Characteristic Classes I

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Before we can talk about characteristic classes, we need a few more constructions in the category of vector bundles.

1 The bundle Hom functor

Let B be any topological space. We denote by **Vec** the category of finite dimensional $\mathbb{R}$-vector spaces and by $\mathbf{Vec}_B$ the category of vector bundles over B. The morphisms in $\mathbf{Vec}_B$ are bundle morphisms covering the identity Id_B.
The Hom sets in **Vec** (respectively $\mathbf{Vec}^{\mathrm{op}} \times \mathbf{Vec}$) have a natural topology and the functor $\mathrm{Hom}_{\mathbf{Vec}} : \mathbf{Vec}^{\mathrm{op}} \times \mathbf{Vec} \to \mathbf{Vec}$ is continuous on all Hom sets with respect to this topology. We can use this to define a corresponding functor $\mathbf{Vec}_B^{\mathrm{op}} \times \mathbf{Vec}_B \to \mathbf{Vec}_B$, which agrees with $\mathrm{Hom}_{\mathbf{Vec}}$ fibrewise.

Definition 1. Let $\pi_i : E_i \to B$ for $i = 1{,}2$ be two vector bundles over B. We define a bundle $\pi : \mathrm{Hom}(E_1, E_2) \to B$. For the total space we take (as a set)

$$\mathrm{Hom}(E_1, E_2) = \coprod_{b \in B} \mathrm{Hom}_{\mathbf{Vec}}((E_1)_b, (E_2)_b),$$

where the projection π to B is the obvious one. To give this the structure of a vector bundle it suffices to define trivializations such that the transition maps are continuous.
Let $U \subseteq B$ be an open subset over which both E_i are trivial. Let $\Phi_i : U \times \mathbb{R}^{n_i} \to \pi_i^{-1}(U)$ be trivializations. Let $V = \mathrm{Hom}_{\mathbf{Vec}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, then we can define a trivialization for $\mathrm{Hom}(E_1, E_2)$ over U by

$$U \times V \to \coprod_{b \in U} \mathrm{Hom}_{\mathbf{Vec}}((E_1)_b, (E_2)_b)\ , (b, f) \mapsto ((\Phi_2)_b \circ f \circ (\Phi_1)_b^{-1}).$$

Let U' be another open subset of B over which both E_i trivialize and let $\rho_i : U \cap U' \to \mathrm{GL}_{n_i}$ be the transition maps of E_i. Then the transition maps of $\mathrm{Hom}(E_1, E_2)$ are given by the composition

$$U \cap U' \xrightarrow{\rho_1 \times \rho_2} \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \xrightarrow{(-)^{-1} \times \mathrm{Id}} \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \xrightarrow{\mathrm{Hom}_{\mathbf{Vec}}} \mathrm{GL}(V).$$

This composition is continuous because $\mathrm{Hom}_{\mathbf{Vec}}$ is continuous on Hom sets. Hom defines a functor $\mathbf{Vec}_B^{\mathrm{op}} \times \mathbf{Vec}_B \to \mathbf{Vec}_B$.

Remark 2. Notice that the trivialization map above can also be written as

$$(b, f) \mapsto \mathrm{Hom}_{\mathbf{Vec}}((\Phi_1)_b^{-1}, (\Phi_2)_b)(f).$$

This removes any reference to what the functor $\mathrm{Hom}_{\mathbf{Vec}}$ actually does. Hence the same construction works for any functor $\mathbf{Vec}^{\mathrm{op}} \times \mathbf{Vec} \to \mathbf{Vec}$, as long as it is continuous on Hom sets. In fact, we can generalize this construction to any multifunctor that is continuous on Hom sets.
In particular, we have tensor products, symmetric powers, dual bundles and exterior powers as functors of vector bundles over a fixed base space. If we apply this construction to the direct sum, the result agrees with the Whitney sum.

Proposition 3. The Hom functor has the following properties.

1. For any two bundles E and F over B the evaluation map $E \oplus \mathrm{Hom}(E, F) \to F$ is continuous.

2. Hom commutes with Whitney sums.

3. Sections of the bundle $\mathrm{Hom}(E, F)$ correspond to bundle morphisms $E \to F$.

4. If E admits a metric $\langle -, - \rangle$, then $E \to \mathrm{Hom}(E, \underline{\mathbb{R}})$, $v \mapsto \langle v, - \rangle$ defines an isomorphism of bundles.

