Characteristic classes II

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The main focus of this talk will be the Chern classes of a complex vector bundle. Similar to the previous talk, one can define them axiomatically and indeed many of the ideas we used for Stiefel-Whitney classes can be used here analogously. But before we get to the definition right away, we need to recap some definitions.

Throughout this talk all topological spaces are considered to be paracompact and Hausdorff (in other words, we are considering the category of paracompact Hausdorff topological spaces).

Recap

Last time we constructed the Thom isomorphism

$$\Phi : H^i(B;\mathbb{Z}_2) \to H^i(E;\mathbb{Z}_2) \xrightarrow{\smile u} H^{i+n}(E,\dot{E};\mathbb{Z}_2)$$

for an n-dimensional vector bundle $\pi : E \to B$, where $\dot{E}$ is the complement of the zero section in E and $u \in H^n(E,\dot{E})$ is the fundamental class called the Thom class. Note that we stated the Thom isomorphism for $\mathbb{Z}_2$ coeffiecents, though there is a generalization for arbitrary commutative rings. For our purpose, we only want to consider integer coefficients, where one needs the bundle to be oriented. The intuition why this is sufficient, is that we need to fix some choice of the generator in each fiber $H^n(F,\dot{F})$. Further details can be found in [1].

Definition 1 (Euler class)**.** *Let $\pi : E \to B$ be real n-dimensional oriented vector bundle, then define the Euler class $e(E) \in H^n(B)$ as the image of the Thom class $u \in H^n(E,\dot{E})$ under the map*

$$H^n(E;\dot{E}) \to H^n(E) \xrightarrow{(\pi^*)^{-1}} H^n(B)$$

Remark. *The Euler class has some very important properties, which are crucial for the construction of the Chern and the Pontrjagin classes. Here are two of the most important ones for the talk:*

Naturality *Given an orientation preserving bundle map*

$$\begin{tikzcd} \phi^*E \arrow[r, "\Phi"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "\phi"] & B \end{tikzcd}$$

between two real n-dimensional oriented vector bundles, we have the relation

$$e(\phi^*E) = \phi^*e(E).$$

Whitney-sum formula *Two oriented vector bundles* E_1 *and* E_2 *over a space* B *induce a orientation on their Whitney sum* $E_1 \oplus E_2$*, such that*

$$e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$$

Reversed orientation *Reversing the orientation of the vector bundle changes the sign of the Euler class.*

Notation. *For a paracompact Hausdorff space* B*,* $\underline{B}$ *denotes the trivial one dimensional bundle over* B*.*

Remark. **Underlying real vectorbundle** *A complex n-dimensional vector bundle has an underlying 2n-dimensional real vector bundle denoted by* $\pi_{\mathbb{R}}$*. Moreover,* $\pi_{\mathbb{R}}$ *has a canonical induced orientation: For any finite complex vector space* V *choose a basis* $z_1, ..., z_n$ *with* $z_i = x_i + i \cdot y_i$*. Then* $z_1, i \cdot z_1, z_2, ..., z_n, i \cdot z_n$ *forms an ordered basis of the underlying real vector space, which determines an orientation. In fact to see that this orientation is independent on the choice of complex basis, only note, that the linear group* $Gl_n(\mathbb{C})$ *is connected. Now apply this construction to each fiber of the underlying vector bundle.*

Complexification of a real vector bundle *One can complexify a real vector bundle* $\pi : E \to B$ *via the tensor product* $E \otimes_{\mathbb{R}} \mathbb{C}$*. Remember that the tensor product of two total spaces over the same base space is given by taking the tensor product fiberwise.*

Hermitian metric *Similarly to a real vector bundle, any complex vector bundle over a paracompact Hausdorff space admits a hermitian metric, i.e. an euclidean metric on its underlying real vector bundle with the additional property* $|v| = |iv| \ \forall v \in V$*. Remember that an euclidean metric on a real vector bundle is a map*

$$\mu : E \to \mathbb{R},$$

such that its restriction to any fiber is a positive definite and quadratic function.

