Bordism Homology

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These are my notes for the 13th talk in the *Graduate Seminar on Topology*, organised by Dr. Daniel Kasprowski, Dominik Kirstein and Simona Veselá in winter 2022/23 at the University of Bonn. If you have questions of comments, please contact me by mail (s6luvall@uni-bonn.de).

Why should you care about bordism? Bordism...

(a) ...gives us a very *geometrical* homology theory that is different from singular homology.

(b) ...provides a surprising connection between differential topology and (stable) homotopy theory.

(c) ...can be used to solve the generalised Poincaré Conjecture in dimensions ≥ 5.

(d) ...has applications in algebraic geometry (Hirzebruch's signature theorem).

The goals of this talk are to introduce the basic definitions, remind ourselves of some results from differential topology, and to prove that bordism yields a homology theory. A concise presentation is given by chapter 21 in tom Dieck's *Algebraic Topology* [Die08]. A very detailed discussion of the oriented case can be found in Conner's and Floyd's *Differentiable periodic maps* [CF64]. If you understand German, Böcker's and tom Dieck's *Kobordismentheorie* [BD70] is a very good reference for the unoriented case.

Contents

Conventions

By a manifold we will always mean a smooth manifold with boundary, unless specified otherwise. We reserve the term *closed manifold* for compact manifolds with empty boundary, while the term *compact manifold* is used for compact manifolds with possibly nonempty boundary. Maps of manifolds are assumed to be smooth, unless specified otherwise. We consider the empty set as a smooth manifold of any dimension.

1 Unoriented bordism

1.1 Bordism over a space

Let us start with a quick motivation. Bordism can be viewed as a generalisation of the construction of homotopy groups. An element $[\alpha] \in \pi_n(X, *)$ can be represented by a continuous map $\alpha\colon S^n \to X$, where two maps $\alpha_1, \alpha_2\colon S^n \to X$ represent the same class if there is a homotopy $H\colon S^n \times [0,1] \to X$ with $H(-,0) = \alpha_1$ and $H(-,1) = \alpha_2$ (all based). We consider the following generalisation:

(a) Instead of maps from S^n, consider maps from any closed manifold M.

(b) Instead of the cylinder, consider any compact manifold for the equivalence relation.

Definition 1.1. Let X be a topological space. A **singular n-manifold** in X is a pair (M, f), where M is a compact n-manifold and $f\colon M \to X$ is continuous. The the singular $(n-1)$-manifold $\partial(M, f) := (\partial M, f|_{\partial M})$ is the **boundary** of (M, f). We call (M, f) **closed** if $\partial M = \emptyset$.

Definition 1.2. Let (M_1, f_1) and (M_2, f_2) be a closed singular n-manifolds in X. Then (M_1, f_1) and (M_2, f_2) are **bordant** if there exists a singular $(n+1)$-manifold (B, F) and a diffeomorphism $\varphi\colon M_1 \amalg M_2 \to \partial B$ such that the diagram

$$\begin{array}{ccc} M_1 \amalg M_2 & \xrightarrow[\sim]{\varphi} & \partial B \\ & {\scriptstyle f_1 \amalg f_2} \searrow \quad \swarrow {\scriptstyle F|_{\partial B}} & \\ & X & \end{array}$$

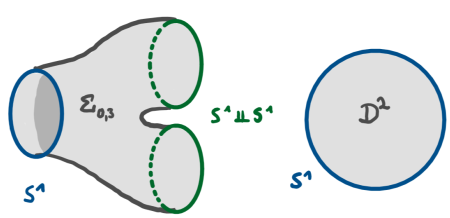
commutes. In this case, we call (B, F, φ) a **bordism** of (M_1, f_1) and (M_2, f_2). We call (M, f) **nullbordant** if it is bordant to $\emptyset$. A **null bordism** of (M, f) is bordism of (M, f) and $\emptyset$. Closed n-manifolds M_1 and M_2 are **bordant** if $(M_1, M_1 \to \mathrm{pt})$ and $(M_2, M_2 \to \mathrm{pt})$ are.

Remark 1.3. In the third part, we identify $M_1 \amalg M_2$ with ∂B and forget the diffeomorphism φ.

Remark 1.4. Singular n-manifolds (M_1, f_1) and (M_2, f_2) are bordant if and only if $(M_1 \amalg M_2, f_1 \amalg f_2)$ is nullbordant.

Example 1.5. For $X = \mathrm{pt}$, this definition coincides with the unoriented version of the bordism that we discussed in Topology II, restricted to smooth manifolds.

Example 1.6. The singular 1-manifold $(S^1, S^1 \to \mathrm{pt})$ is bordant to $(S^1 \amalg S^1, S^1 \amalg S^1 \to \mathrm{pt})$. In fact, $(S^1, S^1 \to \mathrm{pt})$ is null-bordant.



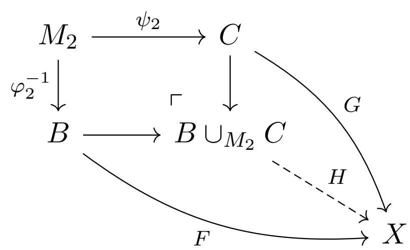
Remark 1.7. Let (B, F, φ) be a bordism between (M_1, f_1) and (M_2, f_2). Then $(\partial B, F|_{\partial B})$ can be written as a disjoint union $(\partial_1 B \amalg \partial_2 B, F|_{\partial_1 B} \amalg F|_{\partial_2 B})$ and φ decomposes into two diffeomorphisms $\varphi_i\colon M_i \to \partial_i B$. We thus also write $(B, F, \varphi_i\colon M_i \to \partial_i B)$ for (B, F, φ).

Proposition 1.8. *Being bordant is an equivalence relation on the set of closed singular n-manifolds.*

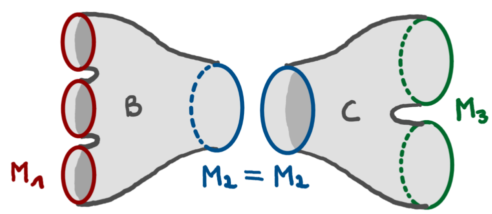
Proof. Reflexivity: Let (M, f) be a closed singular n-manifold. Consider $B := M \times [0,1]$ and define F to be $f \circ \mathrm{pr}_1\, M \times [0,1] \to M \to X$. There $\partial B = M \times \{0\} \cup M \times \{1\}$ and there is a canonical diffeomorphism $g\colon M \amalg M \to M \times \{0\} \cup M \times \{1\}$. We have $F|_{\partial B} \circ g = f \amalg f$ as desired.

