Exotic Spheres

Seminar: Characteristic Classes II

Thibaud van den Hove

June 3, 2020

Goal

We want to show existence of exotic spheres, i.e. construct a smooth n-manifold that is homotopy equivalent, but not diffeomorphic to S^n.

Goal

We want to show existence of exotic spheres, i.e. construct a smooth n-manifold that is homotopy equivalent, but not diffeomorphic to S^n.

This will be done by considering the space of unit vectors of some vector bundle. (Unless specified, all our vector bundles will be real.) First, we must construct this vector bundle.

Proposition

There exists an oriented 4-plane bundle ξ over S^4, with

$$p_1(\xi) = -2u \quad \text{and} \quad e(\xi) = u,$$

where u is a generator of $H^4(S^4, \mathbb{Z})$.

Preparations

Proposition

There exists an oriented 4-plane bundle ξ over S^4, with

$$p_1(\xi) = -2u \quad \text{and} \quad e(\xi) = u,$$

where u is a generator of $H^4(S^4, \mathbb{Z})$.

Proof

Consider the skew-field $\mathbb{H}$ of quaternions. For any $m \geq 1$, we have the projective space $\mathbb{P}^m(\mathbb{H})$ of 1-dimensional subspaces of $\mathbb{H}^{m+1}$. This is a $4m$-dimensional smooth manifold.

Preparations

Proposition

There exists an oriented 4-plane bundle ξ over S^4, with

$$p_1(\xi) = -2u \quad \text{and} \quad e(\xi) = u,$$

where u is a generator of $H^4(S^4, \mathbb{Z})$.

Proof

Consider the skew-field $\mathbb{H}$ of quaternions. For any $m \geq 1$, we have the projective space $\mathbb{P}^m(\mathbb{H})$ of 1-dimensional subspaces of $\mathbb{H}^{m+1}$. This is a $4m$-dimensional smooth manifold.
Let γ be the canonical quaternion line bundle over $\mathbb{P}^m(\mathbb{H})$:

$$E(\gamma) = \{(L, v) \mid L \in \mathbb{P}^m(\mathbb{H}), v \in L\}.$$

The space of unit vectors of $E(\gamma)$ can be identified with $S^{4m+3} \subseteq \mathbb{H}^{m+1}$.

Proof

Consider the underlying real (complex) bundle γ_R (γ_C). As the underlying bundle of γ_C, γ_R is orientable.

We want to compute $H^*(\mathbb{P}^m(\mathbb{H}))$. To do this, we can use the Gysin sequence for γ_R:

$$\ldots \to H^{i+n-1}(E_0) \to H^i(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\cup e} H^{i+n}(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\xi_0^*} H^{i+n}(E_0) \to H^{i+1}(\mathbb{P}^m(\mathbb{H})) \to \ldots$$

Proof

Consider the underlying real (complex) bundle γ_R (γ_C). As the underlying bundle of γ_C, γ_R is orientable.

We want to compute $H^*(\mathbb{P}^m(\mathbb{H}))$. To do this, we can use the Gysin sequence for γ_R:

$$\ldots \to H^{i+3}(S^{4m+3}) \to H^i(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\cup e} H^{i+4}(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\xi_0^*} H^{i+4}(S^{4m+3}) \to H^{i+1}(\mathbb{P}^m(\mathbb{H})) \to \ldots$$

Preparations

Proof

Consider the underlying real (complex) bundle γ_R (γ_C). As the underlying bundle of γ_C, γ_R is orientable.
We want to compute $H^*(\mathbb{P}^m(\mathbb{H}))$. To do this, we can use the Gysin sequence for γ_R:

$$\ldots \to H^{i+3}(S^{4m+3}) \to H^i(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\cup e} H^{i+4}(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\xi_0^*} H^{i+4}(S^{4m+3}) \to H^{i+1}(\mathbb{P}^m(\mathbb{H})) \to \ldots$$

As $H^0(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$, we see that for $k < 4m+3$, we have $H^k(\mathbb{P}^m(\mathbb{H})) = 0$ for $k \not\equiv 0 \mod 4$, and $H^k(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$ for $k \equiv 0 \mod 4$, generated by $e(\gamma_R)^{k/4}$.

Proof

Consider the underlying real (complex) bundle γ_R (γ_C). As the underlying bundle of γ_C, γ_R is orientable.
We want to compute $H^*(\mathbb{P}^m(\mathbb{H}))$. To do this, we can use the Gysin sequence for γ_R:

$$\ldots \to H^{i+3}(S^{4m+3}) \to H^i(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\cup e} H^{i+4}(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\xi_0^*} H^{i+4}(S^{4m+3}) \to H^{i+1}(\mathbb{P}^m(\mathbb{H})) \to \ldots$$

As $H^0(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$, we see that for $k < 4m+3$, we have $H^k(\mathbb{P}^m(\mathbb{H})) = 0$ for $k \not\equiv 0 \mod 4$, and $H^k(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$ for $k \equiv 0 \mod 4$, generated by $e(\gamma_R)^{k/4}$.
For $n = 4m+3, 4m+4$, we have an exact sequence

$$0 \to H^{4m+3}(\mathbb{P}^m(\mathbb{H})) \to H^{4m+3}(S^{4m+3}) \to H^{4m}(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\cup e} H^{4m+4}(\mathbb{P}^m(\mathbb{H})) \to 0.$$

