The h-cobordism theorem and applications

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11. June 2020

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References

1. Surgery Theory: Foundations by Crowley, Lück and Macko.
2. The h-cobordism Theorem by Andrew R. Mackie-Mason.
3. A Concise Course in Algebraic Topology by J. Peter May.
4. Topology and Geometry by Glen Bredon.

h-cobordism Theorem

Definition

Definition 1 (cobordism)

n-dimensional cobordism $(W; M_0, f_0, M_1, f_1)$ *consists of a compact n-dimensional manifold* W *where* $\partial W = \partial_0 W \coprod \partial_1 W$ *a disjoint decomposition, closed* $(n-1)$*-dimensional manifolds* M_0 *and* M_1*, and a diffeomorpshim* $f_0 : M_0 \to \partial_0 W$ *and* $f_1 : M_1 \to \partial_1 W$*.*

Remark 2

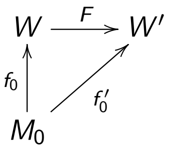
If we want to specify M_0*, we say that* $\underline{W \text{ is a cobordism over } M_0}$*.*

Remark 3

If $\partial_0 W = M_0$*,* $\partial_1 W = M_1$ *and* f_0 *and* f_1 *are given by the identity or if* f_0 *and* f_1 *are obvious from the context, we briefly write* $(W; \partial_0 W, \partial_1 W)$*.*

Definition 4

Two cobordisms $(W; M_0, f_0, M_1, f_1)$ *and* $(W'; M_0, f_0', M_1', f_1')$ *over* M_0 *are diffeomorphic relative* M_0 *if there is a diffeomorphism* $F : W \to W'$ *with* $F \circ f_0 = f_0'$.



Definition 5 (h-cobordism)

A cobordism $(W; M_0, f_0, M_1, f_1)$ *is called h-cobordism, if the inclusion* $\partial_i W \to W$ *for* $i = 0, 1$ *are homotopy equivalence.*

Remark 6

We call an h-cobordism over M_0 *trivial, if it is diffeomorphic relative* M_0 *to the trivial h-cobordism* $(M_0 \times [0,1]; M_0 \times \{0\}, M_0 \times \{1\})$.

Remark 7

The definitions also work with topological manifold.

Theorem 8 (h-Cobordism Theorem)

Every h-cobordism $(W; M_0, f_0, M_1, f_1)$ over a simply connected closed manifold M_0 with $\dim(M_0) \geq 5$ is trivial.

Our goal is to trivialize W: namely show that it is diffeomorphic relative M_0 to

$$(M_0 \times [0,1]; M_0 \times \{0\}, M_0 \times \{1\}).$$

Suffice to show that it is diffeomorphic relative $\partial_0 W$ to

$$(\partial_0 W \times [0,1]; \partial_0 W \times \{0\}, \partial_0 W \times \{1\}).$$

$$
\begin{array}{ccccc}
M_0 \times [0,1] & \xrightarrow{f_0 \times \mathrm{id}} & \partial_0 W \times [0,1] & \xrightarrow{F} & W \\
\uparrow \iota & & \uparrow \iota & \nearrow & \\
M_0 & \xrightarrow{f_0} & \partial_0 W & &
\end{array}
$$

where F is the diffeomorphism between cobordism $\partial_0 W \times [0,1]$ and W relative $\partial_0 W$, and two vertical maps are inclusion maps:

$$\iota : M_0 \to M_0 \times \{0\} \to M_0 \times [0,1]$$

$$\iota : \partial_0 W \to \partial_0 W \times \{0\} \to \partial_0 W \times [0,1].$$

Hence we get a diffeomorphism,

$$(M_0 \times [0,1]; M_0 \times \{0\}, M_0 \times \{1\}) \to (W; M_0, f_0, M_1, f_1).$$

The method of showing $(W; M_0, f_0, M_1, f_1)$ is diffeomorphic relative $\partial_0 W$ to

$$(\partial_0 W \times [0,1]; \partial_0 W \times \{0\}, \partial_0 W \times \{1\}).$$

is to construct a "handlebody decomposition", namely we will reconstruct W from $\partial_0 W \times [0,1]$ by attaching handles and then try to cancel those handles. We will continue the discussion about this in the proof sketch of h-Cobordism Theorem if we have time.

