The Whitney Trick

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Recollections I: Transversality

Definition

Two submanifolds $Y', Y'' \subseteq X^m$ of a given manifold X^m are called **transverse** (or **intersect transversally**) if for any $p \in Y' \cap Y''$ we have $T_pY' + T_pY'' = T_pX$.

Facts:

- If $Y', Y'' \subseteq X$ intersect transversally, then $Y' \cap Y''$ is a submanifold with $\operatorname{codim}(Y' \cap Y'') = \operatorname{codim} Y' + \operatorname{codim} Y''$.
- If $Y', Y'' \subseteq X$ are submanifolds of complementary dimensions that intersect transversally, then by the previous, $Y' \cap Y''$ is a 0-manifold, ie Y' and Y'' intersect in isolated points. If moreover Y', Y'', X are oriented, then we can assign a **sign** for each isolated point of $Y' \cap Y''$ corresponding to the induced orientation.

The Whitney Trick

All manifolds throughout this talk are assumed smooth, compact and oriented.

Theorem (Whitney's Trick)

Consider two submanifolds $N_1^{k_1}$ and $N_2^{k_2}$ of complementary dimensions intersecting transversely inside of M^n, an n-manifold without boundary. Furthermore suppose that N_1 is oriented as well as the normal bundle of N_2 in M. Suppose that $k_1, k_2 \geq 3$. Let P and P' be two intersection points of N_1 and N_2 having opposite signs. Suppose there exists paths γ_1 and γ_2 from P to P' in N_1 and N_2, respectively, such that the loop $\gamma_1^{-1}\gamma_2$ is nullhomotopic in M. Then there is an ambient isotopy of N_1 into a submanifold N_1' transverse to N_2 such that

$$N_1' \cap N_2 = N_1 \cap N_2 - \{P, P'\}$$

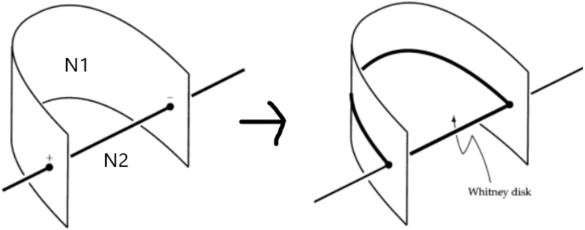
The Whitney Trick

Remark

We can strengthen the above result slightly. In fact we can assume that $k_2 \geq 3$, $n \geq 5$ and in case $k_1 = 1$ or $k_1 = 2$ suppose that the induced map $\pi_1(M - N_2) \to \pi_1(M)$ is 1-1. The proof is essentially the same. The above assumptions simplify the exposition.

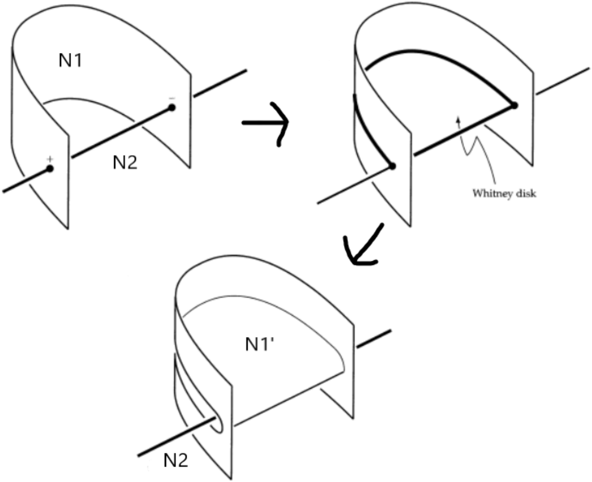
Idea of the Proof

- Assume, using transversality, that γ_1 and γ_2 do not intersect any points of $N_1 \cap N_2$ except P, P'. Since the loop $\gamma_1^{-1}\gamma_2$ is homotopically trivial, it bounds some immersed disk in the complement of N_1 and N_2.
- By the classical Whitney Embedding Theorem, it is known that embeddings are always dense in the space of all (smooth) maps $A^n \to B^{2n+1}$. In particular, this implies that immersions of disks in manifolds of dimension **at least 5** can always be approximated by embeddings. In our case, this results in an embedded disk bounded by our circle, called a *Whitney disk*.