Example *The tangent space of a complex manifold is a complex vector bundle.*

Now we are able to define the Chern classes of complex vector bundle (cf. [3]).

Definition 2 (Chern class)**.** *Let* $\pi : E \to B$ *be a complex vector bundle. Then the chern classes of this vector bundle are the unique classes* $c_i(E) \in H^{2i}(B)$ *for* $i \geq 0$ *satisfying the following axioms:*

Naturality *Given a complex bundle map*

$$\begin{array}{ccc} \phi^*E & \xrightarrow{\Phi} & E \\ \downarrow & & \downarrow \\ N & \xrightarrow{\phi} & M \end{array}$$

between two complex n-dimensional vector bundles, we have the relation

$$c_i(\phi^*E) = \phi^*c_i(E).$$

for any i.

Whitney-sum formula *Denote by* $c(E) := \sum_{i=0} c_i(E) \in H^*(B)$ *the total Chern class, then for two complex vector bundles of the same base space, we have*

$$c(E \oplus E') = c(E) \smile c(E'),$$

where $E \oplus E'$ *denotes the Whitney sum and* $c(E) \smile c(E')$ *the usual cup product.*

Triviality $c_0(E) = 1$ *for any complex vector bundle.*

Normalization *Consider the tautological line bundle over the complex projective plane* $L \to \mathbb{C}P^\infty$. *Then we have*

$$c(L) = 1 + e(L)$$

Remark.

- *The Euler class is only defined for real oriented vector bundles. In the normalization I use implicitely that any n-dimensional complex vector bundle has a underlying oriented 2n-dimensional real vector bundle. Also note that the Euler class* $e(L)$ *is a generator of* $H^2(\mathbb{C}P^\infty)$.

- *From a categorical viewpoint one can think about the i´th chern class* c_i *as a natural transformation of contravariant functors* $F \Rightarrow H^{2i}$, *where*

$$F : Top \to Ab$$

sending a space to the isomorphism classes of complex vector bundles and

$$H^{2i} : Top \to Ab$$

the usual cohomology.

- *There are quite a few ways of defining Chern classes axiomatically, for example one can normalize them differently.*

- *We will later see*
  - $c_0(E) = 1$
  - $c_n(E) = e(E)$
  - $c_k(E) = 0 \; \forall k > n$
- *For a complex manifold and its tangent bundle $TM \to M$ we write $c_i(M)$ and $c(M)$ for $c_i(TM)$ and $c(TM)$ respectively.*

Example (Chern class of $\mathbb{C}P^n$)**.** *We want to calculate the total chern class of $\mathbb{C}P^n$. But the computation can be done analogously to the Stiefel-Whitney class of $\mathbb{R}P^n$, so I omit the details and just sketch the idea. Start with the tautological line bundle $L \to \mathbb{C}P^n$. It is a subbundle of $\mathbb{C}P^n \times \mathbb{C}^{n+1} \to \mathbb{C}P^n$, hence using the canonical hermitian metric we can consider its orthogonal complement $L^\perp = \{(x, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | x \perp v\}$. Now observe that*

$$\mathrm{T}\mathbb{C}P^n \cong Hom(L, L^\perp)$$

and $\underline{\mathbb{C}P^n} \cong Hom(L, L)$ to conclude

$$\mathrm{T}\mathbb{C}P^n \oplus Hom(L, L) \cong Hom(L, L^\perp \oplus L) = Hom(L, \mathbb{C}^{n+1}) = \bar{L},$$

where $\bar{L}$ is the dual of L, which satisfies $c_k(\bar{L}) = (-1)^k c_k(L)$ (cf [1] 14.9). Finally we get

$$c(\mathrm{T}\mathbb{C}P^n) = c(\mathrm{T}\mathbb{C}P^n \oplus Hom(L, L)) = c(\bar{L})^{n+1} = (1 - e(L))^{n+1}.$$

Next I want to continue by proving existence. I have essentially found three different approaches, which all have their advantages and disadvantages. In this talk I decided to follow Kreck´s construction (cf. [2]).