An application of h-cobordism Theorem

Poincaré Conjecture

Theorem (Poincaré Conjecture)

If M is a closed manifold homotopy equivalent to the standard n-sphere S^n, then M homeomorphic to S^n.

For n=1: Classification of closed 1-manifolds.

Every connected closed 1-manifold is homeomorphic to S^1

For n=2: Classification of closed surfaces.

Any connected closed surface is homeomorphic to one of

1. the sphere S^2;
2. a connected sum of tori $\#^g T^2$, for $g \geq 1$;
3. a connected sum of real projective planes $\#^k \mathbb{R}P^2$, for $k \geq 1$.

Neither $\pi_1(\#^g T^2)$ nor $\pi_1(\#^k \mathbb{R}P^2)$ are trivial, but $\pi_1(M) \cong \pi_1(S^2)$ is trivial.

For n=3.

Perelman proved this in 2003 (and won the Fields Medal in 2006) for resolving this case.

For n=4.

Freedman solved this in 1982, and also received a Fields Medal.

For $n \geq 5$.

By theory of cobordisms. In particular, we will prove for $n \geq 6$ using *h*-Cobordism Theorem.

Recap: Homotopy Theory

Definition 9 (n-connected)

Recall that a pair (X, A) is n-connected if

1. *the inclusion $\iota : A \to X$ induces for each base point $a \in A$ a bijection*
   $$\iota_* : \pi_k(A, a) \to \pi_k(X, a)$$
   for $k < n$ and surjection for $q = n$
2. *Or equivalently, if $\pi_0(A) \to \pi_0(X)$ is surjective and $\pi_k(X, A, a) = 0$ for $q \in \{1, ..., n\}$ and each $a \in A$.*

Remark 10

By long exact sequence of homotopy group, the above two definition are indeed equivalent.

Theorem 11

A relative CW-complex (X, A) with no m-cells for $m \leq n$ is n-connected. (cf. May Chp 10.4)

Theorem 12 (The Relative Hurewicz Theorem)

Suppose that X, A are simply connected and that (X, A) is $n-1$-connected, $n \geq 2$. Then $H_k(X, A) = 0$ for all $k < n$ and

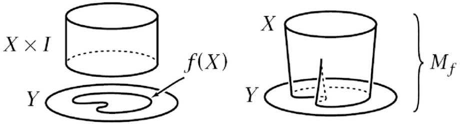
$$h_n : \pi_n(X, A, *) \to H_n(X, A)$$

is an isomorphism. (cf. Bredon Chp VII Thm 9.5)

Theorem 13 (Whitehead)

Weak homotopy equivalence between CW-complexes is indeed a homotopy equivalence. (cf. May Chp 10.3)

Mapping Cylinder for $f : X \to Y$,



$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \tilde{f}} & \nearrow & \\ M_f & & \end{array}$$

$\tilde{f} : x \mapsto (x, 1)$ is an inclusion and $M_f \to Y$ such that $(x, t) \mapsto f(x)$ and $y \mapsto y$ is a homotopy equivalence.

Corollary 14

Suppose that X, A are simply connected CW-complexes, if $f : X \to A$ such that $f_* : H_k(X) \xrightarrow{\cong} H_k(A)$ is an isomorphism for all $k \in \mathbb{Z}$, then f is a homotopy equivalence.

Proof.

1. We pass to the mapping cylinder and assume that f is an inclusion.
2. Then by long exact sequence of homology the hypothesis is then equivalent to $H_k(X, A) = 0$ for all $k \in \mathbb{Z}$.
3. Since both X and A are simply connected, then (X, A) is 1-connected. Inductively using Relative Hurewicz Theorem, we get that $\pi_k(X, A, *) = 0$ for all $k \in \mathbb{Z}$.

