A)$ on $\pi_{n+1}(X, A)$ is trivial if and only if $\pi_1(F)$ acting on $\pi_{n+1}(E, F)$ is trivial. And this statement always holds for fibrations as the image of the action is send to zero under the isomorphism $p_* \colon \pi_{n+1}(E, F) \to \pi_{n+1}(B, b_0)$ as for $\gamma \in \pi_1(F)$ we have $p_*(\gamma) = c_{b_0} \in \pi_1(b_0)$ is nullhomotopic.

$(ii) \implies (i)$: note that $\pi_i(X, A)$ is isomorphic to π_{i-1} of the homotopy fiber of the inclusion $A \to X$ which is by assumption $K(G, n)$. Thus $\pi_i(X, A) \cong 0$ except $\pi_{n+1}(X, A) \cong G$. Thus the pair (X, A) is n-connected, hence X/A is n-connected, thus by Hurewicz $\pi_{n+1}(X/A) \cong H_{n+1}(X/A)$. Furthermore, $\pi_1(A)$ acts trivially on $\pi_{n+1}(X, A)$, hence by relative Hurewicz $\pi_{n+1}(X, A) \cong H_{n+1}(X, A)$. Now $H_{n+1}(X, A) \cong H_{n+1}(X/A)$, thus the quotient map $X \to X/A$ induces isomorphism on π_{n+1}. The quotient space X/A is n-connected by the above observation, now we can attaching cells of dimension $\geq n + 3$ to make X/A a $K(G, n + 1)$. This results in a map $X/A \to K(G, n + 1)$. Define k as the composition of the quotient and this map. Next replace k by a fibration. This results in commutative diagram



The second row is a fibration where $E_k \to K(G, n+1)$ replaces k, hence $X \to E_k$ is a homotopy fibration. Furthermore, $A \to F_k$ is a weak homotopy equivalence by inspecting the map of long exact sequences induced by $(X, A) \to (E_k, F_k)$ and applying the 5-lemma. ☐

Proof. (Theorem 2.5). Consider the long exact sequence of the pair (X_{n-1}, X_n) (here X_n is seen as the homotopy fiber of $X_{n-1} \to K(G_n, n + 1)$),

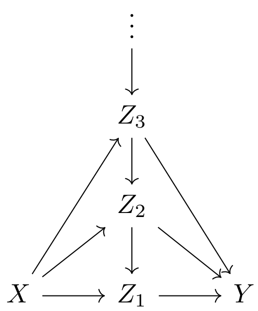
$$\cdots \to \pi_{n+1}(X_{n-1}) \to \pi_{n+1}(X_{n-1}, X_n) \to \pi_n(X_n) \to \pi_n(X_{n-1}) \to \cdots$$

The map $\partial \colon \pi_{n+1}(X_{n-1}, X_n) \to \pi_n(X_n)$ is an isomorphism respecting the action of $\pi_1(X_n) \cong \pi_1(X)$. Thus the action of $\pi_1(X)$ on $\pi_n(X)$ can be identified with the action of $\pi_1(X_n)$ on $\pi_{n+1}(X_{n-1}, X_n)$. Applying Lemma 2.6 gives the required equivalence. ☐

3 Moore-Postnikov tower

A Moore-Postnikov tower is a natural generalization of a Postnikov tower.

Definition 3.1. Let $f \colon X \to Y$ be a map between path-connected spaces. A *Moore-Postnikov tower* for f is a commutative diagram



with each composition $X \to Z_n \to Y$ homotopic to f such that

(MP1) the map $X \to Z_n$ induces an isomorphism on π_i for $i < n$ and a surjection for $i = n$.

(MP2) the map $Z_n \to Y$ induces an isomrophism on π_i for $i > n$ and an injection for $i = n$.

(MP3) the map $Z_{n+1} \to Z_n$ is a fibration with fiber a $K(\pi_n(F), n)$ where F is the homotopy fiber of f.

Remark 3.2. A Moore-Postnikov tower specializes to a Postnikov tower by taking Y to be a point and then setting $X_n = Z_{n+1}$ discarding the space Z_1 which has trivial homotopy groups.

The next theorem shows that everything we have done also holds in the general case. We omit the proof and refer to Hatcher Theorem 4.71.

Theorem 3.3. Every map $f: X \to Y$ between connected CW complexes has a Moore-Postnikov tower, which is unique up to homotopy equivalence. Moreover, a Moore-Postnikov tower of principal fibrations exists $\iff$ $\pi_1(X)$ acts trivially on $\pi_n(M_f, X)$ for all $n \geq 2$, where M_f is the mapping cylinder of f.

Remark 3.4. Taking the map $* \to Y$ results in a tower where Z_n is n-connected and $Z_n \to Y$ induces an isomorphism on π_i for $i > n$. Such a tower is called a *Whitehead tower* of Y.