Existence of the Chern classes

- Remember the cohomology ring of $\mathbb{C}P^N$ for some $N > 0$:

$$H^*(\mathbb{C}P^N) = \mathbb{Z}[e(L)] \big/ e(L)^{N+1}$$

  using $e(L) \in H^2(\mathbb{C}P^N)$ a generator.
- Let $\pi : E \to B$ be a complex n-dimensional vectorbundle. For $N > n$ consider the space $B \times \mathbb{C}P^N$.
- Now consider the two pullbacks

$$\begin{array}{ccccccc} p_1^*E & \longrightarrow & E & & p_2^*L & \longrightarrow & L \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B \times \mathbb{C}P^N & \xrightarrow{p_1} & B & & B \times \mathbb{C}P^N & \xrightarrow{p_2} & \mathbb{C}P^N \end{array}$$

where p_1 and p_2 are the projections.

- Consider their tensor product vector bundle $p_1^*E \otimes p_2^*L \to B \times \mathbb{C}P^N$. It is a complex n-dimensional vector bundle, hence we can consider its Euler class $e(p_1^*E \otimes p_2^*L) \in H^{2n}(B \times \mathbb{C}P^N)$.
- The Künneth isomorphism yields

$$\sum_{i+j=k} H^i(B) \otimes H^j(\mathbb{C}P^N) \to H^k(B \times \mathbb{C}P^N).$$

Now define the Chern classes as the coefficients of the Euler class, i.e. the chern classes are uniquely determined by

$$e(p_1^*E \otimes p_2^*L) = \sum_{i=0}^{k} c_i(E) \otimes e(L)^{k-i}$$

- Note that this construction does not depend on the choice of $N > n$, because the pullback of $\iota : \mathbb{C}P^N \to \mathbb{C}P^{N+1}$ is precisely the tautological bundle over $\mathbb{C}P^N$.

Theorem 3. *This construction defines Chern classes.*

Proof. We have to show that the axioms hold.

Naturality Follows almost directly from naturality of the Euler class. First note that a bundle morphism

$$\begin{array}{ccc} \phi^*E & \xrightarrow{\Phi} & E \\ \downarrow & & \downarrow \\ A & \xrightarrow{\phi} & B \end{array}$$

induces a map

$$\begin{array}{ccc} p_1^*(\phi^*E) \otimes p_2^*L & \xrightarrow{\Phi \otimes id} & p_1^*E \otimes p_2^*L \\ \downarrow & & \downarrow \\ A \times \mathbb{C}P^N & \xrightarrow{\phi \times id} & B \times \mathbb{C}P^N. \end{array}$$

Hence we calculate

$$\sum_{i=0}^{k} \phi^* c_i(E) \otimes e(L)^{k-i} = (\phi \times id)^* \left(\sum_{i=0}^{k} c_i(E) \otimes e(L)^{k-i} \right)$$
$$= e(p_1 \phi^* E \otimes p_2^* L) = \sum_{i=0}^{k} c_i(\phi^* E) \otimes e(L)^{k-i},$$

and since $e(L)$ generates the cohomology ring $H^*(\mathbb{C}P^N)$ we conclude $\phi^* c_i(E) = c_i(\phi^* E)$.

Whitney-sum formula Let E and E' be k- and l-dimensional complex vector bundles over B respectively and take $N > k + l$. Then consider

$$p_1^*(E \oplus E') \otimes p_2^* L \cong (p_1^* E \otimes p_2^* L) \oplus \left(p_1^* E' \otimes p_2^* L\right).$$

Hence the Euler classes agree and moreover comparing its coefficients yield

$$c_i(E \oplus E') = \sum_{r+s=i} c_r(E) \smile c_s(E')$$

and thus

$$c(E \oplus E') = c(E) \smile c(E').$$

Triviality For the trivial bundle over a space $\underline{B}$ one can immediately see that $c(\underline{B}) = 1$. Now consider any complex vector bundle and pull it back to a point

$$\begin{array}{ccc} \iota^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \xrightarrow{\iota} & B. \end{array}$$

The pullback bundle is clearly trivial, thus we conclude

$$\iota^*(c(E)) = c(\iota^*(E)) = 1$$

and hence $c_0(E)$.