2 Axiomatic definition and properties of Stiefel-Whitney classes

From now on $H^*(-)$ will denote cohomology with coefficients in $\mathbb{Z}/2$. We assume all base spaces of vector bundles to be paracompact and Hausdorff. The most important consequence of this is, that all bundles admit metrics and classifying maps. In particular, every subbundle has an orthogonal complement.

Definition 4 (Stiefel-Whitney classes)**.** For any integer $i \geq 0$ the i-th Stiefel-Whitney class w_i is a natural transformation between the functor, which assigns to a space the set of isomorphism classes of vectorbundles, and the i-th cohomology functor $\mathrm{H}^i(-, \mathbb{Z}/2)$ with coefficients $\mathbb{Z}/2$.
This means that w_i assigns to each vector bundle $E \to B$ an element of the cohomology of the base space B.

1. (Naturality) If $f : B^{'} \to B$ is a continuous map, we get $w_i(f^*E) = f^*w_i(E)$.

2. For any vectorbundle $E \to B$, $w_i(E) = 0$ for $i > \mathrm{rank}(E)$ and $w_0(E) = 1$. In particular, we can define the total Stiefel-Whitney class $w(E) = \sum_i w_i(E)$ in the cohomology ring $\mathrm{H}^*(B, \mathbb{Z}/2)$.

3. (Whitney product theorem) Let E, F be bundles over the same base space, then $w(E \oplus F) = w(E) \cup w(F)$.

4. (Normalization) Recall that we have an isomorphism $\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$, where x corresponds to the unique nonzero cohomology class of degree 1. Let γ^1 be the tautological line bundle over $\mathbb{R}\mathrm{P}^\infty$, then $w(\gamma^1) = 1 + x$.

Remark 5. One can also normalize Stiefel-Whitney classes as follows:
Let γ_1^1 be the tautological line bundle over $\mathbb{R}\mathrm{P}^1$ and recall that $\mathrm{H}^*(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}/2[x]/(x^2)$. Now define $w(\gamma_1^1) = 1 + x$. It follows from the other axioms, that these normalizations are equivalent, since γ_1^1 is the pullback of γ^1 along $i : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^\infty$ and the map i induces an isomorphism in cohomology in degrees 0 and 1.

Theorem 6 (main theorem)**.** Stiefel-Whitney classes exist and are uniquely determined by the four axioms.

We will prove this theorem later. Let us first look at some calculations of Stiefel-Whitney classes and see some applications using only the axioms.

Proposition 7. The following are immediate consequences of the axioms:

1. Let $\underline{\mathbb{R}}^n$ be the trivial bundle of rank n over any base space B, then $w(\underline{\mathbb{R}}^n) = 1$.

2. Stiefel-Whitney classes are invariants of stable bundles, i.e. $w(E \oplus \underline{\mathbb{R}}) = w(e)$. In particular, since the tangent bundle of the n-sphere TS^n is trivial we have $w(TS^n) = 1$.

3. Let $\gamma_n^1 \to \mathbb{R}\mathrm{P}^n$ be the tautological line bundle, then $w(\gamma_n^1) = 1 + x$, where x is the unique nonzero degree one element of $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2[x]/(x^{n+1})$.

Proof. 1.The total Stiefel-Whitney class of any bundle over a point must be 1, as all higher cohomology groups are trivial. Thus the total Stiefel-Whitney class of $\underline{\mathbb{R}}^n$ is 1, because the trivial bundle is the pullback of the rank n bundle over a one point space along the constant map.
2. This is a direct consequence of 1. and axiom 3.
3. See Remark **??**. □

Our next goal is to calculate the Stiefel-Whitney class of the tangent bundle $T\mathbb{RP}^n$ of $\mathbb{RP}^n$.

