Brown Representability

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Theorem 1 (Brown Representability)**.** Every reduced cohomology theory on $\mathbf{CW}_*$ is of the form $h^*(X) = [X, K_n]$ for some Ω-spectrum $\{K_n\}$, which is unique up to homotopy equivalence, and all pointed CW-complexes X.

Remark. *Brown Representability supplies us with another proof that singular cohomology is representable by maps into Eilenberg-MacLane-spaces: For $\{K_n\}$ an Ω-spectrum representing $H^*(\_; G)$, we get:*

$$\pi_i(K_n) = [S^i, K_n] = \begin{cases} \tilde{H}^n(S^i; G) & \text{for } i = n, \\ 0 & \text{else.} \end{cases}$$

Definition 2. A *reduced cohomology theory* on $\mathbf{CW}_*$ is a sequence of functors $h^n : \mathbf{CW}_* \to \mathbf{Ab}$ with natural isomorphisms $h^n(X) \cong h^{n+1}(\Sigma X)$ for all X such that:

- *homotopy axiom*: $f \simeq g \colon X \to Y \Rightarrow f^* = g^* \colon h^n(Y) \to h^n(X)$.
- *exactness axiom*: $A \hookrightarrow X$ inclusion $\Rightarrow h^n(X/A) \to h^n(X) \to h^n(A)$ is exact.
- *wedge axiom*: Let $X = \bigvee_\alpha X_\alpha$ with $i_\alpha \colon X_\alpha \to X$ the inclusions, then there is an isomorphism $\Pi_\alpha i_\alpha^* \colon h^n(X) \to \Pi_\alpha h^n(X_\alpha)$.

Remark. *Equivalently, a reduced cohomology theory is a sequence of contravariant functors $\tilde{h}^n \colon \boldsymbol{CW}_* \to \boldsymbol{Ab}$ together with natural homomorphisms $\delta \colon \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A)$ for CW-pairs (X, A) satisfying the homotopy axiom, the wedge axiom as well as admitting a long exact sequence of cohomology groups.*

You can get a (unreduced) cohomology theory h from a reduced one $\tilde{h}$ via defining $h(X, A) = \tilde{h}(X/A)$.

Lemma 3. Let K be fixed. Then $h(X) := [X, K]$ is a contravariant functor $h \colon \mathbf{CW}_* \to \mathbf{sets}$ satisfying the homotopy, exactness and wedge axioms as well as the *Mayer-Vietoris axiom*:

Let $X = A \cup B$ be a CW-complex, where A, B both contain the basepoint. If there exist $a \in h(A)$ and $b \in h(B)$ such that $a|_{A\cap B} = b|_{A\cap B} \in h(A \cap B)$, then there exists an $x \in h(X)$ with $x|_A = a$ and $x|_B = b$.

Definition 4. Let K be a connected CW-complex, $u \in h(K)$, where h satisfies the homotopy, wedge and Mayer-Vietoris axioms. (K, u) is called *n-universal* if $T_u\colon \pi_i(K) \to h(S^i)$, $T_u(f) = f^*(u)$ is an isomorphism for all $i < n$ and subjective for $i = n$.

It is *π_*-universal* if it is n-universal for all n.

Lemma 5. Given (Z, z) with Z conn. CW-complex, $z \in h(Z)$, h as above, there exists a π_*-universal (K, U) such that $Z \subseteq K$ is a subcomplex and $u|_Z = z$.

Lemma 6. Let h as above, (K, u) π_*-universal, (X, A) a CW-pair. Then for all $x \in h(X)$ and $f\colon A \to K$ with $f^*(u) = x|_A$, there exists $g\colon X \to K$ extending f with $g^*(u) = x$.

Theorem 7. For $h\colon \mathbf{CW}_* \to \mathbf{sets}_*$ a contravariant functor satisfying the homotopy, Mayer-Vietoris and wedge axioms, there exists a connected CW-complex K and $u \in h(K)$ such that $T_u : [X, K] \to h(X)$, $T_u(f) = f^*(u)$ is a bijection for all X. (K, u) is called *universal* for h.