Normalization Pick any point $x_0 \in \mathbb{C}P^1$ and consider the pullback bundle of

$$j_0 : \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^1, x \mapsto (x, x_0)$$

where we denote the linebundle over $\mathbb{C}P^\infty$ and $\mathbb{C}P^1$ by E and L respectively, i.e.

$$\begin{array}{ccc} j_0^*(p_1^*E \otimes p_2^*L) & \longrightarrow & p_1^*E \otimes p_2^*L \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty & \xrightarrow{j_0} & \mathbb{C}P^\infty \times \mathbb{C}P^1. \end{array}$$

Pulling the j_0 into the tensor product yields

$$j_0^*(p_1^*E \otimes p_2^*L) = j_0^*(p_1^*E) \otimes j_0^*(p_2^*L)$$

and we conclude

$$\begin{aligned} e(E) &= e(\underbrace{j_0^* p_1^*}_{=id^*} E \otimes \underbrace{j_0^* p_2^*}_{=0^*} L) = e(j_0^*(p_1^*E \otimes p_2^*L)) \\ &= j_0^* e(p_1^*E \otimes p_2^*L) = j_0^*(\sum c_i(E) \otimes e(L)^{k-i}) \\ &= j_0^*(c_1(E) \otimes 1) = c_1(E) \end{aligned}$$

□

Remark. *In the last part we actually didn´t use much of the structure of either complex projective plane. In fact one can analogously show that the top Chern class of a complex vector bundle is the Euler class.*

Lemma 4. *The Chern classes are unique.*

Proof. This can be done analogously to the uniqueness proof of the Stiefel-Whitney classes. Consider the splitting principle (cf. [3]) for an n-dimensional complex vector bundle $\pi : E \to X$, i.e. there exist a map $p : Y \to X$ such that

$$p^* : H^*(X) \to H^*(Y) \text{ is injective}$$
$$p^*(E) \cong L_1 \oplus ... \oplus L_n,$$

where the L_i are complex line bundles over Y.
Now assume there are two choices of natural transformations for the Chern classes c_i and c_i'. Then they agree on the universal line bundle $\mathbb{C}P^\infty$ and hence on any line bundle. We compute

$$\begin{aligned} p^*c(E) &= c(p^*(E)) = c(L_1 \oplus ... \oplus L_n) \\ &= c(L_1) \smile ... \smile c(L_n) = c'(L_1) \smile ... \smile c'(L_n) \\ &= c'(L_1 \oplus ... \oplus L_n) = c'(p^*(E)) = p^*c'(E). \end{aligned}$$

Now conclude that they agree on each complex vector bundle by noting that p^* is injective. □

As mentioned before there are some ways on how to construct Chern classes and I would like to give a quick sketch of the two other constructions.

Remark. *Here is the construction found in Milnor-Stasheff (cf. [1]).*

- *Idea: Construct them inductively starting at the top Chern class.*
- *We already know: $c_n(E) = e(E)$ for any n-dimensional complex vector bundle. Here it is part of the definition.*
- *Remember the complement of the zero section $\dot{E}$ in E from last talk. Continue by inductively constructing new vector bundles out of the previous one via:*

  $B^{(n)} := B, E^{(n)} := E$

  $B^{(n-1)} := \dot{E}$

  $E^{(n-1)} := \{(e, v) \subset \dot{E} \times \mathbb{C}^{n-2} | v \in x^{\perp} \subset F, \text{ where } e = (b, x) \in B \times F_0\}$