4. By long exact sequence of homotopy group $f_* : \pi_k(X) \xrightarrow{\cong} \pi_k(A)$ is an isomorphism. By Whitehead, we get that f is a homotopy equivalence.

□

Proof of Poincaré Conjecture

Theorem 15 (Poincaré Conjecture)

If M is a closed n-manifold, $n \geq 6$, homotopy equivalent to the standard n-sphere S^n, then M homeomorphic to S^n.

Let $D_i^n \hookrightarrow M$ for $i = 0, 1$ be the inclusion of two embedded disjoint disks. Let $W = M \setminus \text{int}(D_0^n) \coprod \text{int}(D_1^n)$, notice that

$$\partial W = \partial D_0^n \coprod \partial D_1^n = S_0^{n-1} \coprod S_1^{n-1}.$$

Note we want to use h-cobordism Theorem, so let us first prove W is indeed a h-cobordism.

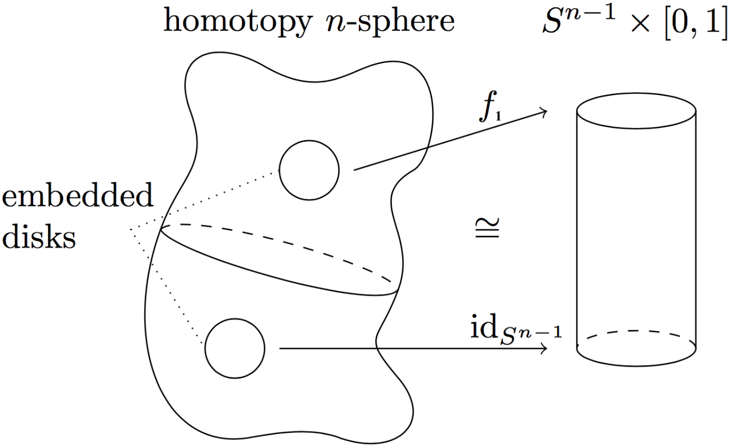
Now suppose that we already prove that W is h-cobordism over a simply connected closed manifold. Hence by h-cobordism theorem we can find a diffeomorphism

$$F : (W; \partial D_0^n, \partial D_1^n) \to (\partial D_0^n \times [0,1]; \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\})$$

By definition we have following commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{F} & \partial D_0^n \times [0,1] \cong S^{n-1} \times [0,1] \,. \\ \uparrow \iota & \nearrow \iota & \\ \partial D_0^n \times \{0\} & & \end{array}$$

Namely, F is identity on $\partial D_0^n \xrightarrow{f_0} \partial D_0^n \times \{0\}$ and induces some (unknown) diffeomorphism $f_1 : \partial D_1^n \to \partial D_0^n \times \{1\}$.


homotopy n-sphere
$S^{n-1} \times [0,1]$
embedded
disks
f_1
$\cong$
$\mathrm{id}_{S^{n-1}}$

Claim 1

W is simply connected.

Proof of Claim 1.

Since (M, W) is a relative CW complex with no m cells for $m \leq n-1$, by **Theorem 11** (M, W) is $n-1$-connected. Hence, W is simply connected. □

Claim 2

$H_k(W, S_i^{n-1}) = 0$ for all $k \in \mathbb{Z}$ and $i = 0, 1$.

Proof of Claim 2.

Step 1:

1. Notice that by definition, we want to show for all $k \in \mathbb{Z}$,

$$H_k(M \setminus \text{int}(D_0^n) \coprod \text{int}(D_1^n), \partial D_0^n) = H_k(W, S_0^{n-1}) = 0.$$

2. By excision (excising $\text{int}(D_0^n)$) we have,

$$H_k(M \setminus \text{int}(D_0^n) \coprod \text{int}(D_1^n), \partial D_0^n) \cong H_k(M \setminus \text{int}(D_1^n), D_0^n).$$

So suffice to show that $H_k(M \setminus \text{int}(D_1^n), D_0^n) = 0$, for all $k \in \mathbb{Z}$.