Lemma 8. Recall that the tautological line bundle γ_n^1 is a subbundle of the trivial bundle $\underline{\mathbb{R}}^{n+1}$ over $\mathbb{RP}^n$. Let

$$\gamma^\perp := \{(x,v) \in \mathbb{RP}^n \times \mathbb{R}^n | v \perp x\}$$

be its orthogonal complement with respect to the canonical metric on $\underline{\mathbb{R}}^{n+1}$. Then

$$T\mathbb{RP}^n \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp).$$

Proof. Let $p : S^n \to \mathbb{RP}^n$ be the quotient map. Since p is a local diffeomorphism, so is its differential $dp : TS^n \to T\mathbb{RP}^n$. In particular, dp is a quotient map. We have seen the isomorphism $TS^n \cong \{(x,v) \in S^n \times R^{n+1} | v \perp x\}$.
Under this isomorphism, the equivalence relation associated to dp on TS^n corresponds to the relation generated by $(x,v) \sim (-x,-v)$. Hence we get a homeomorphism on the quotients

$$T\mathbb{RP}^n \cong \{(x,v) \in S^n \times R^{n+1} | v \perp x\} / \sim .$$

Using this identification we can explicitly give an isomorphism $\mathrm{Hom}(\gamma_n^1, \gamma^\perp) \to T\mathbb{RP}^n$. Let $x \in S^n$ be an element and let $\varphi : (\gamma_n^1)_{p(x)} \to (\gamma^\perp)_{p(x)}$ be a linear map i.e. an element of $\mathrm{Hom}(\gamma_n^1, \gamma^\perp)_{p(x)}$. To φ we assign the element of $T\mathbb{RP}^n$ represented by $(x, \varphi(x))$. This map is clearly well defined, linear and bijective. Continuity follows from the fact that for any two bundles E, F the evaluation map $E \oplus \mathrm{Hom}(E,F) \to F$ is continuous.

□

Proposition 9. The Stiefel-Whitney class of $T\mathbb{RP}^n$ is

$$w(T\mathbb{RP}^n) = (1+x)^{n+1}.$$

Proof. The bundle $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is a line bundle that has a nowhere vanishing section given by the identity. Thus $\mathrm{Hom}(\gamma_n^1, \gamma_n^1) \cong \underline{\mathbb{R}}$. Using this and Lemma **??** we get

$$\begin{aligned} T\mathbb{R}\mathrm{P}^n \oplus \underline{\mathbb{R}} &\cong \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \\ &\cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \\ &\cong \mathrm{Hom}(\gamma_n^1, \underline{\mathbb{R}}^{n+1}) \\ &\cong \mathrm{Hom}(\gamma_n^1, \underline{\mathbb{R}})^{n+1} \\ &\cong (\gamma_n^1)^{n+1}. \end{aligned}$$

This means that $w(T\mathbb{R}\mathrm{P}^n) = w(T\mathbb{R}\mathrm{P}^n \oplus \underline{\mathbb{R}}) = w(\gamma_n^1)^{n+1} = (1+x)^{n+1}$. □

Corollaray 10. The tangent bundle of $\mathbb{R}\mathrm{P}^2$ does not split as a sum of line bundles.

Proof. Suppose $T\mathbb{R}\mathrm{P}^2 = E \oplus F$ for two line bundles E and F, then

$$1+x+x^2 = (1+x)^2 = w(T\mathbb{R}\mathrm{P}^n) = w(E)w(F) = 1+(w_1(E)+w_1(F))+w_1(E)w_1(F).$$

Thus $x = w_1(E) + w_1(F)$ and $x^2 = w_1(E)w_1(F)$. This is impossible in $\mathbb{Z}/2[x]/(x^3)$. □

Recall that an immersion between smooth manifolds is a smooth map for which the induced map on tangent spaces is fibrewise injective. The Whitney immersion theorem states that any smooth n dimensional manifold admits an immersion into $\mathbb{R}^{2n-1}$. A natural question to ask is, whether the dimension of the target $2n-1$ is optimal. Does there exists a number $m(n) < 2n-1$ such that every n manifold has an immersion into $\mathbb{R}^{m(n)}$.
If n is not a power of two, the answer is yes (immersion conjecture)! But using Stiefel-Whitney classes, we can show that for $n = 2^k$ there does exist a smooth manifold that does not admit an immersion into $\mathbb{R}^m$ for $m < 2n-1$.

Theorem 11. For $n = 2^k$, $\mathbb{R}\mathrm{P}^n$ does not immerse into $\mathbb{R}^{2n-2}$.