Symmetry is immediate from the definition.

For transitivity, let $(B, F, \varphi_i\colon M_i \to \partial_i B)$ be a bordism between (M_1, f_1) and (M_2, f_2) and let $(C, G, \psi_i\colon M_i \to \partial_i B)$ be a bordism between (M_2, f_2) and (M_3, f_3). Consider $D := B \cup_{M_2} C$ for the maps $\varphi_2^{-1}\colon M_2 \to B$ and $\psi_2\colon M_2 \to \partial_2 C$. Then D carries a smooth structure, and the canonical maps $B \to D$ and $C \to D$ are smooth embeddings (this will be discussed later). Since $G \circ \psi_2 = f_2$ and $F \circ \varphi_2^{-1} = f_2$, we get an induced map $H\colon D = B \cup_{M_2} C \to X$ such that the diagram



commutes. Then $(D, H, \varphi_1 \amalg \psi_3)$ is a bordism between (M_1, f_1) and (M_3, f_3). $\square$



Definition 1.9. We call an equivalence classes with respect to bordism a **bordism class** and write $[M, f]$ for the bordism class of (M, f). We denote the set of all bordism classes of closed singular n-manifolds in X by $\mathfrak{N}_n(X)$ for $X \neq \emptyset$ and $n \geq 0$.

Remark 1.10. In general, the bordism class depends on the chosen smooth structure. However, there are surprising results such as that all 28 exotic 7-spheres are bordant.

In Topology II, we proved the following result using the Poincaré-Lefschetz duality:

Proposition 1.11. *Boundaries of compact topological manifolds have even Euler characteristic.*

Every smooth manifold is a topological manifold, so we find examples of non-nullbordant manifolds.

Example 1.12. The manifolds $\mathbb{R}P^{2n}$ and $\mathbb{C}P^{2n}$ are not nullbordant. Indeed, the Euler characteristics $\chi(\mathbb{R}P^{2n}) = 1$ and $\chi(\mathbb{C}P^{2n}) = 2n + 1$ are not even.

Remark 1.13. The signature is not an invariant for *unoriented* bordism.

1.2 The bordism groups $\mathfrak{N}_n(X)$

Proposition 1.14. *We have a well-defined binary operation*

$$+\colon \mathfrak{N}_n(X) \times \mathfrak{N}_n(X) \to \mathfrak{N}_n(X), \qquad [M, f] + [M', f'] \coloneqq [M \amalg M', f \amalg f'].$$

that gives $\mathfrak{N}_n(X)$ the structure of an abelian group in which every nontrivial element has order 2. So $\mathfrak{N}_n(X)$ is a vector space over $\mathbb{F}_2$.

Proof. To see that $+$ is well-defined, let $[M_1, f_1] = [M_2, f_2] \in \mathfrak{N}_n(X)$ and $[M_1', f_1'] = [M_2', f_2'] \in \mathfrak{N}_n(X)$. Let (B, F, φ_i) as well as (B', F', φ_i') be bordisms between (M_1, f_1) and (M_2, f_2) as well as (M_1', f_1') and (M_2', f_2'), respectively. Then $(B \amalg B', F \amalg F', \varphi_i \amalg \varphi_i')$ is a bordism between $[M_1 \amalg M_1', f_1 \amalg f_1']$ and $[M_2 \amalg M_2', f_2 \amalg f_2']$.

Let us briefly discuss the group structure. Associativity and commutativity are immediate. The neutral element is given by the class of any nullbordant manifold, e.g. $\emptyset$ or $(S^n, \mathrm{const}\colon S^n \to X)$. Every $[M, f] \in \mathfrak{N}_n(X)$ is bordant to itself, so $[M, f] + [M, f]$ is nullbordant. Thus, every element is self-inverse. □

Definition 1.15. We call $\mathfrak{N}_n(X)$ the **n-th bordism group** over X for $n \in \mathbb{Z}$, where $\mathfrak{N}_n(X) = 0$ if $X = \emptyset$ or $n < 0$. We denote $\mathfrak{N}_n(\mathrm{pt})$ by $\mathfrak{N}_n$ and a bordism class $[M, f] \in \mathfrak{N}_n(\mathrm{pt})$ just by $[M]$.

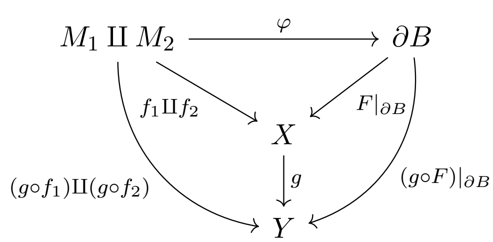
Example 1.16. Since $\mathbb{R}P^2$ is not nullbordant, we have $\mathfrak{N}_2 \neq 0$.

Lemma 1.17. *A continuous map $g\colon X \to Y$ induces a homomorphism*

$$\mathfrak{N}_n(g) = g_*\colon \mathfrak{N}_n(X) \to \mathfrak{N}_n(Y),\ [M, f] \mapsto [M, g \circ f].$$

This turns $\mathfrak{N}_n(-)$ into a functor $\mathsf{Top} \to \mathsf{Ab}$.

Proof. For well-definedness, let (M_1, f_1) and (M_2, f_2) be bordant via the bordism (B, F, φ). Then $(B, F \circ f, \varphi)$ is a bordism between $(M_1, g \circ f_1)$ and $(M_2, g \circ f_2)$.



Moreover, we have

$$\begin{aligned} g_*([M_1, f_1] + [M_2, f_2]) &= [M_1 \amalg M_2, g \circ (f_1 \amalg f_2)] \\ &= [M_1 \amalg M_2, (g \circ f_1) \amalg (g \circ f_2)] \\ &= g_*[M_1, f_1] + g_*[M_2, f_2], \end{aligned}$$

so f_* is a group homomorphism. Certainly, $(g \circ g')_* = g_* \circ g'_*$ and $\mathrm{id}_* = \mathrm{id}$. □

Lemma 1.18. *If $g, g'\colon X \to Y$ are homotopic, then $g_* = g'_*$.*

Proof. Let $[M, f] \in \mathfrak{N}_n(X)$ and let $H\colon X \times [0,1] \to Y$ be a homotopy between g and g'. Then $(M \times [0,1], H \circ (f \times \mathrm{id}))$ is a bordism witnessing $g_*[M,f] = g'_*[M,f]$. □

The main goal of this talk is to prove that the functors $\mathfrak{N}_*(-)$ give rise to a homology theory (we will define the relative groups later).

