Talk 2: Eilenberg-MacLane spaces

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1 Definition, construction and uniqueness

Definition 1. Fix some natural number n and some group G (abelian if $n \geq 2$). Then a connected space X is called an *Eilenberg-MacLane space of type* $K(G,n)$, or simply a $K(G,n)$, if

$$\pi_i(X) \cong \begin{cases} G & i = n \\ 0 & \text{else.} \end{cases}$$

We actually already know a few examples of $K(G,n)$'s:

Examples:

(1) S^1 is a $K(\mathbb{Z},1)$, since $\pi_1(S^1) \cong \mathbb{Z}$ and all higher homotopy groups vanish.

(2) $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2,1)$, using that the universal cover S^∞ is contractible.

(3) $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$, using the LES associated to the fiber bundle $S^1 \to S^\infty \to \mathbb{CP}^\infty$ and the fact that $S^\infty \simeq *$.

Next we're going to give a general construction of $K(G,n)$ spaces in two steps:

Step 1: Construct an $(n-1)$-connected CW-complex X with dimension $n+1$ and $\pi_n(X) \cong G$: we attach $(n+1)$-cells via basepoint-preserving maps φ_β to a wedge of n-spheres, i.e. we define

$$X := \left(\bigvee_\alpha S^n_\alpha \right) \cup_\beta e^{n+1}_\beta.$$

This CW-complex is $(n+1)$-dimensional and from cellular approximation we get that X is $(n-1)$-connected. Using Hurewicz we have an isomorphism $\pi_n(X) \cong H_n(X)$. By viewing X as the pushout of the diagram

$$\bigvee_\alpha S^n_\alpha \xleftarrow{\amalg \varphi_\beta} \coprod_\beta S^n_\beta \hookrightarrow \coprod_\beta D^{n+1}_\beta$$

and by using a Mayer-Vietoris argument, we get that

$$H_n(X) \cong \left(\bigoplus_\alpha \mathbb{Z} \right) \Big/ \Big\langle [\varphi_\beta] \Big\rangle,$$

where $[\varphi_\beta]$ is just the homotopy class of φ_β. By choosing suitable indices α and attaching maps φ_β, this quotient group can be seen to be isomorphic to our group G if we choose some presentation $G = F/R$.

Step 2: Next, we attach higher dimensional cells to kill off the higher homotopy groups, while leaving the groups below unchanged: choose cellular maps $\varphi_\alpha : S^{n+1} \to X$ generating $\pi_{n+1}(X)$ and use these to obtain a CW-complex $Y := X \cup_\alpha e^{n+2}_\alpha$. Cellular approximation tells us that the inclusion $X \hookrightarrow Y$ induces isomorphisms on π_i for $i \leq n$ and moreover $\pi_{n+1}(Y) \cong 0$, since any element in $\pi_{n+1}(Y)$ is essentially a map $S^{n+1} \to X$, so these are null-homotopic by definition of Y. After infinitely many iterations of step 2 we obtain a space X_n with all the properties of a $K(G,n)$. Define $K(G,n) := X_n$.

Proposition 2. *If two CW-complexes K, K' are both $K(G,n)$ spaces, then $K \simeq K'$.*

The proof of this proposition will rely on the following technical lemma, which we will not prove (this is Hatcher, lemma 4.31):

Lemma 3. *If a CW-complex X is of the form $(\bigvee_\alpha S^n_\alpha) \cup_\beta e^{n+1}_\beta$, for $n \geq 1$, then for all homomorphisms $\psi : \pi_n(X) \to \pi_n(Y)$, where Y is some path-connected space, there exists a map $f : X \to Y$ such that $f_* = \psi$.*

