Handle decompositions

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Recall

h-Cobordism Theorem

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s-Cobordism Theorem

There is a more general statement for M_0 connected with possibly nontrivial fundamental group π such that h-cobordisms over M_0 are described by the so called Whitehead group $Wh(\pi)$.

Outline

- Handle Decomposition
- CW-Structures
- Reducing Handle Decomposition

Handles

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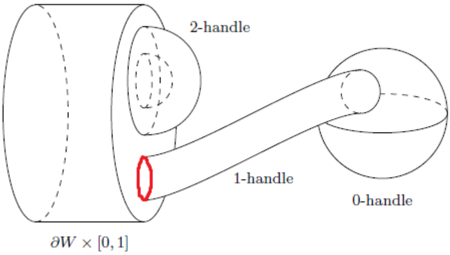

Handlebody Decomposition

In the following, W is a compact manifold of dimension n with boundary $\partial W = \partial_0 W \sqcup \partial_1 W$.

Construction

Consider $W_0 = \partial_0 W \times [0,1]$ and an embedding $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W_0$. Denote $W_1 = W_0 + (\phi^q)$. Iterating this process

$$W_r = W_0 + (\phi^{q_1}) + \cdots + (\phi^q_r)$$

with boundary $\partial W_r = \partial_0 W \sqcup \partial_1 W_r$. We say this is a handle decomposition of W_r relative to $\partial_0 W$.

Handlebody Decomposition

From Morse theory

Lemma

W admits a handlebody decomposition relative to $\partial_0 W$.

$$W = W_0 + (\phi^{q_1}) + \cdots + (\phi^{q_r})$$

where $=$ means diffeomorphic relative to $\partial_0 W$.

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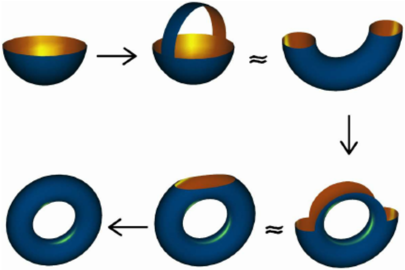

Handle Manipulation

Strategy

We want to decide if W is a trivial cobordism, i.e., we ask whether we can get rid of the handles without changing the diffeomorphism type of W relative to $\partial_0 W$.

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Summary

- Attaching handles via isotopic embeddings ϕ^q and ψ^q gives $W + (\phi^1) = W + (\psi^q)$.
- If $W = W'$ and ϕ^q defines a handle on $\partial_1 W$, then $W + (\phi^q) = W' + (\bar{\phi}^q)$ for some embedding $\bar{\phi}^q$ on $\partial_1 W'$.
- Handles can be ordered by increasing index.

Handle Manipulation

Lemma

Let $\phi^q, \psi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be isotopic embeddings, then $W + (\phi^q) = W + (\psi^q)$.

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Lemma

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Proof.

Let i be an isotopy from ϕ^q to ψ^q. There exists a diffeotopy $H \colon W \times [0,1]$ which is stationary on $\partial_0 W$ and such that $i = H \circ \phi^q \times Id_{[0,1]}$. Then H_1 is a diffeomorphism relative $\partial_0 W$ taking ϕ^q to ψ^q, so it extends to a diffemorphism $W + (\phi^q) \to W + (\psi^q)$. □

Handle Manipulation

Lemma

Let W and W' as in our assumption with boundary $\partial_0 W \sqcup \partial_1 W$ and $\partial_0 W' \sqcup \partial_1 W'$, resp. Let $F\colon W \to W'$ be a diffeomorphism restricting to a diffeomorphism $f_0\colon \partial_0 W \to \partial_0 W'$ and an embedding $\phi^q\colon S^{q-1} \times D^{n-q} \to \partial_1 W$. Then there exists an embedding $\bar{\phi}^q\colon S^{q-1} \times D^{n-q} \to \partial_1 W'$ and a diffeomorphism $F'\colon W + (\phi^q) \to W' + \left(\bar{\phi}^q\right)$ extending F.

Proof.

Pick $\bar{\phi}^q = F \circ \phi^q$ and F' the map induced by F. □

Handle Manipulation

Lemma

Let $V = W + (\psi^r) + (\phi^q)$ with $q \leq r$. Then V is diffeomorphic relative $\partial_0 W$ to $W + (\bar{\phi}^q) + (\psi^r)$ for an embedding $\bar{\phi}^q$.

