Talk 1: Fibrations

Calle Sönne

October 10, 2022

Introduction

In todays talk we will introduce the notion of fibrations, a sort of dual notion to cofibrations which we discussed last term. The structure will be as follows:

1. We introduce fibrations and get a basic feel for their properties
2. We do an explicit example, using fibrations to compute non-trivial homotopy groups
3. Finally we show that, similar to the case of cofibrations, every morphism can be turned into a fibration up to homotopy.

The talk mostly follows the course notes of last years Algebraic Topology 1 course, which are available online at the following link:
`https://github.com/alvgutcac/AT1/blob/main/Algebraic_Topology_I___Stefan_Schwede__Bonn__Winter_2021.pdf`.
In this document I have only included sketches of proofs. For rigorous proofs I will be refering to the notes.

1 Fibrations

In order to define fibrations we first define the homotopy lifting property (HLP), a dual notion to the homotopy extension property (HEP) introduced last term in Topology II.

Definition 1 (HLP)**.** *We say that a continuous map $p : E \to B$ has the HLP for a space X if all solid diagrams below admits a lifting, given by the dotted arrow:*

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ \big\downarrow{\scriptstyle i_0} & \overset{\tilde{H}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & B \end{array}$$

Here $i_0(x) := (x, 0)$, and f, H are any maps such that the diagram commutes. In other words, if given any homotopy $H : X \times I \to B$, and a lift $f : X \to E$ of the initial map $H(-, 0)$ we can lift the whole homotopy H.

One also has a relative version of the above:

Definition 2 (Relative HLP)**.** *We say that $p : E \to B$ has the relative HLP for a pair (X, A) if all solid diagrams below admits a lifting given by the dotted arrow:*

$$\begin{array}{ccc} X \cup A \times I & \xrightarrow{f} & E \\ \big\downarrow{\scriptstyle i_0} & \overset{\tilde{H}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & B \end{array}$$

With this we are ready to define fibrations:

Definition 3 (Fibrations)**.** *We say that $p : E \to B$ is **Hurewicz**, resp. **Serre fibration** if it has the HLP for all spaces X, resp. CW-complexes X*

Before we do anything fun we will state some basic (technical) properties of fibrations. We will not give the proofs as they are not particularly enlightening.

Proposition 1. *For a map $p : E \to B$ TFAE:*

1. *p is a Serre fibration*
2. *p has the absolute HLP for D^n for all n*
3. *p has the relative HLP for $(D^n, \partial D^n)$ for all n*
4. *p has the relative HLP for all relative CW-complexes*

Sketch. 3 $\implies$ 4 follows by induction + colimits from attaching single cells. So essentially, it is true because CW complexes are built by cells D^n. The rest of the equivalences are either immediate or just technical. For a full proof see Lemma II.4 in the notes. $\square$

Proposition 2 (Pullback)**.** *Both Hurewicz and Serre fibrations are stable under pullback. In other words, if in the following cartesian diagram p is a Hurewicz (resp. Serre) fibration, then so is p':*

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & Z \end{array}$$

Sketch. Lift the homotopy via p first, and then use the universal property of pullbacks to further lift the homotopy to $X \times_Z Y$. For a full proof see Theorem II.9 in the notes. $\square$

We already know an example of Hurewicz fibrations, namely covering spaces, which we know to satisfy the homotopy lifting property for all spaces, and hence is an example of a Hurewicz fibration. This generalizes to the fact that all fiber bundles are Serre fibrations, which already gives us a wide class of examples:

Proposition 3. *All fiber bundles are Serre fibrations*

Sketch. The proof is essentially the same as for covering spaces. It is divided into two parts:

1. The trivial case: $E \cong B \times F$ where F is the fiber. In this case one can explicitly find a lift: $\tilde{H}(x,t) := (H(x,t), f_2(x))$ where $f = (f_1, f_2)$.
2. The general case: Check HLP for all disks D^n, or equivalently for all I^n. One can now reduce to the trivial case by decomposing I^n into small enough cubes by the Lebesgue lemma and then glue the lifts along intersections.