Step 2:

1. since D_0^n is contractible, for $k \in \mathbb{Z}$,

   $$H_k(M \backslash \text{int}(D_1^n), D_0^n) \cong H_k(M \backslash \text{int}(D_1^n), \text{pt}) = \tilde{H}_k(M \backslash \text{int}(D_1^n)),$$

   where ˜ denotes the reduced homology.
2. Since $M \setminus \text{int}(D_1^n)$ is path connected, $H_0(M \setminus \text{int}(D_1^n), D_0^n) = 0$. Then suffice to show that

   $$H_k(M \setminus \text{int}(D_1^n)) = 0$$

   for all $k \geq 1 \in \mathbb{Z}$.

Step 3: Consider the long exact sequence in homology for pair of space $(M, M \setminus \text{int}(D_1^n))$, we have

$$\cdots \longrightarrow H_{k+1}(M, M \setminus \text{int}(D_1^n)) \xrightarrow{\partial} H_k(M \setminus \text{int}(D_1^n)) \longrightarrow H_k(M) \longrightarrow$$
$$\longrightarrow H_k(M, M \setminus \text{int}(D_1^n)) \xrightarrow{\partial} H_{k-1}(M \setminus \text{int}(D_1^n)) \longrightarrow \cdots$$

Notice that by excision (excising $M \setminus D_1^n$) we have

$$H_k(M, M \setminus \text{int}(D_1^n)) \cong H_k(D_1^n, \partial D_1^n) \cong H_k(S^n, \text{pt}).$$

Then for $k \geq 1$ and $k \neq n, n-1$, we have

$$\cdots \to 0 \to H_k(M \setminus \text{int}(D_1^n)) \to 0 \to \cdots$$

that is $H_k(M \setminus \text{int}(D_1^n)) = 0$ for $k \geq 1$ and $k \neq n, n-1$.

$$\cdots \longrightarrow H_{k+1}(M, M \setminus \mathrm{int}(D_1^n)) \xrightarrow{\partial} H_k(M \setminus \mathrm{int}(D_1^n)) \longrightarrow H_k(M) \longrightarrow$$
$$\longrightarrow H_k(M, M \setminus \mathrm{int}(D_1^n)) \xrightarrow{\partial} H_{k-1}(M \setminus \mathrm{int}(D_1^n)) \longrightarrow \cdots$$

For $k = n, n-1$, since $H_k(M) \xrightarrow{\cong} H_k(M, M \setminus \mathrm{int}(D_1^n))$ we have

$$0 \to H_n(M \setminus \mathrm{int}(D_1^n)) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \to H_{n-1}(M \setminus \mathrm{int}(D_1^n)) \to 0$$

that is $H_k(M \setminus \mathrm{int}(D_1^n)) = 0$ for $k = n, n-1$. □

Claim 3

$S_i^{n-1} \hookrightarrow W$ is homotopy equivalence for $i = 0, 1$.

Proof of Claim 3.

By **Claim 2** $H_k(W, S_i^{n-1}) = 0$ for all k then by **Corollary 14** we know $S_i^{n-1} \hookrightarrow W$ is homotopy equivalence for $i = 0, 1$. □

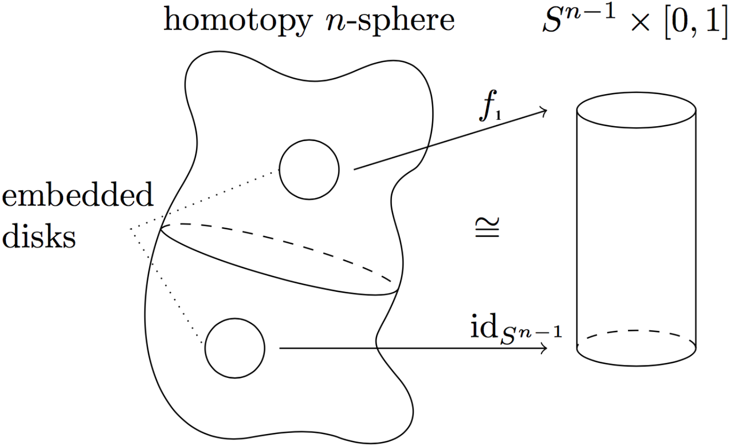
Then W is h-cobordism over a simply connected closed manifold ∂D_0^n where $n \geq 6$. Hence by h-cobordism theorem we can find a diffeomorphism