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Proof.

By the first lemma, it suffices to show ϕ^q is isotopic to an embedding $\bar{\phi}^q$ which doesn't meet the handle ψ^r
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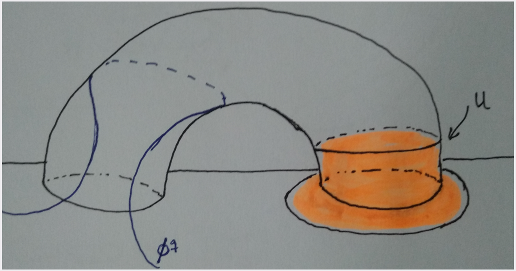
- Since $\dim S^{q-1} \times \{0\} + \dim \{0\} \times S^{n-r-1} = (q-1) + (n-r-1) < n-1$, there is an isotopy of ϕ^q to an embedding whose restriction to $S^{q-1} \times \{0\}$ doesn't meet the transverse sphere of (ψ^r).

□

Proof.

- Since D^{n-q} is contractible, there is a closed neighbourhood $U \subset \partial_1 (W + (\psi^r))$ and an isotopy from ϕ^q to an embedding which does not meet U.
- Take a diffeotopy on $\partial_1 (W + (\psi^r))$ taking all points in $\partial (\psi^r) \setminus U$ outside of the handle (ψ^r). This determines $\bar{\phi}^q$.



Handle Cancellation

Remark

Under certain circumstances, given a handle (ϕ^q) on W we can attach a second handle ψ^{q+1} such that $W + (\phi^q) + \left(\psi^{q+1}\right) = W$.

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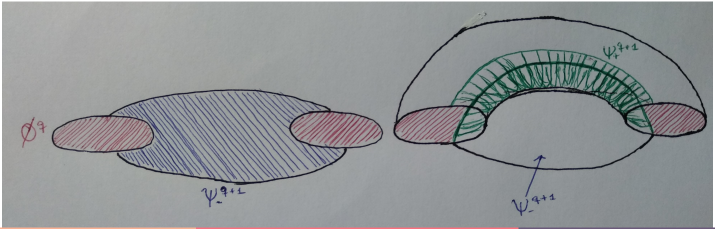
- Pick embedding $\mu\colon S^{q-1} \times D^{n-q} \cup_{S^{q-1} \times S_+^{n-q-1}} D^q \times S_+^{n-q-1} \to \partial_1 W$.
- Let ϕ^q be its restriction to the first factor.
- Let ψ_-^{q+1} be the restriction of μ to the second factor and $\psi_+^{q+1}\colon S_+^q \times S_+^{n-q-1} \to \partial(\phi^q) \subset \partial_1(W + (\phi^q))$ restriction of the characteristic map of ϕ^q. Glue both to the desired ψ^{q+1}.



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Lemma

Let $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ and $\psi^{q+1} \colon S^q \times D^{n-q-1} \to \partial_1 (W + (\phi^q))$ be embeddings such that $\psi^{q+1} (S^q \times \{0\})$ meets the transverse sphere of (ϕ^q) transversely at exactly one point. Then $W = W + (\phi^q) + (\psi^{q+1})$.

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Proof.

Similarly to last lemma, pick $U \subset \partial(\phi^q)$ neighbourhood of the transverse sphere of (ϕ^q) and a diffeotopy on $\partial_1 (W + (\phi^q))$ taking any point in $\partial(\phi^q) \setminus U$ outside the handle (ϕ^q).
Now we are in the situation of the previous example, for which the claim holds. □

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Lemma

Let $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be a trivial embedding. Then there is an embedding $\phi^{q+1} \colon S^q \times D^{n-q-1} \to \partial_1 (W + (\phi^q))$ such that $W = W + (\phi^q) + (\phi^{q+1})$.

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Remark

The Euler characteristic $\chi(W)$ is $\sum_{q \geq 0} (-1)^q p_q$ where p_q is the number of handles of W of index q, so one cannot get rid of a single handle.