$\square$

The following proposition gives a good picture for how fibrations look in general, but as we give no proof as it is not used and only intended for intuition:

Proposition 4. *For $p : E \to B$ with B path-connected we get the following:*

1. *If p is a Hurewicz fibration, then all fibers are homotopy equivalent*
2. *If p is a Serre fibration, then all fibers that are CW-complexes are homotopy equivalent.*

A motivating property for considering fibrations, if being able to lift homotopies isn't enough motivation by itself, is that given a fibration $p : E \to B$ and $b \in B$ we get a long exact sequence:

$$\cdots \to \pi_n(p^{-1}(b), e) \to \pi_n(E, e) \to \pi_n(B, b) \xrightarrow{p_*} \pi_{n-1}(p^{-1}(b), e) \to \cdots \to \pi_0(p^{-1}(b), e) \to \pi_0(E, e) \to \pi_0(B, b) \to 0$$

One can see this LES as being induced by the "sequence" $p^{-1}(b) \to E \to B$. For an immediate, trivial, example of the above is the case where $p : E \to B$ is a covering space. Namely, as the fibers are discrete we get that $\pi_n(p^{-1}(b), x) = 0$ for $n \geq 1$ and hence it recovers the familiar statement that covering maps induce isomorphisms on homotopy groups for $n \geq 2$ and an injection for $n = 1$.

The LES in general follows from the following proposition, combined with the standard LES in homotopy:

Theorem 1. *Let $p : E \to B$ be a Serre fibration, $Y \subset B$ a subspace and $x \in p^{-1}(Y)$, then the projection induces an isomorphism:*

$$p_* : \pi_n(E, p^{-1}(Y), x) \cong \pi_n(B, Y, p(x))$$

Proof. The proof in class was mostly via pictures. One shows surjectivity by using the (relative) HLP to lift the loop itself. This is possible by describing the elements as maps $(I, \partial I, J) \to (B, Y, p(x))$. One shows injectivity by lifting the homotopy between two maps. This is slightly trickier than surjectivity, as one needs to make sure that the lift satisfies the right condition around the edges (i.e. that the lifted homotopy is still a homotopy between the two loops). See Lemma II.5 in the notes for a proof and nice pictures.

We now use this sequence to compute some non-trivial homotopy groups. Recall the fiber bundles:

$$\mathbb{C}^\times \to \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n_{\mathbb{C}}$$

As a special case for $n = 1$ we get the Hopf fibration:

$$S^1 \to S^3 \to S^2$$

Here we have restricted the projection from $C^{n+1}\backslash\{0\}$ to the subspace S^3 and used the homeomorphism $\mathbb{P}^1_{\mathbb{C}} \cong S^2$. Now using the associated LES, and since $C^\times \simeq S^1$, we get that for $m \geq 3$:

$$\pi_m(\mathbb{C}^{n+1}\backslash\{0\}) \cong \pi_m(\mathbb{P}^n_{\mathbb{C}})$$

And for $m = 2$ we get an injection. The special case of the Hopf fibration then lets us compute:

$$\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$$

We could not do this so easily last term!

2 Turning maps into fibrations

Our next goal is to show how to turn maps into fibrations, up to homotopy. Recall the "dual" statement that any map can be turned into a cofibration up to homotopy. This relies on factoring through the mapping cylinder Cf. The proof for fibrations is also very similar and relies on a "dual" of the mapping cylinder which we denote by Ef. In order to define Ef we need to introduce path spaces and for this we might as well talk about loop spaces and mapping spaces.

2.1 Mapping spaces

A very useful property of **Top** is the fact that one can endow the set of maps between two topological spaces with a useful topology. In other words, we can make sense of $X^Y := \mathrm{Hom}(X, Y)$ as an object in **Top**. We now define this topology: for every open $O \subset Y$ and $K \subset X$ compact, define:

$$W(K, O) := \{f \in Y^X : f(K) \subset O\}$$

The topology is generated by the above sets as a subbasis. For us, the main use of this topology is that it gives the following adjunction for "compactly generated spaces" (these include CW-complexes and basically all spaces we care about):

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y)$$

This can also be written as an "exponential law":

$$Z^{Y \times X} \cong (Z^Y)^X$$

In complete (CG-space) generality, the above is just a bijection, but for locally compact Hausdorff spaces its actually a homeomorphism! The above is similar to the tensor-hom adjuntion from algebra, which also uses that the set of module homomorphism can be given the structure of a module.