Proof. Let $\varphi : \mathbb{R}\mathrm{P}^n \to \mathbb{R}^{2n-2}$ be an immersion, then $d\varphi$ is a bundle morphism covering φ. The morphism $d\varphi$ factors through the pullback $\varphi^* T\mathbb{R}^{2n-2}$. Since φ is an immersion, $d\varphi$ is fibrewise injective. Thus the induced map $T\mathbb{R}\mathrm{P}^n \to \varphi^* T\mathbb{R}^{2n-2}$ is also fibrewise injective.
Because the tangent bundle of $\mathbb{R}^{2n-2}$ is trivial, we have $\varphi^* T\mathbb{R}^{2n-2} \cong \underline{\mathbb{R}}^{2n-2}$. So $T\mathbb{R}\mathrm{P}^n$ is a subbundle of a trivial bundle of rank $2n-2$.
Now let $N \to \mathbb{R}\mathrm{P}^n$ be the orthogonal complement of $T\mathbb{R}\mathrm{P}^n$ in $\mathbb{R}^{2n-2}$. N is called the normal bundle of the immersion φ.

By definition of N we know that $T\mathbb{R}\mathrm{P}^n \oplus N \cong \mathbb{R}^{2n-2}$ and $\operatorname{rank} N = n-2$. The first property implies that $1 = w(\mathbb{R}^{2n-2} = w(T\mathbb{R}\mathrm{P}^n)w(N)$, so $w(N)$ is the multiplicative inverse of $w(T\mathbb{R}\mathrm{P}^n)$. But

$$w(T\mathbb{R}\mathrm{P}^n) = (1+x)^n = (1+x)(1+x)^{2^k} = (1+x)(1^{2^k} + x^{2^k}) = 1 + x + x^n.$$

The multiplicative inverse of this element in $\mathbb{Z}/2[x]/(x^{n+1})$ is $\sum_{i=0}^{n-1} x^i$. This is a contradiction to $\operatorname{rank} N = n-2$. □

3 Existence and uniqueness of Stiefel-Whitney classes

Existence of Stiefel-Whitney Classes. There are different methods of explicitly constructing Stiefel-Whitney classes. We follow [3] and [1]. A different approach using the Leray-Hirsch theorem can be found in [2]. We will need the following theorem.

Theorem (Thom Isomorphism)**.** Let $E \to B$ be a vector bundle of rank n over a paracompact space B. We denote by $\dot{E}$ the complement of the zero section in E. Then $H^i(E, \dot{E}) = 0$ for $i < n$ and $H^n(E, \dot{E})$ contains a unique element u, such that for the inclusion of any fibre $i : F \to E$, the element i^*u is the unique generator of $H^n(F, \dot{F})$. Here $\dot{F}$ denotes the complement of zero in F.
Aditionally the map

$$H^i(E) \to H^{i+n}(E, \dot{E}), \quad x \mapsto x \cup u$$

is an isomorphism. The composition

$$\Phi : H^i(B) \to H^i(E) \to H^{i+n}(E, \dot{E})$$

is called the Thom isomorphism. The cohomology class u is called the Thom class.

We can now define

$$w_i(E) = \Phi^{-1}(\mathrm{Sq}^i(u)).$$

Let us check that this definition satisfies the axioms.
1. *Naturality:*
Consider the following pullback diagram, where $E \to B$ is a rank n bundle with Thom class u, and f is any continuous map.

$$\begin{array}{ccc} E' & \xrightarrow{\hat{f}} & E \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

Now we show that the Thom element of E' is $\hat{f}^*u$. Let $b' \in B'$ be any point. Let $F' = (\pi')^{-1}(b')$ and $F = \pi^{-1}(f(b'))$. The map $\hat{f}$ restricts to a homeomorphism between F' and F. We get the following commutative diagram in cohomology.

$$\begin{array}{ccc}
\mathrm{H}^n(E, \dot{E}) & \xrightarrow{\hat{f}^*} & \mathrm{H}^n(E', \dot{E}') \\
\downarrow & & \downarrow \\
\mathrm{H}^n(F, \dot{F}) & \xrightarrow{\hat{f}\restriction_F^*} & \mathrm{H}^n(F', \dot{F}')
\end{array}$$