Remark. *We note the following facts:*

i) The universal pair is unique up to homotopy.

ii) The wedge axiom implies that $h(pt) = 0$.

iii) The homotopy, wedge and Mayer-Vietoris axioms together imply the exactness axiom.

iv) We can use the bijection $T_u\colon [\Sigma Y, K] \to h(\Sigma Y)$ to define a group structure on $h(\Sigma Y)$.

Proofs

Proof of Theorem 1

As $h^n(X) \cong h^{n+1}(\Sigma X)$ in any cohomology theory and ΣX connected, it is enough to prove that the statement holds true for connected CW-complexes. In the end, we can then use the properties of an Ω-spectrum to see the following:

$$h^n(X) \cong h^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}] \cong [X, \Omega K_{n+1}] \cong [X, K_n].$$

As each h^n satisfies the requirements of Theorem 7, we get CW-complexes K_n with $h^n(X) \cong [X, K_n]$. It remains to show that these CW-complexes actually form an Ω-spectrum, i.e. that we get weak homotopy equivalences $K_n \to \Omega K_{n+1}$.

The isomorphism $h^n(X) \cong h^{n+1}(\Sigma X)$ corresponds to a natural isomorphism $\phi\colon [X, K_n] \to [\Sigma X, K_{n+1}] \to [X, \Omega K_{n+1}]$. Naturality means that we get the following commutative diagram for all $f\colon X \to K_n$.

$$\begin{array}{ccc} [K_n, K_n] & \xrightarrow{f^*} & [X, K_n] \\ \downarrow \phi & & \downarrow \phi \\ [K_n, \Omega K_{n+1}] & \xrightarrow{f^*} & [X, \Omega K_{n+1}]. \end{array}$$

We define $\varepsilon_n := \phi(id)\colon K_n \to \Omega K_{n+1}$ and calculate:

$$\phi(f) = \phi(f \circ id) = \phi \circ f^*(id) = f^* \circ \phi(id) = f^*(\varepsilon_n) = \varepsilon_n \circ f.$$

Thus $\phi\colon [K_n, K_n] \to [K_n, \Omega K_{n+1}]$ is given by composition with ε_n. We use $X = S^k$ and the fact that ϕ is a bijection to conclude that ε_n induces isomorphisms on all homotopy groups, so it is a weak homotopy equivalence. One still has to check that we preserve group structure throughout. $\square$

Proof of Lemma 5 (Sketch)

We construct (K, u) from (Z, z) inductively.

We set $K_1 := Z \vee (\bigvee_\alpha S^1)$, where the α range over $h(S^1)$. Via the wedge axiom, we get an isomorphism $i_Z \times (\Pi_\alpha i_\alpha)\colon h(K_1) \to h(Z) \times (\Pi_\alpha h(S^1))$. Thus, there exists $u_1 \in h(K_1)$ such that $u|_Z = z$ and $u|_{S^1_\alpha} = \alpha$. Then, (K_1, u_1) is 1-universal.

We assume now we have constructed (K_n, u_n) n-universal with $Z \subseteq K_n$ and $u_n|_Z = z$. We represent elements in the kernel of $T_{u_n}\colon \pi_n(K_n) \to h(S^n)$ by $f_\alpha\colon S^n \to K_n$ and define $f := \bigvee_\alpha f_\alpha\colon \bigvee_\alpha S^n_\alpha \to K_n$. The reduced mapping cylinder Mf deformation retracts onto K_n, so we can regard $u_n \in h(Mf)$, but then clearly $u_n|_{\bigvee_\alpha S^n_\alpha} = 0$ by the definition of f. Via the exactness axiom and using that $Cf = Mf \diagup \bigvee_\alpha S^n_\alpha$ the following sequence is exact:

$$h(Cf) \to h(Mf) \to h(\bigvee_\alpha S^n_\alpha).$$

We get $\omega \in h(Cf)$ with $\omega \mapsto u_n$ and set $K_{n+1} := Cf \vee (\bigvee_\beta S^{n+1}_\beta)$, where $\beta \in h(S^{n+1})$. Using the wedge axiom, we can find $u_{n+1} \in h(K_{n+1})$ that restricts to $\omega \in h(Cf)$ and $\beta \in h(S^{n+1}_\beta)$. We claim that this is $(n+1)$-connected. For this, consider the following commutative diagram:

$$\begin{array}{ccc} \pi_i(K_n) & \longrightarrow & \pi_i(K_{n+1}) \\ & {\scriptstyle T_{u_n}} \searrow \quad \swarrow {\scriptstyle T_{u_{n+1}}} & \\ & h(S^i) & \end{array}$$

The upper map is an isomorphism for $i < n$ and surjective for $i = n$, as we construct K_{n+1} from K_n by attaching $(n+1)$-cells. The same properties hold for T_{u_n}, as (K_n, u_n) is n-universal, so by commutativity the same holds for $T_{u_{n+1}}$. We can also see that $T_{u_{n+1}}$ is injective for $i = n$: An element in the kernel of $T_{u_{n+1}}$ pulls back to an element in the kernel of $T_{u_n} \subseteq \pi_i(K_n)$ via surjectivity of the upper map and commutativity of the diagram. However, we have constructed K_{n+1} by attaching cells for all elements in the kernel of T_{u_n}, so this is trivial. Also, for $i = n+1$, $T_{u_{n+1}}$ is surjective by construction.

We can now define $K := \bigcup K_n$. We use a mapping telescope argument to show that there exists $u \in h(K)$ such that $u|_{K_n} = u_n$. Via a similar argument as above, we see that (K, u) is π_*-universal. □

Proof of Lemma 6

Wlog we can assume that K is the reduced mapping cylinder, and thus f is the inclusion of a subcomplex. We define $Z := X \cup_A K$. Via Mayer-Vietoris, we get $z \in h(Z)$ such that $z|_X = x$ and $z|_K = u$.

We can embed (Z, z) into (K', u') which is π_*-universal. The inclusion $(K, u) \to (K', u')$ induces isomorphisms on homotopy groups as both are π_*-universal, so K' deformation retracts onto K. The deformation retract induces a homotopy relative A of $X \hookrightarrow K'$ to $g\colon X \to K$. By the homotopy, clearly $g^*(u) = x$ holds since $u'|_K = u$ and $u'|_X = x$. □

Proof of Theorem 7

It suffices to show that if (K, u) is π_*-universal, then (K, u) is already universal, as we know by Lemma 5 that there exists a π_*-universal (K, u), i.e. such that $T_u\colon \pi_i(K) \to h(S^i)$, $f \mapsto f^*(u)$ is an isomorphism for all i. We apply Lemma 6 with $A = pt$ and varying $x \in X$ to see that $T_u\colon [X, K] \to h(X)$ is surjective.

For injectivity, suppose $T_u(f_0) = T_u(f_1)$, i.e. $f_0^*(u) = f_1^*(u)$. Now, we apply Lemma 6 with $(X \times I/(pt \times I), X \times \delta I/(pt \times \delta I))$ with map $f_0 \sqcup f_1$ on $X \times \delta I$, and $x = p^* f_0^*(u) = p^* f_1^*(u)$ where $p\colon X \times I/(pt \times I) \to X$ is the reduced projection:

$$\begin{array}{ccc} (X \times \delta I \diagup (pt \times \delta I), x) & \xrightarrow{f_0 \sqcup f_1} & (K, u) \\ \big\downarrow & \nearrow & \\ (X \times I \diagup (pt \times I), x) & & \end{array}$$

The dashed map that we get is exactly the desired homotopy from f_0 to f_1. $\square$