A special instance of the mapping space, is the path space: $\mathrm{Hom}(I, X) =: X^I$. This is what we need to define Ef. Before we return to fibrations, we migt as well talk about the loop space, another very useful construction that arises as a special case of mapping spaces. To define it, we need to go to the category of pointed topological spaces $\mathbf{Top}_*$. The above adjunction descends also to this category, except now we need to take the "pointed product", i.e. the smash product $- \wedge -$, and also only consider pointed maps. The adjunction then becomes (where by abuse of notation the exponential now only consists of pointed maps):

$$\mathrm{Hom}_{\mathbf{Top}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}_*}(X, Z^Y)$$

Again, whenever all spaces are locally compact Hausdorff we get that its even a homeomorphism. For pointed topological spaces, we can also make sense of the "loop space" defined as: $\Omega(X, x_0) := (X, x_0)^{(S^1, 1)}$. This can also be shown to be homeomorphic to the subspace of X^I consisting of loops at x_0. As an instance of the above adjunction for $X = S^n, Y = S^1, Z = Z$ we get the following *bijection*:

$$\pi_{n+1}(Z, z_0) \cong [S^{n+1}, Z] \cong [S^n, Z^{S^1}] \cong \pi_n(\Omega Z, *)$$

This actually turns out to be a group isomorphism (by using the LES of some fibration!), but we will not prove this.

Now we finally define Ef for a morphism $f : X \to Y$. Recall first that the mapping cylinder is defined as the following pushout:

$$\begin{array}{ccc} X & \xrightarrow{i_0} & X \times I \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & Cf \end{array}$$

As a "dual" to this, we then define Ef as the following pullback:

$$\begin{array}{ccc} Ef & \longrightarrow & Y^I \\ \downarrow & \lrcorner & \downarrow{\scriptstyle ev_0} \\ X & \xrightarrow{f} & Y \end{array}$$

Here ev_0 sends a path g to $g(0)$. Thus the points of this space arepaths in Y which have starting point at some $f(x)$ for $x \in X$. We also get maps $X \to Ef \to Y$ given by $x \mapsto (x, \mathrm{const}_{f(x)})$ and $(x, g) \mapsto g(1)$ respectively. Now we can state the final theorem:

Theorem 2. *Every map $f : X \to Y$ can be, naturally and functorially, factored as a homotopy equivalence followed by a fibration as $X \to Ef \to Y$ where the maps are as defined above.*

Sketch. See theorem III.8 in the notes for a full proof. The fact that the first map is a homotopy equivalence is because all points of Ef are paths starting at some $f(x)$ with $x \in X$ and hence one can define the homotopy by simultaneously collapsing all of these to the constant path at $f(x)$.

The reason that the second map is a fibration comes from describing Ef as the following pullback:

$$\begin{array}{ccc} Ef & \longrightarrow & Y^I \\ \downarrow & \lrcorner & \downarrow \\ X \times Y & \longrightarrow & Y \times Y \end{array}$$

Now if one can show that $Y^I \to Y \times Y$ is a fibration we are done, since $Ef \to Y = Ef \to X \times Y \to Y$ and hence as composition and pullback preserves fibrations we are done. To show that $Y^I \to Y \times Y$ is a fibration we write it in the following way: $Y^I \to Y^{\{0,1\}}$ where the map is now induced by restriction from I to the subset $\{0, 1\} \subset I$. Our result then follows by the next lemma: $\square$

Lemma 1. *For any topological space Z and any relative CW-complex (X, A) we get that the map $Z^X \to Z^A$ given by $f \mapsto f|_A$ is a Serre fibration*

Sketch. The key idea is to utilize the mapping space adjunction between $- \times I$ and $(-)^I$ in order to redue the theorem to the fact that (X, A) is a relative CW-complex and hence satisfies the HEP. $\square$