Remark 1.19. Given topological spaces X and Y, we can define a bilinear pairing

$$\cdot\colon \mathfrak{N}_p(X) \times \mathfrak{N}_q(Y) \to \mathfrak{N}_{p+q}(X \times Y), \qquad [M,f]\cdot[N,g] \coloneqq [M \times N, f \times g]$$

In particular, we get maps $\mathfrak{N}_p \times \mathfrak{N}_q \to \mathfrak{N}_{p+q}$ and $\mathfrak{N}_p \times \mathfrak{N}_q(X) \to \mathfrak{N}_{p+q}(X)$. With these and addition as before, $\mathfrak{N}_* \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathfrak{N}_n$ becomes a graded $\mathbb{F}_2$-algebra and $\mathfrak{N}_*(X) \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathfrak{N}_n(X)$ becomes a graded $\mathfrak{N}_*$-module. In fact $\mathfrak{N}_n(-)$ is a functor from Top to the category of graded $\mathfrak{N}_*$-modules.

Let us end this section with some important theorems in basic bordism theory, which are nevertheless far out of reach for us at this point.

Thom proved that is a natural isomorphism

$$\mathfrak{N}_*(X) \cong \mathfrak{N}_* \otimes_{\mathbb{F}_2} H_*(X;\mathbb{F}_2)$$

and furthermore, that we have

$$\mathfrak{N}_* \cong \mathbb{F}_2[u_2, u_4, u_5, \ldots]$$

where u_i represents a manifold of dimension i for $i \neq 2^j - 1$. We can take $u_{2k} = [\mathbb{R}P^{2k}]$.

2 Some differential topology

In this section and in this section only, manifolds are not automatically assumed to be smooth.

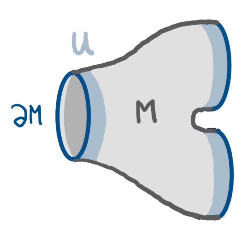
To prove that Bordism yields a homology theory, we need some differential topology. In this section, we state some elementary properties of smooth manifolds.

Collars and gluing

Remark 2.1. This section is not part of the oral presentation of the talk.

While disjoint unions of smooth manifolds carry a canonical smooth structure, it is not so obvious how gluing smooth manifolds (in nice ways) yields smooth manifolds.

Definition 2.2. A **collar** of a smooth manifold is a diffeomorphism $\kappa\colon \partial M \times [0,1) \to M$ onto an open neighbourhood U of ∂M in M such that for all $x \in \partial M$ we have $\kappa(x,0) = x$. (Instead of $[0,1)$, we can also take $\mathbb{R}_-$ or $\mathbb{R}_+$.)



Proposition 2.3 ([Die08](15.7.8)). *Every smooth n-manifold has a collar.*

Proposition 2.4 ([Die08](15.10.1)). *Let M_1, M_2 be smooth n-manifolds and let N be a smooth $(n-1)$-manifold with embeddings $f_i\colon N \to M_i$ as union of components of ∂M_i for $i \in \{1,2\}$. Then there is a smooth structure on $M_1 \cup_N M_1$, unique up to diffeomorphism, such that the inclusions $M_i \to M_1 \cup_N M_2$ are smooth for $i \in \{1,2\}$.*

2.1 Smooth approximation

Continuous maps can be approximated arbitrarily well by smooth maps. To make this precise, we should first argue that we can talk about distances on smooth manifolds.

Theorem 2.5 (Whitney, [Die08](15.7.1)). *A smooth n-manifold has a closed embedding into $\mathbb{R}^{2n+1}$.*

Corollary 2.6. *Every smooth n-manifold is metrisable.*

For the rest of this subsection, we will assume that our manifolds carry a metric induced by an embedding into some $\mathbb{R}^p$

Definition 2.7. Let M and N be smooth manifolds. Let $f\colon M \to N$ and $\varepsilon\colon M \to (0,\infty)$ be continuous. A continuous map $g\colon M \to N$ is an **ε-approximation** of f, if for all $x \in M$, we have $|f(x) - g(x)| < \varepsilon(x)$.

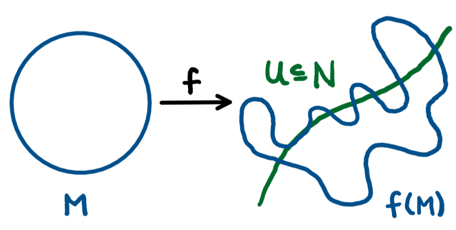
Although this is not very surprising, it is an important result that if a map $g\colon N \to M$ is *sufficiently close* to a map $f\colon N \to M$, then f and g are homotopic.

Lemma 2.8 ([Die08](15.8.3)). *Let $f\colon M \to N$ and $\varepsilon\colon M \to (0,\infty)$ be continuous. Then there exists a continuous map $\delta\colon M \to (0,\infty)$ such that the following holds: If g is a δ-approximation of f, then g is homotopic to f via a homotopy $H\colon M \times [0,1] \to N$ such that $H(x,t) = f(x)$ if $f(x) = g(x)$ and $H(-,t)$ is an ε-approximation for f for all t.*

Theorem 2.9 ([Die08](15.8.1), Approximation theorem). *Let $f\colon M \to N$ be a continuous map of smooth manifolds and let $A \subseteq M$ be a closed subset such that $f|_A$ is smooth. For any continuous $\varepsilon\colon M \to (0,\infty)$ there is a smooth ε-approximation g of f with $f|_A = g|_A$. In particular, there is a smooth map homotopic to f relative A.*

2.2 Transversality

As a motivation, let $f\colon M \to N$ be a map of manifolds without boundary and let $U \subseteq N$ be a submanifold. We cannot expect that $f^{-1}(U)$ is a submanifold of M.



In fact, it can be proved that every closed subset of M can be realised as the preimage of a point in N under a smooth map (this is due to Whitney, for a proof, cf. [BJ90] 14.1.). Transversality provides a framework in which we can avoid such pathological phenomena.