$$F : (W; \partial D_0^n, \partial D_1^n) \to (\partial D_0^n \times [0,1]; \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\})$$

By definition we have following commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad F \quad} & \partial D_0^n \times [0,1]\,. \\ {\scriptstyle \iota}\Big\uparrow & \nearrow{\scriptstyle \iota} & \\ \partial D_0^n \times \{0\} & & \end{array}$$

Namely, F is identity on $\partial D_0^n \xrightarrow{f_0} \partial D_0^n \times \{0\}$ and induces some (unknown) diffeomorphism $f_1 : \partial D_1^n \to \partial D_0^n \times \{1\}$.


homotopy n-sphere
$S^{n-1} \times [0,1]$
f_1
embedded
disks
$\cong$
$\mathrm{id}_{S^{n-1}}$

Claim 4 (Alexander trick)

Any homeomorphism $f_1 : \partial D_1^n = S^{n-1} \to S^{n-1} = \partial D_0^n \times \{1\}$ *can be extended to a homeomorphism* $\bar{f}_1 : D_1^n = D^n \to D^n = D_0^n \times \{1\}$.

Proof of Claim 4: radical extension.

Note that we can think of D^n as the product $S^{n-1} \times [0, 1]$ with $S^{n-1} \times \{0\}$ identified to a single point. Define $\bar{f}_1$ as follows

$$\bar{f}_1(x, t) = (t \cdot f_1(x), t).$$

The fact that $\bar{f}_1$ is a homeomorphism follows directly from f_1 being a homeomorphism.

Note that we cannot extend this lemma to diffeomorphisms, because problems will arise near $t = 0$. Thus diffeomorphisms $f_1' : S^{n-1} \to S^{n-1}$ only extend to homeomorphisms $D^n \xrightarrow{\bar{f}_1'} D^n$. □

Now define a homeomorphism

$$h : M = D_0^n \bigcup_{j_0} W \bigcup_{j_1} D_1^n \to D_0^n \times \{0\} \bigcup_{i_0} \partial D_0^n \times [0,1] \bigcup_{i_1} D_0^n \times \{1\}$$

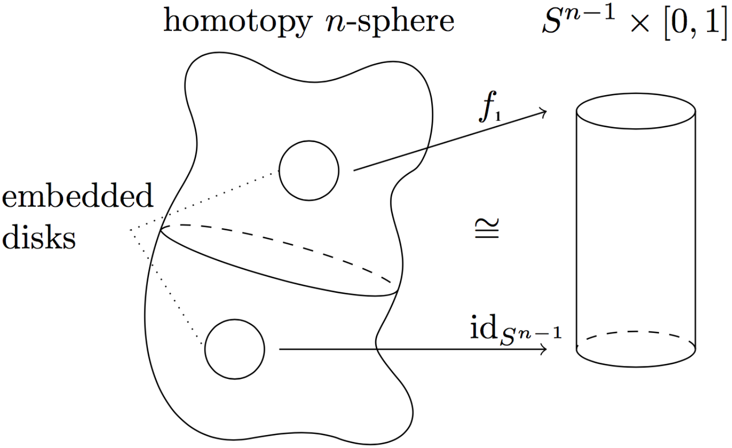
where $i_k : \partial D_0^n \times \{k\} \to D_0^n \times [0,1]$ and $j_k : \partial D_k^n \to W$ for $k = 0, 1$ are the canonical inclusion maps.