Proof. (of the Proposition) Since homotopy equivalence is an equivalence relation, we can assume that K is the $K(G,n)$ constructed from X as in the above lemma. So by the lemma we get that there exists a map $f : X \to K'$ such that f_* is an isomorphism on π_n. For each e^{n+2} that we attach, we have that the composition $S^{n+1} \to X \to K'$, where the first map is the attaching map of e^{n+2}, is null-homotopic, since $\pi_{n+1}(K') \cong 0$. By a result from Topology I we know that this composition extends to the cell e^{n+2}. Repeating this, we get an extension $\bar{f} : K \to K'$ of f. Since $\bar{f}_*$ is an isomorphism on all homotopy groups, it follows from Whitehead that $\bar{f}$ is a homotopy equivalence. □

2 Main theorem, connection with cohomology

Now we state the main theorem of todays talk:

Theorem 4. *There exists a natural isomorphism $T : [X, K(G,n)]_* \to H^n(X;G)$ for all CW-complexes, G any abelian group and for all $n > 0$. Moreover, T has the form $T([f]) = f^*(\alpha)$ for some fundamental class $\alpha \in H^n(K(G,n);G)$ which is independent of f.*

Definition 5 (Loopspace)**.** The *loopspace* ΩK of a pointed space (K, k_0) is defined as the set $\{f : I \to K \mid f(0) = f(1) = k_0\}$. As discussed in the previous talk, it is topologised by the compact-open topology on K^I.

Definition 6. An Ω*-spectrum* is a sequence of CW-complexes $K_1, K_2, \dots$ together with weak homotopy equivalences (WHE) $K_n \to \Omega K_{n+1}$ for all n.

Before giving the prime example of an Ω-spectrum, let me remind you about the *adjoint relation*

$$[\Sigma X, K]_* \cong [X, \Omega K]_*,$$

where X, K are pointed CW-complexes. In class I gave a quick geometric „proof“ of this relation.

Example. $\{K(G,n)\}_{n\geq 0}$ is an Ω-spectrum: Using the adjoint relation, we just compute

$$\pi_k(\Omega K(G,n+1)) = [S^k, \Omega K(G,n+1)]_* \cong [S^{k+1}, K(G,n+1)]_* = \pi_{k+1}(K(G,n+1)),$$

so it follows from our uniqueness statement, that $K(G,n) \simeq \Omega K(G,n+1)$ and in particular we have a WHE between these spaces.

Up to this point it is not entirely clear why the two sets in the adjoint relation are indeed groups, so let me state the group structure for both:

- $[\Sigma X, K]_*$: Addition is defined similar to that of homotopy groups, namely $f + g$ is the compostion $\Sigma X \to \Sigma X \vee \Sigma X \xrightarrow{f \vee g} K$, where the first map collapses the „equator“.
- $[X, \Omega K]_*$: Addition is given by composition of loops, i.e. $(f+g)(x) := f(x) * g(x)$.

An axiom of a cohomology theory in Hatcher is that it has values in **Ab**, so we have to check whether $[X, K_n]_*$ is abelian, where $K_n := K(G,n)$. From the previous example we have WHE's $K_n \to \Omega K_{n+1}$ and $K_{n+1} \to \Omega K_{n+2}$. Since $\Omega(-)$ is a functor, we get a WHE $K_n \to \Omega^2 K_{n+2}$ and by a result from Topology II we can write $[X, K_n]_* \cong [X, \Omega^2 K_{n+2}]_*$. Since $\Omega^2 K \subseteq (K^I)^I \cong K^{I^2}$, we can view $\Omega^2 K$ as maps $I^2 \to K$ such that ∂I^2 is mapped to k_0 and hence $[X, \Omega^2 K_{n+2}]_*$ is abelian by a similar proof which shows that $\pi_2(K)$ is abelian.

Theorem 7. *If $\{K_n\}$ is an Ω-spectrum, then the functor $X \mapsto h^n(X) := [X, K_n]_*$, $n \in \mathbb{Z}$, defines a reduced cohomology theory on the category of pointed CW-complexes and basepoint-preserving maps.*

Proof. (1) (Homotopy invariance) A map $f: X \to Y$ induces a map $f^*: [Y, K_n]_* \to [X, K_n]_*$ which depends only on the basepoint-preserving homotopy class. It can be checked that f^* is indeed a homomorphism, by replacing K_n with ΩK_{n+1}.