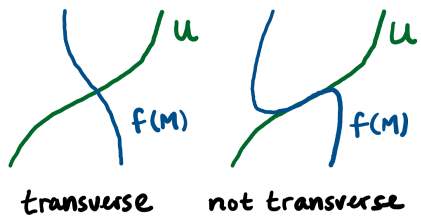
Definition 2.10. Let $f\colon M \to N$ be a map of manifolds with empty boundary and let $U \subseteq N$ be a submanifold with empty boundary. Let $a \in M$. We call f **transverse to U in a** if, provided that $f(a) \in U$, we have

$$T_a f(T_a M) + T_{f(a)} U = T_{f(x)} N,$$

or if $f(a) \notin U$. The map f is **transverse to U** if it is transverse to U in every $a \in M$. If $U = \{u\}$ is a point and f is transverse to U, then we call u a **regular value** of f.

Remark 2.11. The points $x \in M$ in which f is transverse to U are an open subset of M.

Example 2.12. Let $N = \mathbb{R}^2$ and $M = (0,1)$.



We should be careful with such pictures. The question whether f is transverse to U in a cannot be settled just by looking at the points in the image.

Remark 2.13. The map f is transverse to $U \subseteq N$ in $a \in f^{-1}(U)$ of and only if the composition

$$T_a M \xrightarrow{T_a f} T_{f(a)} N \to T_{f(a)} N / T_{f(a)} U$$

is surjective. In particular, if $U = \{u\}$, then f is transverse to U in $a \in f^{-1}(u)$ if and only if the induced map $T_a f\colon T_a M \to T_{f(a)} U$ is surjective, and u is a regular value of f if and only if all $a \in f^{-1}(u)$ are regular points of f.

Theorem 2.14 ([Die08](15.9.2,15.9.8))**.** *Let $f\colon M \to N$ and let $U \subseteq N$ be a submanifold. Suppose that U and N have empty boundary.*

(a) Let U have codimension k. If f and $f|_{\partial M}$ are transverse to U, then $f^{-1}(U)$ is a submanifold of M of codimension k or empty.

(b) (Transversality theorem) Let $A \subseteq M$ be a closed subset. Suppose that f is transverse to U in every $x \in A$ and $f|_{\partial M}$ is transverse to U in every $x \in \partial M \cap A$. Let $\varepsilon\colon M \to (0,\infty)$. Then there is a smooth ε-approximation $g\colon M \to N$ of f with $g|_A = f|_A$, and that is transverse to U on M and ∂M. In particular, we find such a map g with $g \simeq f$ relative A.

3 Bordism homology

Starting from now, all manifolds are again smooth manifolds with (possibly empty) boundary.

We prove that bordism gives rise to a homology theory. However, we postpone the definition of relative homology groups. We first show exactness of the *absolute* Mayer-Vietoris sequence. The advantage of this approach is that it will be easier to draw pictures and to give geometric intuition.

3.1 Absolute bordism homology

We already proved that we have homotopy invariant functors $\mathfrak{N}_n(-)\colon \mathsf{Top} \to \mathsf{Ab}$.

We will now prove that we also get an exact Mayer-Viertoris sequence. What should the boundary operator look like? Suppose that X is the union of open subsets X_0 and X_1, and keep these spaces fixed throughout this subsection. We have to construct a homomorphism

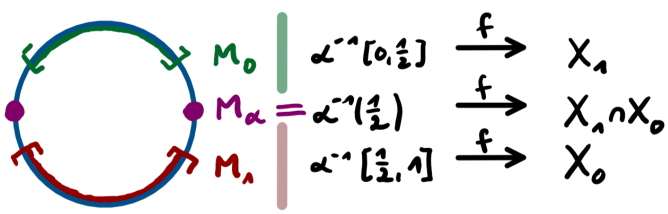
$$\partial\colon \mathfrak{N}_n(X) \to \mathfrak{N}_{n-1}(X_0 \cap X_1).$$

Given a singular n-manifold (M, f) in X for $n \geq 1$, let $M_i \coloneqq f^{-1}(X \setminus X_i)$ for $i \in \{0, 1\}$.

Definition 3.1. Let (M, f) be a singular n-manifold in X. A **separating function** is a smooth function $\alpha\colon M \to [0, 1]$ such that $M_i \subseteq \alpha^{-1}(i)$ for $i \in \{0, 1\}$ and such that $1/2$ is a regular value of α. Given a separating function α for (M, f), we define $M_\alpha \coloneqq \alpha^{-1}(1/2)$. Then f induces a map $f_\alpha \coloneqq f|_{M_\alpha}\colon M_\alpha \to X_0 \cap X_1$ and (M_α, f_α) is a singular $(n-1)$-manifold in $X_0 \cap X_1$.

Lemma 3.2. *Every singular n-manifold admits a separating function.*

Proof. The M_i are two disjoint closed subsets of M, so by the Lemma of Urysohn (manifolds are metrisable, hence normal), there is a continuous map $\alpha\colon M \to [0, 1]$ such that $\alpha^{-1}(i)$ is a neighbourhood of M_i for $i \in \{0, 1\}$. In particular, α is locally constant on a neighbourhood of the closed subset $M_0 \cup M_1$ of M, hence smooth on $M_1 \cup M_2$. By the approximation theorem (2.9), we can assume that α is smooth. By the transversality theorem, considering the closed submanifold $\{1/2\}$ of $[0, 1]$, we can arrange that $1/2$ is a regular value of α. □



Remark 3.3. Let α be a separating function for (M, f). Then M is the union of the manifolds $B_0 = \alpha^{-1}[1/2, 1]$ and $B_1 = \alpha^{-1}[0, 1/2]$ with $\partial B_0 = \partial B_1 = M_\alpha$. Conversely, given $(\tilde{M}, \tilde{f})$ with $\tilde{M} \subseteq M$, such that (M, f) is the union of manifolds (B_0, F_0) and (B_1, F_1) with common boundary $\tilde{M}$, then with collars $\tilde{M} \times (0, 1/2] \to B_1$ and $\tilde{M} \times [1/2, 1) \to B_0$, we obtain an embedding $\tilde{M} \times (0, 1) \to M$ which is the identity on $\tilde{M} \times \{1/2\}$. By smooth approximation, we can choose $\alpha\colon M \to [0, 1]$ such that $\alpha(\tilde{m}, t) = t$ for $\tilde{m} \in \tilde{M}$ and $1/4 \leq t \leq 3/4$. Then, by construction, $(M_\alpha, f_\alpha) = (\tilde{M}, \tilde{f})$.