Idea of the Proof

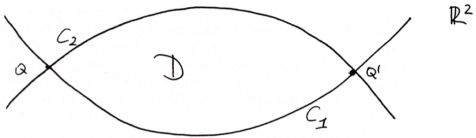
By using this Whitney disk as a guide, we can now push N_1 past N_2, till the intersection points disappear, as in the following figure:



Proof

Proof of the Whitney Trick:

- Without loss of generality suppose that the sign at P is $+1$, while the sign at P' is -1.
- By slightly deforming our paths, assume (using transversality) that γ_1 and γ_2 do not intersect any points of $N_1 \cap N_2$ except P, P'.
- Idea: We embed a "standard model" within which the required isotopy is easy to write. For our standard model, choose two curves C_1, C_2 in $\mathbb{R}^2$ intersecting transversally at Q, Q' and enclosing a disc D.
- Choose an embedding $\varphi_1 : C_1 \cup C_2 \to N_1 \cup N_2$ such that $\varphi_1(C_1) = \gamma_1$ and $\varphi_1(C_2) = \gamma_2$. We will find an isotopy F_t from $F_0 = id$ to F_1 such that $N_1' = F_1(N_1)$ meets the required conditions.

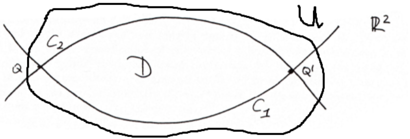


Proof

In order to construct F_t we will need the following lemma:

Lemma

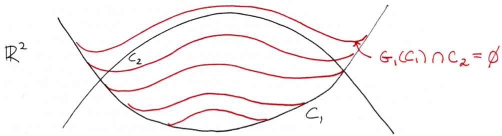
For some neighborhood U of D, we can extend $\varphi_1|_{U\cap(C_1\cup C_2)}$ to an embedding $\varphi : U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1} \to M$ such that $\varphi^{-1}(N_1) = (U \cap C_1) \times \mathbb{R}^{k_1-1} \times 0$ and $\varphi^{-1}(N_2) = (U \cap C_2) \times 0 \times \mathbb{R}^{k_2-1}$.



- Let $W = \varphi(U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1})$.
- We define F_t to be the identity outside of our embedded standard model(ie outside of W).

Proof

- Choose an isotopy $G_t : U \to U$ of our standard model such that:
  - $G_0 = id$
  - G_t is the identity in a neighborhood of the boundary $\overline{U} - U$ of U, $0 \leq t \leq 1$
  - $G_1(U \cap C_1) \cap C_2 = \emptyset$



- Let $\rho : \mathbb{R}^{k_1 - 1} \times \mathbb{R}^{k_2 - 1} \to [0, 1]$ be a *smooth* function such that
  - $\rho(x, y) = 1$ if $|x|^2 + |y|^2 \leq 1$
  - $\rho(x, y) = 0$ if $|x|^2 + |y|^2 \geq 2$

  for $x \in \mathbb{R}^{k_1 - 1}, y \in \mathbb{R}^{k_2 - 1}$.

Proof

- Define an isotopy $H_t : U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1} \to U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1}$ by $H_t(u, x, y) = (G_{t\rho(x,y)}(u), x, y)$, $u \in U$.
- It is easy to see that $F_t(w) = \varphi \circ H_t \circ \varphi^{-1}(w)$, $w \in W$, defines the required isotopy on W. This finishes the proof of Whitney trick, modulo the lemma.

Recollections II: The exponential map

Let M be a Riemannian manifold. It is well known (by the existence and uniqueness of geodesics) that for any $p \in M$, $V \in T_pM$ there exists a unique maximal geodesic $\gamma : I \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$. Denote it by γ_V.

Definition

Define $\mathcal{E} = \{V \in TM : \gamma_V$ is defined on an interval containing $[0,1]\}$. Now we define the **exponential map** $\exp : \mathcal{E} \to M$ by $\exp(V) = \gamma_V(1)$. The restricted exponential $\exp_p$ is the restriction to $\mathcal{E}_p = \mathcal{E} \cap T_pM$.

Lemma

1. *$\mathcal{E}$ is an open subset of TM, containing the zero section, and $\mathcal{E}_p$ is star shaped with respect to 0.*
2. *For each $V \in TM$, the geodesic γ_V is given by $\gamma_V(t) = \exp(tV)$.*
3. *The exponential map $\exp : \mathcal{E} \to M$ is smooth.*

Recollections II: The exponential map

Lemma (Normal Neighborhood Lemma)

For any $p \in M$, there is a neighborhood V of the origin in T_pM and a neighborhood U of $p \in M$ such that $\exp_p : V \to U$ is a diffeomorphism, ie the exponential map is a local diffeomorphism.