Notation

Let $W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} \left(\phi_i^0\right) + \sum_{i=1}^{p_1} \left(\phi_i^1\right) + \cdots + \sum_{i=1}^{p_n} \left(\phi_i^n\right)$ We denote:

- $W_q = \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} \left(\phi_i^0\right) + \sum_{i=1}^{p_1} \left(\phi_i^1\right) + \cdots + \sum_{i=1}^{p_q} \left(\phi_i^q\right)$.
- $\partial_1 W_q = \partial W_q \setminus (\partial_0 W \times \{0\})$.
- $\partial_1^0 W_q = \partial_1 W_q \cap \partial_1 W_{q+1}$

Elimination Lemma

Lemma

Let $1 \leq q \leq n-3$ such that
$W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} \left(\phi_i^q\right) + \sum_{i=1}^{p_{q+1}} \left(\phi_i^{q+1}\right) + \cdots + \sum_{i=1}^{p_n} \left(\phi_i^n\right)$.
Suppose there is i_0 with $1 \leq i_0 \leq p_q$ and an embedding
$\psi^{q+1} \colon S^q \times D^{n-q-1} \to \partial_1^0 W_q$ satisfying:

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1. $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_q$ to an embedding $\psi_1^{q+1} \colon S^q \times \{0\} \to \partial_1 W_q$ meeting transversally the transverse sphere of $\left(\phi_{i_0}^q\right)$ in exactly one point and disjoint to the transverse sphere of any other q-handle.

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2. $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to a trivial embedding $\psi_2^{q+1} \colon S^q \times \{0\} \to \partial_1^0 W_{q+1}$.

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2. $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to a trivial embedding $\psi_2^{q+1} \colon S^q \times \{0\} \to \partial_1^0 W_{q+1}$.

Then $W = \partial_0 W \times [0,1] + \sum_{i \neq i_0} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} \left(\bar{\phi}_i^{q+1}\right) + (\psi^{q+2}) + \sum_{i=1}^{p_{q+2}} \left(\bar{\phi}_i^{q+2}\right) \cdots + \sum_{i=1}^{p_n} (\bar{\phi}_i^n)$

Elimination Lemma

Proof.

The embeddings ψ_1^{q+1} and ψ_2^{q+1} can be extended to handle defining embeddings $\psi_1^{q+1} \colon S^q \times D^{n-q-1} \to \partial_1 W_q$ and $\psi_2^{q+1} \colon S^q \times D^{n-q-1} \to \partial_1^0 W_{q+1}$ such that

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- ψ_1^{q+1} is isotopic to ψ^{q+1} in $\partial_1 W_q$ and ψ_2^{q+1} is isotopic to ψ^{q+1} in $\partial_1 W_{q+1}$
- The conditions of the lemma still hold

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Then the Elimination Lemma follows by suitably applying the previous lemmas. □

Cellular chain complex

Let (X, A) be a relative CW-complex with X connected, fundamental group π and filtration $A \subset X_0 \subset X_1 \subset \cdots \subset X$. Consider $p\colon \widetilde{X} \to X$ universal covering of X and write $\widetilde{A} = p^{-1}(A)$ and $\widetilde{X_q} = p^{-1}(X_q)$. Then we have

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- $\left(\widetilde{X}, \widetilde{A}\right)$ has CW-structure given by $\widetilde{A} \subset \widetilde{X_0} \subset \cdots \subset \widetilde{X}$
- Cellular $\mathbb{Z}\pi$-chain complex $H_q\left(\widetilde{X_q}, \widetilde{X_{q-1}}\right)$ with π action via deck transformations with differential

$$H_q\left(\widetilde{X_q}, \widetilde{X_{q-1}}\right) \xrightarrow{\partial_q} H_{q-1}\left(\widetilde{X_{q-1}}\right) \xrightarrow{i_*} H_{q-1}\left(\widetilde{X_{q-1}}, \widetilde{X_{q-2}}\right)$$

Cellular chain complex

- Construction of $\mathbb{Z}\pi$-basis of $C_q\left(\widetilde{X}, \widetilde{A}\right)$:
  For each $i \in I_q$ q-cell in X given by characteristic map $\left(\Phi_i^q, \phi_1^q\right)$ pick a lift $\left(\widetilde{\Phi_i^q}, \widetilde{\phi_1^q}\right) : \left(D^q, S^{q-1}\right) \to \left(\widetilde{X_q}, \widetilde{X_{q-1}}\right)$ and a generator of $H_q\left(D^q, S^{q-1}\right) \cong \mathbb{Z}$. Write b_i for its image by $\left(\widetilde{\Phi_i^q}, \widetilde{\phi_1^q}\right)$ in $H_q\left(\widetilde{X_q}, \widetilde{X_{q-1}}\right)$. This defines a basis $\{b_i | i \in I_q\}$.