If we follow the Thom class $u \in \mathrm{H}^n(E, \dot{E})$ through the diagram we see that $\hat{f}^*u$ maps to the generator of $\mathrm{H}^n(F', \dot{F}')$. Since the fibre F' was arbitrary, this means that $\hat{f}^*u$ is the Thom class of E'. If Φ (respectively Φ') deontes the Thom isomorphism of E (respectively E'), it follows that $\Phi' \circ f^* = \hat{f}^* \circ \Phi$. Hence

$$w_i(E') = (\Phi')^{-1} \operatorname{Sq}^i(\hat{f}^*u) = (\Phi')^{-1} \hat{f}^* \operatorname{Sq}^i(u) = f^* \Phi \operatorname{Sq}^i(u) = f^* w_i(E).$$

2. *Axiom:*
This follows immediately from the properties of Steenrod squares.
3. *Whitney product formula:*
Let $\pi_i : E_i \to B$ for $i = 1,2$ be two bundles over B of rank n_i. Recall that we have the following pullback diagram, where Δ is the diagonal map.

$$\begin{array}{ccc}
E_1 \oplus E_2 & \longrightarrow & E_1 \times E_2 \\
\downarrow & & \downarrow \\
B & \xrightarrow{\Delta} & B \times B
\end{array}$$

It suffices to show that $w(E_1 \times E_2) = w(E_1) \times w(E_2)$, because then

$$w(E_1 \oplus E_2) = w(\Delta^*(E_1 \times E_2) = \Delta^* w(E_1 \times E_2) = \Delta * (w(E_1) \times w(E_2)) = w(E_1) \cup w(E_2).$$

Just like in the proof of naturality, we first consider the Thom elements and show that $u_{E_1 \times E_2} = u_{E_1} \times u_{E_2}$. Let $(b_1, b_2) \in B \times B$ and let F_i be the fibre of E_i over b_i. Then the fibre of $E_1 \times E_2$ over (b_1, b_2) is $F_1 \times F_2$.

$$\begin{array}{ccc}
\mathrm{H}^{n_1}(E_1, \dot{E}_1) \otimes \mathrm{H}^{n_2}(E_2, \dot{E}_2) & \xrightarrow{\times} & \mathrm{H}^{n_1+n_2}(E_1 \times E_2, (E_1 \times E_2)^{\cdot}) \\
\downarrow & & \downarrow \\
\mathrm{H}^{n_1}(F_1, \dot{F}_1) \otimes \mathrm{H}^{n_2}(F_2, \dot{F}_2) & \xrightarrow{\times} & \mathrm{H}^{n_1+n_2}(F_1 \times F_2, (F_1 \times F_2)^{\cdot})
\end{array}$$

The above diagram commutes, because the cross product is natural. The arrow on the bottom is an isomorphism by the Künneth theorem. Thus, if we follow $u_{E_1} \otimes u_{E_2}$ we see that $u_{E_1} \times u_{E_2}$ maps to the generator of $\mathrm{H}^{n_1+n_2}(F_1 \times F_2, (F_1 \times F_2)^{\cdot})$, which means that $u_{E_1 \times E_2} = u_{E_1} \times u_{E_2}$. From this it follows that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{H}^i(B) \times \mathrm{H}^j(B) \arrow[r] \arrow[d, "\times"] & \mathrm{H}^i(E_1) \times \mathrm{H}^j(E_2) \arrow[r, "(-\cup u_{E_1}, -\cup u_{E_2})"] \arrow[d, "\times"] & \mathrm{H}^{i+n_1}(E_1, \dot{E}_1) \times \mathrm{H}^{j+n_2}(E_2, \dot{E}_2) \arrow[d, "\times"] \\
\mathrm{H}^{i+j}(B \times B) \arrow[r] & \mathrm{H}^{i+j}(E_1 \times E_2) \arrow[r, "-\cup u_{E_1 \times E_2}"] & \mathrm{H}^{i+n_1+j+n_2}(E_1 \times E_2, (E_1 \times E_2)^{\cdot})
\end{tikzcd}$$