Reference: Lee, John. *Riemannian Manifolds: An Introduction to Curvature*. Springer, 1997

Lemma

For some neighborhood U of D, we can extend $\varphi_1|_{U\cap(C_1\cup C_2)}$ to an embedding $\varphi : U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1} \to M$ such that $\varphi^{-1}(N_1) = (U \cap C_1) \times \mathbb{R}^{k_1-1} \times 0$ and $\varphi^{-1}(N_2) = (U \cap C_2) \times 0 \times \mathbb{R}^{k_2-1}$.

Technical Lemma

Lemma

There exists a Riemannian metric on M such that:

1. N_1 *and* N_2 *are totally geodesic submanifolds of* M *(ie if a geodesic in* M *is tangent to* N_1 *or to* N_2 *at any point then it is entirely in* N_1 *or* N_2*, respectively).*
2. *There exist coordinate neighborhoods* $B_P, B_{P'}$ *about* P, P' *in which the metric is the Euclidean metric and so that* $B_P \cap \gamma_1$, $B_P \cap \gamma_2$, $B_{P'} \cap \gamma_1$, $B_{P'} \cap \gamma_2$ *are straight line segments. ("Metric is Euclidean near* P, P'*")*

Proof of the Lemma

Lemma

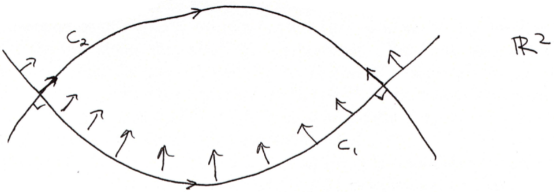
For some neighborhood U of D, we can extend $\varphi_1|_{U\cap(C_1\cup C_2)}$ to an embedding $\varphi : U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1} \to M$ such that $\varphi^{-1}(N_1) = (U \cap C_1) \times \mathbb{R}^{k_1-1} \times 0$ and $\varphi^{-1}(N_2) = (U \cap C_2) \times 0 \times \mathbb{R}^{k_2-1}$.

Proof of the Lemma:

- Fact: We can endow M with a Riemmanian metric which is Euclidean near P, P' and under which N_1, N_2 are totally geodesic submanifolds.
- Let $\tau_2(P), \tau_2(P')$ be the unit tangent vectors to $\gamma_2 \subseteq N_2$ at P, P'. Note that $\tau_2(P)$ is orthogonal to N_1, by the definition of our metric.
- Consider the bundle over γ_1 of vectors orthogonal to N_1. This bundle is trivial since γ_1 is contractible.

Proof of the Lemma

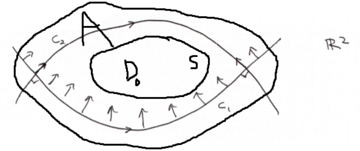
- Hence we can extend $\tau_2(P)$ to a smooth field of unit vectors along γ_1 orthogonal to N_1, equal to the parallel translates of $\tau_2(P)$ near P (in $B_P \cap \gamma_1$) and equal to the parallel translates of $-\tau_2(P')$ near P' (in $B_{P'} \cap \gamma_2$).
- We also construct a corresponding vector field on our model in $\mathbb{R}^2$.



- Since the exponential map is a local diffeomorphism we can find a neighborhood of C_1 in the plane and an extension of $\varphi_1|_{C_1}$ to an embedding of this neighborhood into M. Repeating this process we can extend $\varphi_1|_{C_2}$ to an embedding of a neighborhood of C_2 into M.

Proof of the Lemma

- Since our metric is Euclidean near P and P', we see that the two embeddings agree at these points and thus combine to form an embedding $\varphi_2 : A \to M$ with $\varphi_2^{-1}(N_1) = A \cap \gamma_1$ and $\varphi_2^{-1}(N_2) = A \cap \gamma_2$, where A is an annular neighborhood of $C_1 \cup C_2$ in



the plane.

- Next we extend φ_2 to a neighborhood U of the entire disc D. Let S denote the inner boundary of the annulus A. Since $\gamma_1^{-1}\gamma_2$ is homotopic to $\varphi_2(S)$, $\varphi_2(S)$ is also contractible in M.

Proof of the Lemma

- Actually $\varphi_2(S)$ is contractible in $M - (N_1 \cup N_2)$ as the following lemma shows:

Lemma

If V^n, $n \geq 5$, is a smooth manifold, M_1 a smooth submanifold of codimension at least 3, then a loop in $V - M_1$ that is contractible in V is also contractible in $V - M_1$.