  In other words: $E^{(n-1)}$ is the complex (n-1)-dimensional vector bundle over $\dot{E}$, with fiber the orthogonal complement of "the" vector in the original fiber F. The local trivialization are build up from the local trivialization of the previous bundle, i.e. given a point $e \in \dot{E}$ consider a trivialization $U \subset B$ around $\pi_0(e)$ and consider

  $$\begin{array}{ccccc} \pi_0^{-1}(U) \times \mathbb{C}^{n-2} & \subseteq & E^{(n-1)} & \longrightarrow & E^{(n)} \\ \downarrow & & \downarrow & & \downarrow \\ \pi_0^{-1}(U) & \subseteq & \dot{E} & \xrightarrow{\pi_0} & B. \end{array}$$

  Note: This is not a pullback.
- *This defines a bundle map, hence one can define the (n-1)'th Chern class by $c_{n-1}(E) := (\pi_0^*)^{-1} e(E^{(n-1)})$.*
- *For example, naturality follows from the fact, that the inductive construction as well as the Euler class is natural (in some sense).*

Here´s a more "universal" viewpoint from [4].

- *Again let $\pi : E \to B$ a complex n-dimensional vector bundle and consider its hermitian metric (a priori it depends on choice of trivializations).*
- *With that one can reduce the structure group $Gl_n(\mathbb{C})$ to U_n, the unitary group.*
- *The classificiation of G-prinicpal bundles gives us a unique bundle map*

$$\begin{array}{ccc}
E & \longrightarrow & EU_n \cong \underset{k\to\infty}{colim}\ V_n(\mathbb{C}^k) \\
\big\downarrow & & \big\downarrow \\
B & \longrightarrow & BU_n \cong \underset{k\to\infty}{colim}\ Gr_n(\mathbb{C}^k).
\end{array}$$

with $V_n(\mathbb{C}^k)$ and $Gr_n(\mathbb{C}^k)$ the Stiefel and Grassmannian manifold respectively.

- *This construction is stable, meaning that the inclusion*

$$\iota : U_n \to U_{n+1}, A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

gives rise to a map $B\iota : BU_n \to BU_{n+1}$ via functoriality, such that the composition of the lower maps is again the classifying map

$$\begin{array}{ccccc}
E \oplus \underline{B} & \longrightarrow & EU_n & \longrightarrow & EU_{n+1} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
B & \longrightarrow & BU_n & \xrightarrow{\iota} & BU_{n+1}.
\end{array}$$

- *Consider the directed system*

$$BU_1 \hookrightarrow BU_2 \hookrightarrow ...$$

- *Stability implies that we get a well-behaved map into the colimit of this directed system, denoted by $BU := \underset{n\to\infty}{colim}\ BU_n$.*

- *At last we use the following fact:*

$$H^*(BU) \cong \mathbb{Z}[c_1, c_2, ...],$$

where $|c_i| = 2i$.

- *Define the Chern classes of BU to be the c_i.*

- *For any other complex vector bundle we can define the Chern classes to be the pullbacks from BU using the classifying map.*

Remark. *The fact about the cohomology ring of BU is highly non-trivial. In fact its proof usually uses Chern classes (see for example [5]).*

To finish the talk, I want to define Pontrjagin classes of a real vector bundle.

Definition 5. *Let $\pi : E \to B$ be a real vector bundle and consider its complexification $E \otimes_{\mathbb{R}} \mathbb{C} \to B$. Define the i'th Pontryagin class by*

$$p_i(E) \coloneqq (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(B)$$

Remark. *Some properties of the Pontrjagin classes.*

- *For a 2n-dimensional real oriented bundle E, we have*

$$p_n(E) = e(E) \smile e(E)$$

- *Let $\pi : E \to B$ a real vector bundle. Then $p_i(E)$ maps to $w_{2i}(E)^2$ under coefficient homomorphism $H^{4i}(B; \mathbb{Z}) \to H^{4i}(B; \mathbb{Z}_2)$. A proof of this can be found in [3].*

- *Similar to the remark on how to define Chern classes using the universal bundle BU_n, one can define the Pontrjagin classes using the unversal bundle BO_n. In fact p_k is the pullback of c_{2k} under the complexification inclusion $BO_n \to BU_n$.*

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