We will show that $(M, f) \mapsto (M_\alpha, f_\alpha)$ induces a homomorphism $\mathfrak{N}_n(X) \to \mathfrak{N}_{n-1}(X_0 \cap X_1)$. First, we have to show that this map is well-defined on bordism classes.

Lemma 3.4. *Let $[M, f] = [N, g] \in \mathfrak{N}_n(X)$ and let α and β be separating functions for (M, f) and (N, g), respectively. Then $[M_\alpha, f_\alpha] = [N_\beta, g_\beta] \in \mathfrak{N}_{n-1}(X_0 \cap X_1)$.*

Proof. Let (B, F) be a bordism between (M, f) and (N, g). Choose a collar

$$\partial B \times [0, 1) = (M \times [0, 1)) \amalg (N \times [0, 1)) \subseteq B.$$

We can adjust F, so that

$$F|_{M\times[0,\frac{1}{2}]} = f\circ \mathrm{pr}_1 \quad \text{and} \quad F|_{N\times[0,\frac{1}{2}]} = g\circ \mathrm{pr}_1\,.$$

Let $\lambda\colon [0,\frac{1}{2}] \to [0,1]$ be smooth with $\lambda|_{[0,\frac{1}{6}]} = 0$ and $\lambda|_{[\frac{2}{6},\frac{1}{2}]} = 1$. By the Lemma of Uryson and smooth approximation, we find a smooth function

$$\psi\colon B \to [0,1]$$

such that $\psi^{-1}(0)$ is a neighbourhood of $F^{-1}(X\setminus X_0)$ and $\psi^{-1}(1)$ is one of $F^{-1}(X\setminus X_1)$. We define

$$\gamma\colon B\to[0,1] \quad y\mapsto \begin{cases}\lambda(t)\psi(x,t)+(1-\lambda(t))\alpha(x) & \text{if } y=(x,t)\in M\times[0,\frac{1}{2}],\\ \lambda(t)\psi(x,t)+(1-\lambda(t))\beta(x) & \text{if } y=(x,t)\in N\times[0,\frac{1}{2}],\\ \psi(y) & \text{otherwise.}\end{cases}$$

Then γ is a smooth function $B\to[0,1]$ and satisfies the following conditions:

(a) We have $F^{-1}(X\setminus X_0)\subseteq\gamma^{-1}(0)$ and $F^{-1}(X\setminus X_0)\subseteq\gamma^{-1}(1)$.

(b) We have $\gamma|_{M\times[0,\frac{1}{6}]} = \alpha\circ\mathrm{pr}_1$ and $\gamma|_{N\times[0,\frac{1}{6}]} = \beta\circ\mathrm{pr}_1$.

(c) In particular, $\gamma|_{\partial B\times[0,\frac{1}{6}]}$ is transverse to $\frac{1}{2}$.

By the Transversality theorem, we find $\delta\colon B\to[0,1]$ such that (a) and (b) hold and such that δ is transverse to $1/2$. Then $(\delta^{-1}(1/2), F|_{\delta^{-1}(1/2)})$ is a bordism between (M_α, f_α) and (N_β, f_β). □

Corollary 3.5. *We have a well defined homomorphism*

$$\partial\colon \mathfrak{N}_n(X)\to\mathfrak{N}_{n-1}(X_0\cap X_1)$$
$$[M,f]\mapsto[M_\alpha,f_\alpha],$$

which is natural with respect to maps of triads.

Proof. It remains to prove that ∂ is additive. Indeed, if $\alpha_i\colon M_i\to[0,1]$ are separating functions of (M_i,f_i) for $i\in\{1,2\}$, then $\alpha_1\amalg\alpha_2\colon M_1\amalg M_2\to[0,1]$ is a separating function of $(M_1,f_1)+(M_2,f_2) = (M_1\amalg M_2, f_1\amalg f_2)$ and $(M_1\amalg M_2)_{\alpha_1\amalg\alpha_2} = (M_1)_{\alpha_1}\amalg(M_2)_{\alpha_2}$.

Moreover, if $g\colon (X,X_0,X_1)\to(Y,Y_0,Y_1)$ is a map of triads and $[M,f]\in\mathfrak{N}_n(X)$, then

$$\partial g_*[M,f] = \partial[M,g\circ f] = [M_\alpha,(g\circ f)|_{M_\alpha}] = [M_\alpha, g\circ f|_{M_\alpha}] = g_*[M_\alpha, f|_{M_\alpha}] = g_*\partial[M,f],$$

and ∂ is indeed natural. □

Proposition 3.6. *Let $j^\nu\colon X_0\cap X_1\to X_\nu$ and $k^\nu\colon X_\nu\to X$ for $\nu\in\{0,1\}$ be the inclusions and let ∂ be defined as above. Then the sequence*

$$\cdots\xrightarrow{\partial}\mathfrak{N}_n(X_0\cap X_1)\xrightarrow{j_*:=j^0_*\oplus j^1_*}\mathfrak{N}_n(X_0)\oplus\mathfrak{N}_n(X_1)\xrightarrow{k_*:=k^0_*\oplus k^1_*}\mathfrak{N}_n(X)$$
$$\xrightarrow{\partial}\mathfrak{N}_{n-1}(X_0\cap X_1)\xrightarrow{j_*}\mathfrak{N}_{n-1}(X_0)\oplus\mathfrak{N}_{n-1}(X_1)\xrightarrow{k_*}\cdots$$

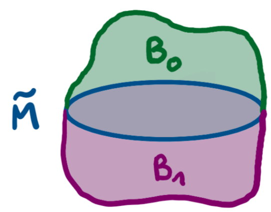
is exact and ends with $\mathfrak{N}_0(X)\to 0$.

Proof. We have to prove exactness at three positions.

Step 1. Exactness at $\mathfrak{N}_{n-1}(X_0 \cap X_1)$.

Suppose that $[M, f] \in \mathfrak{N}_n(X)$ is given. Then we can decompose M into the parts $B_1 = \alpha^{-1}[0, \frac{1}{2}]$ and $B_0 = \alpha^{-1}[\frac{1}{2}, 1]$ with common boundary $\alpha^{-1}(\frac{1}{2}) = M_\alpha$. Since $f(B_1) \subseteq X_1$, we see that $(B_1, f|_{B_1})$ is a null-bordism of (M_α, f_α) in X_1, thus $j^1_*\partial[M, f] = 0 \in \mathfrak{N}_{n-1}(X_1)$. Similarly, we have $j^0_*\partial[M, f] = 0$, and thus $j_* \circ \partial = 0$.