$$h|_{D_0^n} = \mathrm{id};$$

$$h|_W = F;$$

$$h|_{D_1^n} = \bar{f}_1.$$

Since $D_0^n \times \{0\} \bigcup_{i_0} \partial D_0^n \times [0,1] \bigcup_{i_1} D_0^n \times \{1\}$ clearly homeomorphic to S^n Poincaré Conjuncture for $n \geq 6$ follows. □


homotopy n-sphere
$S^{n-1} \times [0,1]$
embedded disks
f_1
$\cong$
$\mathrm{id}_{S^{n-1}}$

A proof sketch of h-Cobordism Theorem

Theorem (h-Cobordism Theorem)

Every h-cobordism $(W; M_0, f_0, M_1, f_1)$ *over a simply connected closed manifold* M_0 *with* $\dim(M_0) \geq 5$ *is trivial.*

Suffice to show that it is diffeomorphic relative $\partial_0 W$ to

$$(\partial_0 W \times [0,1]; \partial_0 W \times \{0\}, \partial_0 W \times \{1\}).$$

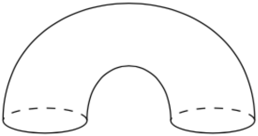
The method for doing this is to construct a "handlebody decomposition"

Definition 16 (Handlebody)

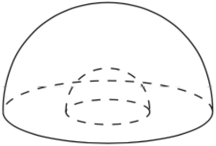
An n-dimensional handle of index q is a structure diffeomorphic to $D^q \times D^{n-q}$. *Where* D^q *and* D^{n-q} *are the closed disk in* $\mathbb{R}^q$ *and* $\mathbb{R}^{n-q}$ *respectively.*

Remark 17

We will refer to this as an (n, q)*-handle or, if the dimension is clearly, simply a q-handle.*



$D^1 \times D^2$



$D^2 \times D^1$

Definition 18 (Attach Handle)

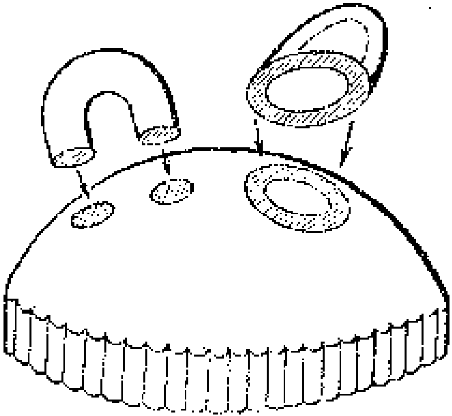
Given a n-dimensional manifold M with boundary ∂M and a smooth embedding $\phi^q : S^{q-1} \times D^{n-q} \to \partial M$, we can attach a q-handle $D^q \times D^{n-q}$ to M. Namely,

$$M \cup_{\phi^q} D^q \times D^{n-q} = (M \coprod D^q \times D^{n-q})/\sim$$

where $x \sim \phi^q(x)$ for $x \in S^{q-1} \times D^{n-q}$. This operation generates a new manifold denoted by $M + (\phi^q)$.

Remark 19

$(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (D^q, S^{q-1})$ is homotopy equivalence, we can think ϕ^q as the attaching map of a q-cell.



Remark 20

Note that $M + (\phi^q)$ is obviously a topological manifold, but one can use some technique to get rid of the corners and get smooth attaching.

Definition 21 (handlebody decomposition)

A handlebody decomposition of a manifold W with $\partial W = \partial_0 W \coprod \partial_1 W$ (relative to $\partial_0 W$) is another manifold W' diffeomorphic to W relative to $\partial_0 W$ with

$$W' = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_n^{q_n})$$

where image of $\phi_i^{q_i}$ is contained in $\partial_1(\partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_{i-1}^{q_{i-1}}))$. (note: q_i not necessarily distinct and increasing)

Lemma 22

If W is a compact manifold with $\partial W = \partial_0 W \coprod \partial_1 W$, then there exists a handlebody decomposition of W rel $\partial_0 W$.