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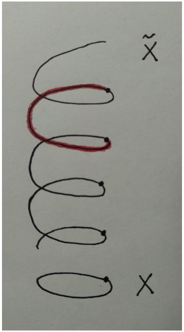
- To make this independent of choices, we say $\{\alpha_j | j \in I_q\}$ and $\{\beta_k | k \in I_q\}$ two such basis are equivalent if there is a bijection $\phi\colon I_q \to I_q$ and elements $\varepsilon_i \in \{\pm 1\}$ and $\gamma_i \in \pi$ for $i \in I_q$ such that $\varepsilon_i \gamma_i \alpha_i = \beta_{\phi(i)}$.



Goal

Let $(W, \partial_0 W)$ as before. We want to find an n-dimensional CW-complex $(X, \partial_0 W)$ and a homotopy equivalence $(f, Id) : (W, \partial_0 W) \xrightarrow{\simeq} (X, \partial_0 W)$.

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Construct inductively spaces $X_{-1} = \partial_0 W \subset X_0 \subset \cdots \subset X_n = X$ and homotopy equivalences $f_q \colon W_q \to X_q$ such that $f_q|_{W_{q-1}} = f_{q-1}$ as follows:

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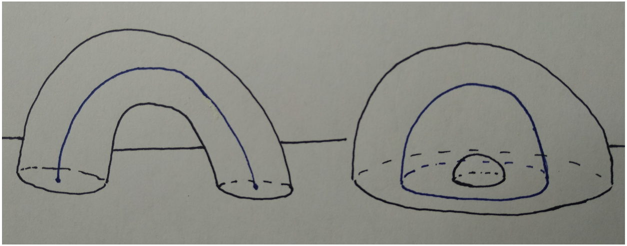
- For $q = -1$ let $f_1 \colon W_1 = \partial_0 W \times [0,1] \to X_1 = \partial_0 W$ the projection.
- Assume we have constructed X_{q-1} and f_{q-1}. For each handle (ϕ_i^q) of W of index q, attach a q-cell to X_{q-1} by $f_{q-1} \circ \phi_i^q|_{S^{q-1} \times \{0\}}$.



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Let $p\colon \widetilde{W} \to W$ universal covering of W and π its fundamental group and write $\widetilde{W_q} = p^{-1}(W)$. The handlebody $\mathbb{Z}\pi$-chain complex $C_*\left(\widetilde{W}, \widetilde{\partial_0 W}\right)$ is given in degree q by $H_q\left(\widetilde{W_q}, \widetilde{W_{q-1}}\right)$ with differential

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Definition

Let $p\colon \widetilde{W} \to W$ universal covering of W and π its fundamental group and write $\widetilde{W_q} = p^{-1}(W)$. The handlebody $\mathbb{Z}\pi$-chain complex $C_*\left(\widetilde{W}, \widetilde{\partial_0 W}\right)$ is given in degree q by $H_q\left(\widetilde{W_q}, \widetilde{W_{q-1}}\right)$ with differential

$$H_q\left(\widetilde{W_q}, \widetilde{W_{q-1}}\right) \xrightarrow{\partial_q} H_{q-1}\left(\widetilde{W_{q-1}}\right) \xrightarrow{i_*} H_{q-1}\left(\widetilde{W_{q-1}}, \widetilde{W_{q-2}}\right)$$

Remark

The homotopy equivelence f induces an isomorphism of $\mathbb{Z}\pi$-chain complexes $C_*\left(\widetilde{W}, \widetilde{\partial_0 W}\right) \cong C_*\left(\widetilde{X}, \widetilde{\partial_0 W}\right)$.

Handlebody chain complex

Remark

- $C_* \left(\widetilde{W}, \widetilde{\partial_0 W} \right)$ has a $\mathbb{Z}\pi$-basis defined in a complete analogous way to CW-complex.
- If W doesn't have handle of index 0 or 1 we can describe the handlebody $\mathbb{Z}\pi$-chain complex in terms of homotopy groups by $\pi_q \left(\widetilde{W_q}, \widetilde{W_{q-1}} \right) \cong H_q \left(\widetilde{W_q}, \widetilde{W_{q-1}} \right)$.