- The proof of this lemma follows from the following 2 lemmas due to Whitney.

Lemma (Whitney I)

Let $f : M_1 \to M_2$ be a continuous map of smooth manifolds which is smooth on a closed subset A of M_1. Then there exists a smooth map $g : M_1 \to M_2$ such that $g \simeq f$ (g is homotopic to f) and $g|_A = f|_A$.

Proof of the Lemma

Lemma (Whitney II)

Let $f : M_1 \to M_2$ be a smooth map of smooth manifolds which is an embedding on the closed subset $A \subseteq M_1$. Assume that $\dim M_2 \geq 2\dim M_1 + 1$. *Then there exists an embedding $g : M_1 \to M_2$ approximating f such that $g \simeq f$ and $g|_A = f|_A$.*

- We now choose a continuous extension of φ_2 to $U = A \cup D_0$, $\varphi_2' : U \to M$, that maps $Int(D)$ into $M - (N_1 \cup N_2)$.
- Applying the above lemmas to $\varphi_2'|_{Int(D)}$ we can obtain a smooth embedding $\varphi_3 : U \to M$ coinciding with φ_2 on a neighborhood of $U - Int(D)$, and such that $\varphi_3(u) \notin N_1 \cup N_2$ for $u \notin C_1 \cup C_2$.

Proof of the Lemma

- It remains to extend φ_3 to $U \times \mathbb{R}^{k_1 - 1} \times \mathbb{R}^{k_2 - 1}$. We will find an obstruction to this extension in general, which will be obviated by our assumption that the signs of P and P' are opposite.
- In particular, we use the following intermediate lemma whose statement and proof are due to Milnor. Henceforth let $U' := \varphi_3(U)$ and $\gamma_i := U' \cap \gamma_i, C_i = U \cap C_i$ for $i = 1, 2$.

Lemma (Milnor)

There exist smooth vector fields $\xi_1, \ldots, \xi_{k_1 - 1}, \eta_1, \ldots, \eta_{k_2 - 1}$ on U' such that:

1. ξ_i, η_j *are orthonormal and orthogonal to* U'.
2. ξ_i *are tangent to* N_1 *along* γ_1.
3. η_j *are tangent to* N_2 *along* γ_2.

Proof of the Lemma

Lemma (Milnor)

There exist smooth vector fields $\xi_1, \dots, \xi_{k_1-1}, \eta_1, \dots, \eta_{k_2-1}$ on U' such that:

1. *ξ_i, η_j are orthonormal and orthogonal to U'.*
2. *ξ_i are tangent to N_1 along γ_1.*
3. *η_j are tangent to N_2 along γ_2.*

We shall use the vector fields constructed to define a map $U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1} \to M$ by

$$(u, x_1, \dots, x_{k_1-1}, y_1, \dots, y_{k_2-1}) \mapsto \exp\left[\sum_{i=1}^{k_1-1} x_i \xi_i(\varphi_3(u)) + \sum_{j=1}^{k_2-1} y_j \eta_j(\varphi_3(u))\right]$$

Proof of the Lemma

- It follows from the lemma and the fact that this map is a local diffeomorphism that there exists an open ε-neighborhood B_ε about the origin in $\mathbb{R}^{k_1+k_2-2} = \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1}$ such that if $\varphi_4 : U \times B_\varepsilon \to M$ denotes the restriction of the above map to $U \times B_\varepsilon$, then φ_4 is an embedding. (U may have to be replaced by a slightly smaller neighborhood, which we still denote by U.)
- Define $\varphi : U \times \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1} \to M$ by $\varphi(u, z) = \varphi_4(u, \frac{\varepsilon z}{\sqrt{1+|z|^2}})$.

Proof of the Lemma

- Then $\varphi(C_1 \times \mathbb{R}^{k_1-1} \times 0) \subseteq N_1$ (*This follows because N_1 is a totally geodesic submanifold by choice of metric; on $C_1 \times \mathbb{R}^{k_1-1} \times 0$, φ only deals with vectors $v \in T_p(N_1) \subseteq T_p(M)$, and $\exp(tv)$ is a geodesic in M, tangent to N_1 at $t = 0$, and thus entirely within N_1*).
  Similarly $\varphi(C_2 \times 0 \times \mathbb{R}^{k_2-1}) \subseteq N_2$.
- Moreover, since $\varphi(U \times 0) = U'$ intersects N_1 and N_2 transversely in γ_1 and γ_2, it follows that for ε small enough, $Im(\varphi)$ intersects N_1 and N_2 transversely. Thus $\varphi^{-1}(N_1) = C_1 \times \mathbb{R}^{k_1-1} \times 0$ and $\varphi^{-1}(N_2) = C_2 \times 0 \times \mathbb{R}^{k_2-1}$, as desired.