(2) (Wedge sum axiom) Let $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ be the inclusion. We want to show that the map

$$\prod_\alpha i_\alpha^* : h^n\left(\bigvee_\alpha X_\alpha\right) \to \prod_\alpha h^n(X_\alpha)$$

is an isomorphism for all n. But this follows immediately, since a map $\bigvee_\alpha X_\alpha \to K$ is the same as a collection of maps $X_\alpha \to K$.

(3) (LES of a CW-pair (X, A)) Checking this axiom is quite tedious, so the following is only a rough sketch:

- Construct the *Puppe sequence* $A \hookrightarrow X \twoheadrightarrow X/A \xrightarrow{\star} \Sigma A \hookrightarrow \Sigma X \twoheadrightarrow \Sigma(X/A) \to \Sigma^2 A \hookrightarrow \cdots$, where the map $\star$ is given by the composition

$$X/A \to X \cup CA \hookrightarrow (X \cup CA) \cup CX \twoheadrightarrow SA \twoheadrightarrow \Sigma A$$

  The first map in this composition is a homotopy equivalence which we have since CA is contractible (a result from Topology I).

- Next, apply the functor $[-, K]_*$ to the Puppe sequence. We get a LES

$$[A, K]_* \leftarrow [X, K]_* \leftarrow [X/A, K]_* \leftarrow [\Sigma A, K]_* \leftarrow [\Sigma X, K]_* \leftarrow \cdots$$

  It suffices to show exactness at $[X, K]_*$: the restriction of a map $f: X \to K$ to A is equivalent to f extending to a map $X \cup CA \to K$, where we identify $[X/A, K]_* \cong [X \cup CA, K]_*$.

- For a WHE $K \to \Omega K'$ we can extend the above LES a few terms to the left:

$$[A, K']_* \leftarrow [X, K']_* \leftarrow [X/A, K']_* \leftarrow [\Sigma A, K']_* \xleftarrow{\cong} [A, \Omega K']_* \xleftarrow{\cong} [A, K]_* \leftarrow [X, K]_* \leftarrow [X/A, K]_* \leftarrow \cdots$$

This finishes the proof. □

We need one more theorem in order to prove the main theorem for today:

Theorem 8. *If h^* is an unreduced cohomology theory on the category of CW-pairs and if $h^n(\mathrm{pt}) \cong 0$ for $n \neq 0$, then there exists a natural isomorphism $h^n(X, A) \cong H^n(X, A; G)$ for all CW-pairs (X, A) and for all n, where $G := h^0(\mathrm{pt})$. The corresponding statement for such homology theories is also true.*

Proof. We will show the theorem only for homology theories. When considering cohomology the main difference will be that we consider direct products instead of sums, so one has to be careful. To get the idea of the proof across, it suffices to check the homology case: since we have an isomorphism $h_n(X, A) \cong \tilde{h}_n(X/A)$ we only check the absolute case.
We have the two cellular chain complexes

$$\cdots \to h_n(X^n, X^{n-1}) \xrightarrow{d_n} h_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

and

$$\cdots \to H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots .$$

Just as for singular homology, we have $h_n^{\mathrm{CW}}(X) \cong h_n(X)$. The individual groups are isomorphic, since

$$h_n(X^n, X^{n-1}) \cong \bigoplus_{I_n} G \cong H_n(X^n, X^{n-1}).$$

So it remains to show that $d_n = \partial_n$: In the case that $n = 1$ we note that $\Sigma^2 X$ has no 1-cells, so immediately $d_1 = 0 = \partial_1$, since taking suspension gives an isomorphism in any homology theory. In the case that $n > 1$, d_n and ∂_n are incidence matrices, so each entry is essentially the mapping degree of some map between spheres. Take some $f: S^n \to S^n$ with $\deg(f) = m$ (deg in the usual sense) . From an exercise from Topology I (or Hatcher lemma 4.60) we know that f also induces multiplication by m on $h_n(S^n) \cong G$, so indeed $d_n = \partial_n$. □