Reducing the Handlebody Decomposition

Lemma

Let W be a compact manifold of dimension $n \geq 6$ with boundary $\partial W = \partial_0 W \sqcup \partial_1 W$. Then the following are equivalent:

1. The inclusion $\partial_0 W \to W$ is 1-connected,
2. We have
$W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\phi_i^3) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n)$

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Proof.

2) $\Rightarrow$ 1) Follows since $W_1 \to W$ is 1-connected and $\partial_0 W \to W_0 = W_1$ is a homotopy equivalence. □

Reducing the Handlebody Decomposition

Proof.

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- By assumption, the inclusion $\partial_0 W \to W_1$ induces an isomorphism on the set of path connected components.
- There is a 1-handle $\left(\phi^1_{i_1}\right)$ such that $\phi^1_{i_1}|_{D^1\times\{0\}}$ connects $\partial_0 W \times \{1\}$ and the handle $\left(\phi^0_{i_0}\right)$.

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- By the Cancellation Lemma, $\left(\phi^1_{i_1}\right)$ cancels $\left(\phi^0_{i_0}\right)$.

$\square$

Proof.

Next we cancel 1-handles via Ellimination Lemma. Let (ϕ_1^1) be a 1-handle. We need to construct an embedding $\psi^2 \colon S^1 \times D^{n-2} \to \partial_1^0 W_1$ satisfying the required conditions:

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- Let $\psi_+^2\colon S_+^1 = D^1 \times \{x\} \subset \left(\phi_1^1\right)$ for some fixed x in the transversal sphere of $\left(\phi_1^1\right)$.

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- We can find an embedding $\psi_-^2 : S_-^1 \to \partial_1^0 W_0$ which glues with ψ_+^2 to $\psi_0^2 \colon S^1 \to \partial_1 W_1$ which is nullhomotopic in W.

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- Since $\dim S^1 + \dim \psi_0^2(S^1) < \dim \partial_1 W_1$, one can isotope the embeddings $\phi_i^2 \colon S^1 \times D^{n-2} \to \partial_1 W_1$ of the 2-handles such that they don't meet $\psi_0^2(S^1)$. Hence ψ_0^2 lies in $\partial_1^0 W_1$

□

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- By connectivity, ψ_0^2 must already be homotopic in $\partial_1 W_2$, say via $h\colon D^2 \to \partial_1 W_2$.

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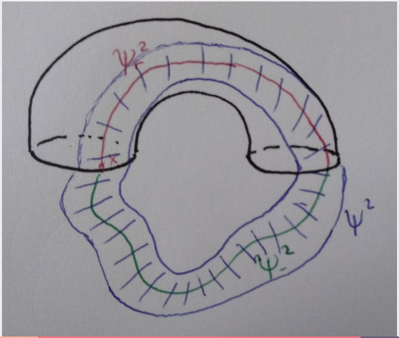
- We can modify h relative to S^1 such that it is an embedding
- Then the normal bundle of h is trivial, hence also for ψ_0^2
- Thus ψ_0^2 extends to an embedding $\psi^2\colon S^1 \times D^{n-1} \to \partial_1^0 W_1$. This satisfies the required conditions by construction



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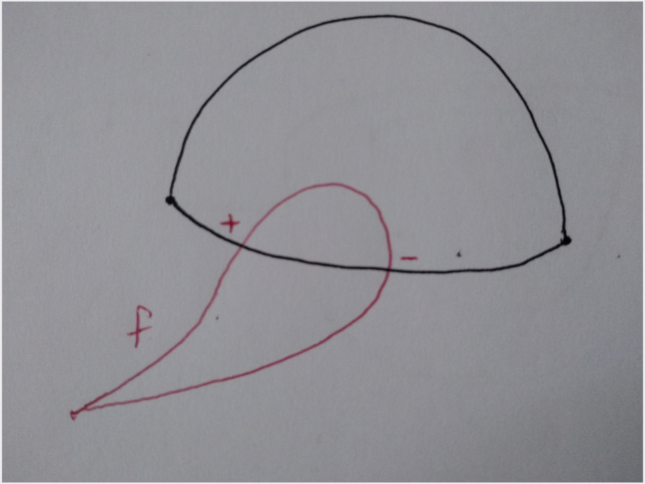
Motivation



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