Milnor's Lemma

Lemma (Milnor)

There exist smooth vector fields $\xi_1, \ldots, \xi_{k_1-1}, \eta_1, \ldots, \eta_{k_2-1}$ on U' such that:

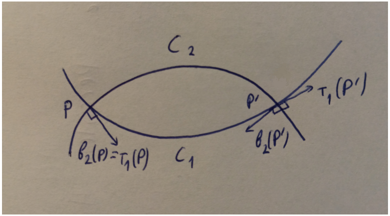
1. *ξ_i, η_j are orthonormal and orthogonal to U'.*
2. *ξ_i are tangent to N_1 along γ_1.*
3. *η_j are tangent to N_2 along γ_2.*

Proof:

- We construct the ξ's in steps: first along γ_1, then extending to $\gamma_1 \cup \gamma_2$ and then finally to all of U'.
- Let τ_1 and τ_2 be the normalized velocity vector fields along γ_1 and γ_2. Let β_2 be the field of unit vectors along γ_2 which are tangent to $U' \subseteq M$ and inward orthogonal to γ_2. Finally, let $\nu(N_2)$ denote the normal bundle of $N_2 \subseteq M$.

Proof of Milnor's Lemma

- Note that $\beta_2(P) = \tau_1(P)$ and $\beta_2(P') = -\tau_1(P')$.



- Choose $k_1 - 1$ vectors $\xi_1(P), \ldots, \xi_{k_1-1}(P)$ which are tangent to N_1 at P, orthogonal to U', and such that the k_1-frame $\tau_1(P), \xi_1(P), \ldots, \xi_{k_1-1}(P)$ is positively oriented in $T_P(N_1)$.
- We parallel translate these vectors to define the ξ's along γ_1. These vectors automatically satisfy (2) because parallel translation along a curve in a a totally geodesic submanifold sends tangent vectors (to N_1) to tangent vectors.

Proof of Milnor's Lemma

- By continuity, the k_1-frame constructed is positively oriented in TN_1 along γ_1. In small neighborhoods of P and P' ($B_P \cap \gamma_2$, $B_{P'} \cap \gamma_2$), we can extend the ξ's along γ_2 by parallel translation.
- We wish to extend the ξ's along the whole of γ_2, however. We have assumed that the intersection numbers of N_1 and N_2 are $+1$ and -1 at P and P', respectively. In other words, $\tau_1(P), \xi_1(P), \ldots, \xi_{k_1-1}(P)$ is positively oriented in $\nu(N_2)$ at P, while negatively oriented in $\nu(N_2)$ at P'.
- But, since $\beta_2(P) = \tau_1(P)$ and $\beta_2(P') = -\tau_1(P')$, at all points of γ_2 near P and P' the frames $\beta_2, \xi_1, \ldots, \xi_{k_1-1}$ are positively oriented in $\nu(N_2)$.

Proof of Milnor's Lemma

- We wish to extend the ξ's to γ_2 – i.e. to find a moving $(k_1 - 1)$-frame over γ_2 agreeing with the frame already defined over γ_1.
- Instead of looking for $(k_1 - 1)$ independent sections over γ_2, we look for a single (nonzero) cross section of the frame-bundle of $(k_1 - 1)$-frames $\zeta_1, \ldots, \zeta_{k_1 - 1}$, orthogonal to N_2 and to U', and such that $\beta_2, \zeta_1, \ldots, \zeta_{k_1 - 1}$ is positively oriented in $\nu(N_2)$.
- This frame bundle is trivial with fiber $SO(n - k_2 - 1) = SO(k_1 - 1)$. Since the fiber is connected (and we are trying to extend over a 1-dimensional manifold γ_2), we can extend $\xi_1, \ldots, \xi_{k_1 - 1}$ to a smooth field of $(k_1 - 1)$-frames on $\gamma_1 \cup \gamma_2$ satisfying conditions (1) and (2).