Now we can finally give the proof of the main theorem:

Proof.
- From theorem 7 we know that $h^n(-) := [-, K(G,n)]_*$ defines a reduced cohomology theory on the category of pointed CW-complexes.
- The reduced version of theorem 8 gives a natural isomorphism $T : [X, K(G,n)]_* \to H^n(X;G)$.
- Set $\alpha := T(\mathbb{1})$, where $\mathbb{1}$ is the identity on $K(G,n)$. Then, using naturality of T, we can write $T([f]) = T(f^*(\mathbb{1})) = f^*(T(\mathbb{1})) = f^*(\alpha)$, as claimed.

□

Corollary 9. *For any space X the map $H^1(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$, $x \mapsto x^2$, is identically zero.*

Proof. By taking a CW-approximation of X, we can assume that X is a CW-complex. Now we can use the main theorem to write $x = f^*(\alpha)$, for some $f : X \to K(\mathbb{Z}, 1)$ and $\alpha \in H^1(K(\mathbb{Z}, 1);\mathbb{Z})$. Then we have by naturality of T that $x^2 = f^*(\alpha)^2 = f^*(\alpha^2) = f^*(0) = 0$, since $\alpha^2 \in H^2(K(\mathbb{Z}, 1);\mathbb{Z}) \cong H^2(S^1;\mathbb{Z}) \cong 0$. □

3 Cross- and cup-product

As in Topology I, we can define a cross-product and afterwards a cup-product for our cohomology theory $h^n(-) = [-, K(G,n)]_*$. Let R be a ring and set $K_n := K(R,n)$. This just means $K_n = K(G_R, n)$ for G_R the underlying abelian group. Consider maps $f : X \to K_m$ and $g : Y \to K_n$ for X, Y pointed CW-complexes. Define $\Phi_{f,g}$ as the composition

$$X \times Y \xrightarrow{(f,g)} K_m \times K_n \xrightarrow{q} K_m \wedge K_n \xrightarrow{\mu} K_{m+n},$$

where μ is defined as follows: we have isomorphisms

$$\begin{aligned}[K_m \wedge K_n, K_{m+n}]_* &\cong H^{m+n}(K_m \wedge K_n; R)\\ &\cong \mathrm{Hom}_R(H_{m+n}(K_m \wedge K_n; R), R)\\ &\cong \mathrm{Hom}_R(R \otimes R, R),\end{aligned}$$

where we used the main theorem, the UCT and Künneth in this order. Now simply define μ to be the unique map $K_m \wedge K_n \to K_{m+n}$ which is mapped to the map $(r \otimes t \mapsto rt)$ under these isomorphisms. We define the cross product by

$$[X, K_m]_* \times [Y, K_n]_* \to [X \times Y, K_{m+n}]_*, \qquad ([f],[g]) \mapsto f \times g := [\Phi_{f,g}].$$

In a similar fashion we define the cup-product as the map

$$[X, K_m]_* \times [X, K_n]_* \to [X, K_{m+n}]_*, \qquad ([f],[g]) \mapsto f \cup g := \Delta_X^*(f \times g) = [\Phi_{f,g} \circ \Delta_X],$$

where $\Delta_X : X \to X \times X$ is the diagonal map. This cup-product satisfies the three properties naturality, graded commutativity and associativity. We will only show how to check naturality: Take some map $\varphi : X \to Y$ between pointed CW-complexes. Then we can write

$$\varphi^*(f \cup g) = [\mu \circ q \circ (f,g) \circ \Delta_Y \circ \varphi] = [\mu \circ q \circ (f \circ \varphi, g \circ \varphi) \circ \Delta_X] = \varphi^*(f) \cup \varphi^*(g),$$

where the second equality follows from a suitable commutative diagram.