Conversely, let $[\tilde{M}, \tilde{f}] \in \mathfrak{N}_{n-1}(X_0 \cap X_1)$ with $j^0_*[\tilde{M}, \tilde{f}] = 0$ and $j^1_*[\tilde{M}, \tilde{f}] = 0$. Let (B_0, F_0) and (B_1, F_1) be singular manifolds in X_0 and X_1 respectively with $\partial B_0 = \partial B_1 = \tilde{M}$ and $F_0|_{\tilde{M}} = F_1|_{\tilde{M}} = \tilde{f}$. Considering $M \coloneqq B_1 \cup_{\tilde{M}} B_2$ and the induced map $f\colon M \to X$, we have $\partial[M, f] = [\tilde{M}, \tilde{f}]$ (cf. (3.3)).



Step 2. Exactness at $\mathfrak{N}_n(X_0) \oplus \mathfrak{N}_n(X_1)$.

Let $[M, f] \in \mathfrak{N}_n(X_0 \cap X_1)$. We denote the inclusion $X_0 \cap X_1 \to X$ by i. We have

$$(k_* \circ j_*)[M, f] = 2i_*[M, f] = 0,$$

Conversely, let $[M_0, f_0] \in \mathfrak{N}_n(X_0)$ and $[M_1, f_1] \in \mathfrak{N}_n(X_1)$ such that

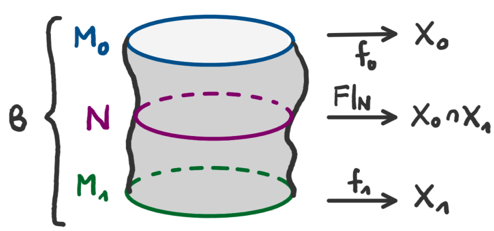
$$k_*([M_0, f_0], [M_1, f_1]) = [M_0, k_0 \circ f_0] + [M_1, k_1 \circ f_1] = 0$$

Let (B, F) be a bordism between $[M_0, k_0 \circ f_0]$ and $[M_1, k_1 \circ f_1]$. Then $F^{-1}(X \setminus X_0) \cup M_1$ and $F^{-1}(X \setminus X_1) \cup M_0$ are disjoint closed subsets of B and, we find a separating function for B (3.2), i.e. $\alpha\colon B \to [0, 1]$ such that

(a) we have $F^{-1}(X \setminus X_0) \cup M_1 \subseteq \alpha^{-1}(0)$ and $F^{-1}(X \setminus X_1) \cup M_0 \subseteq \alpha^{-1}(1)$

(b) the map α is transverse to $1/2$.

Let $(N, f) \coloneqq (\alpha^{-1}(1/2), F|_{\alpha^{-1}(1/2)})$. Then $(\psi^{-1}[0, 1/2], F|_{\alpha^{-1}[0,1/2]})$ is a bordism between (N, f) and (M_1, f_1) in X_1 and, similarly, we get a bordism between (N, f) and (M_0, f_0) in X_0. Hence, $(j^0_* \oplus j^1_*)[N, f] = ([M_0, f_0], [M_1, f_1])$.



Step 3. Exactness at $\mathfrak{N}_n(X)$.

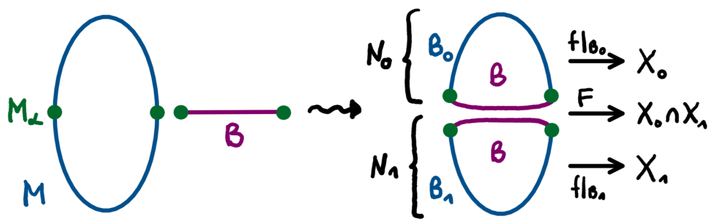
Let $([M_0, f_0], [M_1, f_1]) \in \mathfrak{N}_n(X_0) \oplus \mathfrak{N}_n(X_1)$, then

$$(k^0_* \oplus k^1_*)([M_0, f_0], [M_1, f_1]) = [M_0, k_0 \circ f_0] + [M_1, k_0 \circ f_1] = [M_0 \amalg M_1, k_0 \circ f_0 \amalg k_0 \circ f_1] \in \mathfrak{N}_n(X).$$

The sets $M_0 \cup (f_0 \amalg k_0 \circ f_1)^{-1}(X \setminus X_1)$ and $M_1 \cup (f_0 \amalg k_0 \circ f_1)^{-1}(X \setminus X_0)$ are disjoint closed subsets of $M_0 \amalg M_1$, thus there is a seperating function α for $M_0 \amalg M_1$ such that $M_\alpha = \alpha^{-1}(1/2) = \emptyset$. Hence $\partial \circ k_* = 0$.

Conversely, let $[M, f] \in \mathfrak{N}_n(X)$ with separating function α and let (B, F) be a null bordism of (M_α, f_α). Then M is the union of $B_1 = \alpha^{-1}[0, 1/2]$ and $B_0 = \alpha^{-1}[1/2, 1]$ with $\partial B_0 = \partial B_1 = M_\alpha$. We obtain singlular manifolds

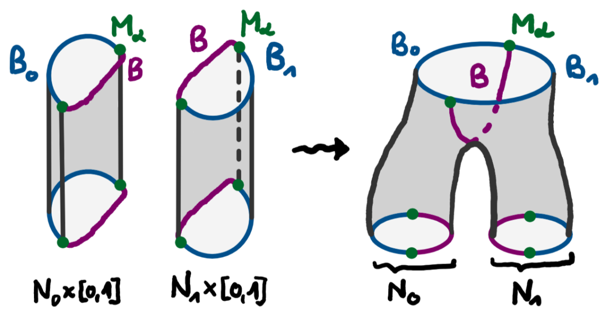
$$(N_0, f_0) := (B_0 \cup_{M_\alpha} B, f|_{B_0} \cup_{M_\alpha} F) \text{ in } X_0$$
$$(N_1, f_1) := (B_1 \cup_{M_\alpha} B, f|_{B_1} \cup_{M_\alpha} F) \text{ in } X_1.$$



We want to show that $[N_0, k_0 \circ f_0] + [N_1, k_1 \circ f_1] = [M, f] \in \mathfrak{N}_n(X)$. For this, first consider $(N_1 \amalg N_2) \times [0, 1]$ and glue $N_0 \times \{1\}$ with $N_1 \times \{1\}$ along $B \times \{1\}$ to obtain

$$L := (N_0 \times [0, 1]) \cup_{B \times \{1\}} (N_1 \times [0, 1]).$$

In a picture:

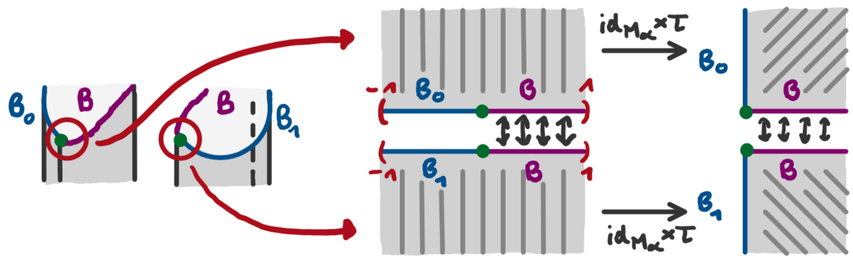


Now L has boundary $\partial L = N_0 \amalg N_1 \amalg M$ and the map $(f_0 \amalg f_1) \circ \mathrm{pr}_1 : (N_0 \amalg N_2) \times [0, 1] \to X$ induces a suitable map $L \to X$, proving that L is a bordism between $(N_0 \amalg N_1, k_0 \circ f_0 \amalg k_1 \circ f_1)$ and $[M, f]$.

There is, however, a subtlety here: A priori it is not clear why L is a *smooth* manifold, because by gluing along $B \times \{1\}$ we do not glue connected components of boundaries (cf. (2.4)). Specifically, there is no smooth structure given for the points $x \in \partial B \times \{1\} = M_\alpha$. We can solve this problem as foloows: By choosing a collars for $M_\alpha = \partial B \hookrightarrow N_i$ for $i \in \{1, 2\}$, we see that we glue two copies of $M_\alpha \times (-1,1) \times [0,1)$ along $M_\alpha \times [0,1) \times \{0\}$. We can introduce a differentiable structure by considering polar coordiantes and the homeomorphism

$$\tau\colon (-1,1)\times[0,1) \cong \{(r,\vartheta) : r \geq 0, 0 \leq \theta \leq \pi\} \to \{(r,\vartheta) : r \geq 0, 0 \leq \theta \leq \pi/2\} \cong [0,1)\times[0,1)$$
$$(r,\vartheta) \mapsto (r,\vartheta/2).$$

In a picture:



Indeed, $\mathrm{id}_{M_\alpha} \times \tau\colon M_\alpha \times (-1,1) \times [0,1] \to M_\alpha \times [0,1) \times [0,1)$ is smooth where this makes sense (everywhere except in $M_\alpha \times \{0\} \times \{0\}$) and on the manifold that we get from gluing two copies of $M_\alpha \times [0,1) \times [0,1)$ along $M_\alpha \times [0,1) \times \{0\}$, we have a smooth structure that agrees with the smooth structure on L in all the points not in $M_\alpha \times \{0\} \times \{0\}$. This technique is known as *straightening the angle*.

This completes the proof of the exactness of the Mayer-Vietoris sequence. □

Definition 3.7. A **one-space homology theory** is a family of functors $h_n\colon \mathsf{Top} \to \mathsf{Ab}$ with $h_n(\emptyset) = 0$ and homomorphisms $\partial\colon h_n(X) \to h_{n-1}(X_0 \cap X_1)$ for triads (X, X_0, X_1) with $X_0, X_1 \subseteq X$ open such that the following holds: The homomorphisms ∂ are natural with respect to maps of such triads and the Mayer-Vietoris sequence is exact.

Theorem 3.8. *The absolute bordism groups $\mathfrak{N}_n(-)$ yield a one-space homology theory.*

3.2 Relative bordism homology

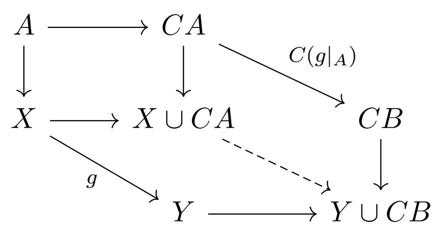
We want to get a relative homology theory as we know it. There are two ways to do this from the point we are at. We could define relative bordism groups from scratch, as it is, for example, done by Dieck [Die08] and also very well-explained in an expository paper by Hopkins [Hop16]. This is a very geometric approach that, again, requires a lot of differential topology. We take a different approach and demonstrate how to get a relative homology theory from a one-space homology theory.

Fix a one-space homology theory given by functors $h_n\colon \mathsf{Top} \to \mathsf{Ab}$ and connecting homorphisms ∂. This will only require a minor restriction.

Definition 3.9. Let (X, A) be a pair of spaces. The **relative groups associated to** h_* are

$$h_n(X, A) \coloneqq \mathrm{coker}(h_n(CA) \to h_n(X \cup CA)),$$

where CA denotes the cone of A. Then a map of pairs $g\colon (X,A) \to (Y,B)$ induces a map $g_*\colon h_n(X,A) \to h_n(Y,B)$ as can be seen by applying $h_n(-)$ to the diagram



We obtain a functor $h_n(-,-)\colon \mathsf{Top}^2 \to \mathsf{Ab}$.

We have $h_n(X,\emptyset) \cong h_n(X)$. To verify the Eilenberg-Steenrod axioms, it suffices verify that we get natural long exact sequences of pairs and the excision (homotopy invariance for pairs follows from the absolute case, the LES and the 5-lemma).

Proposition 3.10. *Let (X,A) be a pair of spaces. Then we have a natural long exact sequence*

$$\cdots \to h_n(A) \xrightarrow{j_*} h_n(X) \xrightarrow{\bar{\ell}_*} h_n(X,A) \xrightarrow{\bar{\partial}} h_{n-1}(A) \xrightarrow{j_*} \cdots$$

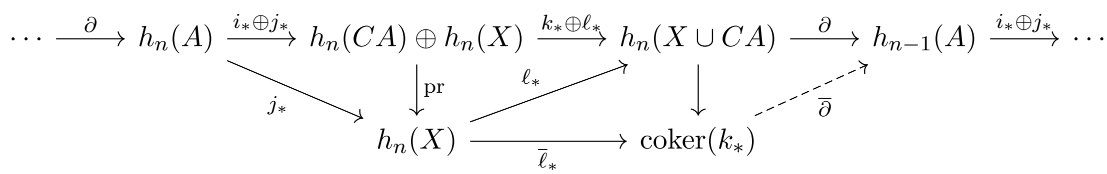
where $j\colon A \to X$ and $\bar{\ell}\colon (X,\emptyset) \to (X,A)$ are the inclusions.