Now in order to show that the W is diffeomorphic to the trivial h-cobordism, we need we find a way to smoothly remove the handles. Luckily we will see a bunch of lemmas which allow us to do lots of operation on the handlebody decomposition, for instance rearrange handles, cancel some handles. Then have the following normal form lemma.

Lemma 23 (Normal form)

Take an h-cobordism $(W; \partial_0 W, \partial_1 W)$ *with* $\dim(W) \geq 6$ *and* $\partial_0 W$ *simply connected. Then for any* $2 \leq q \leq n-3$*, we have*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

Think $(W, \partial_0 W)$ as CW-complex, then

$$W_q = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q)$$

and

$$W = W_{q+1} = W_q + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Then we have the following CW-chain complex for $(W, \partial_0 W)$

$$\cdots \to 0 \to C^{\text{cell}}_{p+1}(W, \partial_0 W) \xrightarrow{d_{p+1}} C^{\text{cell}}_{p}(W, \partial_0 W) \to 0 \to \cdots$$

Namely,

$$\cdots \to 0 \to H_{q+1}(W_{q+1}, W_q) \xrightarrow{d_{p+1}} H_q(W_q, W_{q-1}) \to 0 \to \cdots$$

$H_{q+1}(W_{q+1}, W_q)$ has $\mathbb{Z}$-basis defined by attaching $\{[\phi_i^{q+1}]\}_i$ and $H_q(W_q, W_{q-1})$ has $\mathbb{Z}$-basis defined by attaching $\{[\phi_i^{q}]\}_i$. Then d_{p+1} is given as $p_{q+1} \times p_q$ matrix.

$$\cdots \to 0 \to H_{q+1}(W_{q+1}, W_q) \xrightarrow{d_{p+1}} H_q(W_q, W_{q-1}) \to 0 \to \cdots$$

Remark 24

The definition of $[\phi_i^q]$ are similar with CW-complex. Consider the following map

$$(\Phi_1^q, \phi_1^q) : (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (W^q, W^{q-1})$$

where Φ_1^q is characteristic map.
Then we have the induced map on homology, namely

$$H_q(\Phi_1^q, \phi_1^q) : H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to H_q(W^q, W^{q-1}).$$

Note $H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \cong \mathbb{Z}$, $[\phi_i^q]$ the image of preferred generator. Then d_{p+1} is given as $p_{q+1} \times p_q$ matrix.

Remark 25

Since, $(W; \partial_0 W, \partial_1 W)$ is h-cobordisms then $H_i(W, \partial_0 W)$ vanishes; hence,

$$\cdots \to 0 \to H_{q+1}(W_{q+1}, W_q) \xrightarrow{d_{p+1}} H_q(W_q, W_{q-1}) \to 0 \to \cdots$$

d_{p+1} isomorphism.
Also $p_q = p_{q+1}$, and d_{p+1} is given as $p_q \times p_q$ matrix, we call this matrix representative matrix.

If matrix representative matrix of d_{p+1} is the empty matrix, then W trivial.

Lemma 26

Take an h-cobordism $(W; \partial_0 W, \partial_1 W)$ with $\dim(W) \geq 6$ and $\partial_0 W$ simply connected. Let A be its representative matrix. If B be any matrix formed from A using any of the following operation, Then there is another handlebody decomposition of W with B as its representative matrix.

1. *B is obtained from A by adding a multiple of the k_{th} row to the l_{th} row, for $k \neq l$;*
2. *B is obtained from A by multiplying the k_{th} row by -1;*
3. *B is obtained from A by interchanging two rows or two columns;*

4. B is of the form $A \oplus I_1$ i.e.

$$B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix};$$

5. A is of the form $B \oplus I_1$ i.e.

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

Using previous lemma rule $1-3$ we can change the matrix representative matrix of d_{p+1} to identity matrix. Then by rule 5, we can change the identity matrix to trivial matrix.