In other words

$$\Phi_{E_1 \times E_2}(- \times -) = \Phi_{E_1}(-) \times \Phi_{E_2}(-).$$

Finally we get

$$\begin{aligned}
\Phi_{E_1 \times E_2} w(E_1 \times E_2) &= \mathrm{Sq}\, u_{E_1 \times E_2} \\
&= \mathrm{Sq}(u_{E_1} \times u_{E_2}) \\
&= \mathrm{Sq}\, u_{E_1} \times \mathrm{Sq}\, u_{E_2} \\
&= \Phi_{E_1} w(E_1) \times \Phi_{E_2} w(E_2) \\
&= \Phi_{E_1 \times E_2}(w(E_1) \times w(E_2))
\end{aligned}$$

and applying $\Phi_{E_1 \times E_2}$ gives $w(E_1 \times E_2) = w(E_1) \times w(E_2)$.
4. *Normalization*:
As explained in Remark **??**, it suffices to show that $w(\gamma_1^1) = 1 + x \in \mathrm{H}^*(\mathbb{RP}^1) \cong \mathbb{Z}/2[x]/(x^2)$ and by axiom 2. this is eqivalent to $w_1(\gamma_1^1) = x$. By definition we have $w_1(\gamma_1^1) = \Phi^{-1}\,\mathrm{Sq}^1(u) = \Phi^{-1}(u \cup u)$, where $u \in \mathrm{H}^1(\gamma_1^1, \dot{\gamma}_1^1)$ is the Thom class of γ_1^1. Since Φ is an isomorphism and there are only two elements in $\mathrm{H}^1(\mathbb{RP}^1)$, the assertion is equivalent to $u \cup u \neq 0$.
The map

$$I \times \mathbb{R} \to \mathbb{RP}^1 \times \mathbb{R}^2\ ,\ (\alpha, t) \mapsto ([e^{i\pi\alpha}], te^{i\pi\alpha})$$

descends to a bundle isomorphism from the Moebius bundle

$$M = (I \times \mathbb{R})/(1, t) \sim (0, -t)$$

to γ_1^1. Recall from Topology 2, that the Thom space $\mathrm{Th}(M)$ is homeomorphic to $\mathbb{RP}^2$. Thus we get the following chain of isomorphisms in cohomology.

$$\mathrm{H}^*(\gamma_1^1, \dot{\gamma}_1^1) \cong \mathrm{H}^*(M, \dot{M}) \cong \mathrm{H}^*(\mathrm{Th}(M), *) \cong \mathrm{H}^*(\mathbb{RP}^2, *)$$

Notice that each of these isomorphisms is induced by a map and hence the composition is compatible with the cup product.The map $(\mathbb{R}\mathrm{P}^2, \emptyset) \to (\mathbb{R}\mathrm{P}^2, *)$ induces an injective homomorphism of (non-unital) rings

$$\mathrm{H}^*(\mathbb{R}\mathrm{P}^2, *) \hookrightarrow \mathrm{H}^*(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/2[x]/(x^3).$$

The Thom class u cannot be zero because it maps to a generator of $\mathrm{H}^1(F, \dot{F})$ for any fibre F. Thus it maps to x in $\mathbb{Z}/2[x]/(x^3)$. In particular, u^2 maps to x^2, which is nonzero, so $u^2 \neq 0$.

□

Uniqueness of Stiefel-Whitney Classes. In order to prove uniquness of Stiefel -Whitney classes we will need some deeper theorem. One approach is to study the cohomology ring of the Grassmanian. The details of this approach can be found in [3]. We have decided to deduce uniqueness from the splitting principle.

Theorem (Splitting principle)**.** Let $E \to B$ be a vector bundle. Then there exists a space $F(E)$ and a map $f : F(E) \to B$ such that

1. f^*E splits as a sum of line bundles and
2. the induced map $f*$ in cohomology is injective.

A proof of the splitting principle can be found in [2].
Let w and w' be two transformations, that satisfy the axioms of the Stiefel-Whitney classes. By the normalization axiom w and w' agree on the universal bundle γ^1. Since any line bundle is a pullback of γ^1, it follows from naturality that w and w' agree on line bundles. By axiom 3. they also agree on any bundle that is the sum of line bundles.
Now let $E \to B$ be any bundle and let $f : F(E) \to B$ be the map we get from the splitting principle. Then

$$f^*w(E) = w(f^*E) = w'(f^*E) = f^*w'(E).$$

Since f^* is injective we get $w(E) = w'(E)$.

□

References

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[2] Allen Hatcher. *Vector Bundles and K-theory*. 2003.

[3] J. Milnor and J. Stasheff. *Characteristic Classes*. Princeton University Press, 1974.