Proof of Milnor's Lemma

- Aiming to extend to all of U', we must consider the frame-bundle over U' of orthonormal $(k_1 - 1)$-frames orthogonal to U', which is trivial with fiber $O(k_1 + k_2 - 2)/O(k_2 - 1) = V_{k_1-1}(\mathbb{R}^{k_1+k_2-2})$, the Stiefel manifold of orthonormal $(k_1 - 1)$-frames in $\mathbb{R}^{k_1+k_2-2} = \mathbb{R}^{k_1-1} \times \mathbb{R}^{k_2-1}$.
- We already have defined a smooth section of this bundle over $\gamma_1 \cup \gamma_2$. Composing this section with projection onto the fiber gives us a smooth map of $S^1 = \gamma_1 \cup \gamma_2$ into $O(k_1 + k_2 - 2)/O(k_2 - 1)$.
- Thus, the obstruction to extending this section lies in $\pi_1(V_{k_1-1}(\mathbb{R}^{k_1+k_2-2}))$ which, as $k_2 \geq 3$ is trivial.[1] Thus we can perform the required extension to all of U', satisfying (1) and (2).

[1] Recall that we have fibration $V_{k-1}(\mathbb{R}^{n-1}) \to V_k(\mathbb{R}^n) \to S^{n-1}$

Proof of Milnor's Lemma

- Finally, to define the η's, consider the bundle over U' of orthonormal frames $\eta_1, \ldots, \eta_{k_2-1}$ in TM such that each η_i is orthogonal to U' and to the ξ's. This bundle is trivial by the contractibility of U'.
- Hence we can find the field of frames $\eta_1, \ldots, \eta_{k_2-1}$ (cross section of the bundle), which, together with the ξ's satisfy the conditions (1), (2), and (3) of our lemma. Of course, condition (3) is satisfied by the η's because they are orthogonal to the ξ's, which were constructed to be orthogonal to N_2 along γ_2.

Technical Lemma

Lemma

There exists a Riemannian metric on M such that:

1. N_1 *and* N_2 *are totally geodesic submanifolds of* M *(ie if a geodesic in* M *is tangent to* N_1 *or to* N_2 *at any point then it is entirely in* N_1 *or* N_2*, respectively).*
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Proof(Sketch):

Suppose that N_1 and N_2 intersect transversely in points $P_1 = P, P_2 = P', \ldots, P_k$.

Proof of the technical lemma

Cover $N_1 \cup N_2$ by coordinate neighborhoods $W_1, \dots, W_m$ in M with coordinate diffeomorphisms $h_i : W_i \to \mathbb{R}^{k_1+k_2} = \mathbb{R}^n$, $i = 1, \dots, m$, such that:

1. There are disjoint coordinate neighborhoods $B_1, \dots, B_k$ with $P_i \in B_i \subseteq \overline{B_i} \subseteq W_i$ and $N_i \cap W_j = \emptyset$ for $i = 1, \dots, k$ and $j = k+1, \dots, m$.
2. $h_i(W_i \cap N_1) \subseteq \mathbb{R}^r \times 0$ and $h_i(W_i \cap N_2) \subseteq 0 \times \mathbb{R}^r$
3. $h_i(W_i \cap \gamma_1)$ and $h_i(W_i \cap \gamma_2)$ $(i = 1, 2)$ are straight line segments in $\mathbb{R}^{k_1+k_2} = \mathbb{R}^n$

Construct a Riemannian metric $< v, w >$ on the open set $W_0 = W_1 \cup \cdots \cup W_m$ by piecing together the metrics on the W_i induced by the h_i, using a partition of unity. Note that because of (1) this metric is Euclidean in the B_i, $i = 1 \dots, k$.

Proof of the technical lemma

- With this metric construct open tubular neighborhoods T and T' of N_1 and N_2 in W_0 using the exponential map. By choosing them thin enough we may assume that $T \cap T' \subseteq B_1 \cup \cdots \cup B_k$.
- Let $A : T \to T$ be the smooth involution ($A^2 = id$) which is the antipodal map on each fiber of T. Define a new Riemannian metric $< v, w >_A$ on T by $< v, w >_A = \frac{1}{2}(< v, w > + < A_* v, A_* w >)$
- Claim: With respect to this new metric, N_1 is a totally geodesic submanifold of T.
- Similarly define a new metric $< v, w >_{A'}$, on T'. It follows from property (2) and the form of $T \cap T'$ that these two new metrics agree with the old metric on $T \cap T'$ and hence together define a metric on $T \cup T'$.
- Extending to all of M the restriction of this metric to an open set O, with $N_1 \cup N_2 \subseteq O \subseteq \overline{O} \subseteq T \cup T'$, completes the construction of a metric on M satisfying conditions (1) and (2).

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Thank you all for your attention! :)