Proof. We can write $X \cup CA$ as the union of open subsets $X_0 := (X \cup CA) \setminus X$ and $X_1 := (X \cup CA) \setminus \{*\}$, where $*$ is represented by any element in $A \times \{1\}$. Then $X_0 \simeq CA \simeq \mathrm{pt}$, $X_1 \simeq X$ and $X_0 \cap X_1 \simeq A$.

Let $i\colon A \to CA$, $k\colon CA \to X \cup CA$ and $\ell\colon X \to X \cup CA$ be the inclusions. By Mayer-Vietoris, we get a natural long exact sequence

$$\cdots \xrightarrow{\partial} h_n(A) \xrightarrow{i_* \oplus j_*} h_n(CA) \oplus h_n(X) \xrightarrow{k_* \oplus \ell_*} h_n(X \cup CA) \xrightarrow{\partial} h_{n-1}(A) \xrightarrow{i_* \oplus j_*} \cdots$$

We obtain a diagram



which we can chase to see that taking the lower composition yields the desired long exact sequence. $\square$

For excision, we need a minor restriction. To make use of the following lemma, we have to assume that the spaces we consider are normal, i.e. that disjoint closed subsets have disjoint open neighbourhoods. This is, for example, true for all CW-complexes, so in particular we will get a functor $h_n(-,-)\colon \mathsf{CW}^2 \to \mathsf{Ab}$. In fact, in the case of bordism homology, this assumption can be dropped.

Lemma 3.11. *Let $U \subseteq A \subseteq X$ and suppose that there exists $\tau\colon X \to [0,1]$ such that we have $U \subseteq \tau^{-1}(0)$ and $\tau^{-1}[0,1) \subseteq A$. Then the inclusion $(X \setminus U) \cup C(A \setminus U) \to X \cup CA$ is a pointed homotopy equivalence.*

Proof. This is (7.2.5) in [Die08]. You can check that $(x, t) \mapsto (x, \max\{2\tau(x) - 1, 0\}t)$ is a pointed homotopy inverse (for the point represented by an element in $A \times \{1\}$). $\square$

Proposition 3.12. *The functors $h_n(-,-)$ satisfy the excision axiom for $U \subseteq A \subseteq X$ with $\overline{U} \subseteq A^\circ$, provided that X is normal.*

Proof. Consider $U \subseteq A \subseteq X$ such that $\overline{U} \subseteq A^\circ$. We have to prove that the map induced by the inclusion $h_n(X \setminus U, A \setminus U) \to h_n(X, A)$ is an isomorphism. We have

$$\begin{aligned} h_n(X \setminus U, A \setminus U) &= \operatorname{coker}(h_n(C(A \setminus U)) \to h_n(X \setminus U \cup C(A \setminus U))) \text{ and} \\ h_n(X, A) &= \operatorname{coker}(h_n(CA) \to h_n(X \cup CA)). \end{aligned}$$

The cones are contractible and it suffices to prove that the inclusion $X \setminus U \cup C(A \setminus U) \to (X \cup CA)$ is an is a homotopy equivalence relative to the point represented by an element in $A \times \{1\}$. But $\overline{U}$ and $X \setminus A^\circ$ are disjoint closed subsets of the normal space X, so by the Lemma of Urysohn there is a function $\tau \colon X \to [0, 1]$ such that $\tau|_{\overline{U}} = 0$ and $\tau|_{X \setminus A^\circ} = 1$. If $\tau(x) \in [0, 1)$, then $x \notin X \setminus A^\circ$, so $x \in A^\circ \subseteq A$. Thus, we have $\tau^{-1}[0, 1) \subseteq A^\circ \subseteq A$. The previous Lemme yields the desired result. $\square$

Corollary 3.13. *Bordism gives us a homology theory $\mathfrak{N}_n(-,-) \colon \mathsf{CW}^2 \to \mathsf{Ab}$.*

Remark 3.14. Bordism homology does *not* satisfy the dimension axiom. For example, we have $\mathfrak{N}_2(\mathrm{pt}) \neq 0$, since $\mathbb{R}P^2$ is not nullbordant.

In partiular, we have found a homology theory that is *different* from singular homology.

In the next talk, we will combine the theories of bordism and vector bundles to obtain the alluded connection to (stable) homotopy theory. The following is not part of the oral presentation.

Bordism homology and singular homology

There is an interesting way to relate bordism homology and singular homology with $\mathbb{F}_2$-coefficients. For simplicity, we only consider the absolute case. For a closed connected n-manifold M, we can consider its fundamental class $z_M \in H_n(M, \mathbb{F}_2)$. If M is closed, but not necessarily connected, say $M = \coprod_{i=1}^n M_i$, then we can define z_M to be the unique element that is the sum of the z_{M_i} via $H_n(M, \mathbb{F}_2) \cong \bigoplus_{i=1}^n H_n(M_i, \mathbb{F}_2)$.

Proposition 3.15. *For every space X, there is a well defined homomorphism*

$$\mu \colon \mathfrak{N}_n(X) \to H_n(X, \mathbb{F}_2), \quad [M, f] \mapsto f_*(z_M)$$

Proof. We only have to check that $f_* z_M$ does not depend on the representative of the bordism class. Recall from Topology II that an inclusion $\partial B \to B$ induces the zero map on $H_n(-, \mathbb{F}_2)$. Thus, if (B, F) is a null-bordism of (M, f), then the diagram

$$\begin{array}{ccc} H_n(B, \mathbb{F}_2) & & \\ \uparrow{\scriptstyle 0} & \searrow^{F_*} & \\ H_n(M, \mathbb{F}_2) & \xrightarrow{f_*} & H_n(X, \mathbb{F}_2) \end{array}$$

commutes, and $f_*(z_M) = 0$. $\square$

Remark 3.16. You can check that the μ is in fact give natural transformations $\mathfrak{N}_n(-) \Rightarrow H_n(-, \mathbb{F}_2)$.

Remark 3.17. If we believe that $\mathfrak{N}_*(X) \cong \mathfrak{N}_* \otimes_{\mathbb{F}_2} H_*(-, \mathbb{F}_2)$ (1.19), we see that μ is surjective.

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