Preparations

Proof

Consider the underlying real (complex) bundle γ_R (γ_C). As the underlying bundle of γ_C, γ_R is orientable.
We want to compute $H^*(\mathbb{P}^m(\mathbb{H}))$. To do this, we can use the Gysin sequence for γ_R:

$$\ldots \to H^{i+3}(S^{4m+3}) \to H^i(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\cup e} H^{i+4}(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\xi_0^*} H^{i+4}(S^{4m+3}) \to H^{i+1}(\mathbb{P}^m(\mathbb{H})) \to \ldots$$

As $H^0(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$, we see that for $k < 4m+3$, we have $H^k(\mathbb{P}^m(\mathbb{H})) = 0$ for $k \not\equiv 0 \mod 4$, and $H^k(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$ for $k \equiv 0 \mod 4$, generated by $e(\gamma_R)^{k/4}$.
For $n = 4m+3, 4m+4$, we have an exact sequence

$$0 \to H^{4m+3}(\mathbb{P}^m(\mathbb{H})) \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{\cup e} H^{4m+4}(\mathbb{P}^m(\mathbb{H})) \to 0.$$

Preparations

Proof

Consider the underlying real (complex) bundle γ_R (γ_C). As the underlying bundle of γ_C, γ_R is orientable.

We want to compute $H^*(\mathbb{P}^m(\mathbb{H}))$. To do this, we can use the Gysin sequence for γ_R:

$$\ldots \to H^{i+3}(S^{4m+3}) \to H^i(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\cup e} H^{i+4}(\mathbb{P}^m(\mathbb{H})) \xrightarrow{\xi_0^*} H^{i+4}(S^{4m+3}) \to H^{i+1}(\mathbb{P}^m(\mathbb{H})) \to \ldots$$

As $H^0(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$, we see that for $k < 4m+3$, we have $H^k(\mathbb{P}^m(\mathbb{H})) = 0$ for $k \not\equiv 0 \mod 4$, and $H^k(\mathbb{P}^m(\mathbb{H})) = \mathbb{Z}$ for $k \equiv 0 \mod 4$, generated by $e(\gamma_R)^{k/4}$.

For $n = 4m+3, 4m+4$, we have an exact sequence

$$0 \to H^{4m+3}(\mathbb{P}^m(\mathbb{H})) \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{\cup e} H^{4m+4}(\mathbb{P}^m(\mathbb{H})) \to 0.$$

In particular, $H^{4m+4}(\mathbb{P}^m(\mathbb{H}))$ is generated by $e(\gamma_R)^{m+1}$.

Proof

Consider the section

$$[z_0 : z_1 : \ldots : z_m] \mapsto \frac{(\overline{z_0}z_0, \overline{z_0}z_1, \ldots, \overline{z_0}z_m)}{\sum_i |z_i|^2}$$

of γ_R.

Preparations

Proof

Consider the section

$$[\lambda z_0 : \lambda z_1 : \ldots : \lambda z_m] \mapsto \frac{(\overline{z_0}\overline{\lambda}\lambda z_0, \overline{z_0}\overline{\lambda}\lambda z_1, \ldots, \overline{z_0}\overline{\lambda}\lambda z_m)}{|\lambda|^2 \sum_i |z_i|^2}$$

of γ_R.

Proof

Consider the section

$$[z_0 : z_1 : \ldots : z_m] \mapsto \frac{(\overline{z_0}z_0, \overline{z_0}z_1, \ldots, \overline{z_0}z_m)}{\sum_i |z_i|^2}$$

of γ_R. This is a well-defined, smooth section, which can only vanish when $z_0 = 0$.

Preparations

Proof

Consider the section

$$[z_0 : z_1 : \ldots : z_m] \mapsto \frac{(\overline{z_0} z_0, \overline{z_0} z_1, \ldots, \overline{z_0} z_m)}{\sum_i |z_i|^2}$$

of γ_R. This is a well-defined, smooth section, which can only vanish when $z_0 = 0$. Using this, we can construct a nowhere vanishing section of $\bigoplus_{i=0}^m \gamma_R$, and hence $e(\bigoplus_{i=0}^m \gamma_R) = 0$. Consequently, $e(\gamma_R)^{m+1} = 0$.

Proof

Consider the section

$$[z_0 : z_1 : \ldots : z_m] \mapsto \frac{(\overline{z_0} z_0, \overline{z_0} z_1, \ldots, \overline{z_0} z_m)}{\sum_i |z_i|^2}$$

of γ_R. This is a well-defined, smooth section, which can only vanish when $z_0 = 0$.
Using this, we can construct a nowhere vanishing section of $\bigoplus_{i=0}^m \gamma_R$, and hence $e(\bigoplus_{i=0}^m \gamma_R) = 0$. Consequently, $e(\gamma_R)^{m+1} = 0$.
So $H^{4m+4}(\mathbb{P}^m(\mathbb{H})) = 0$, and by exactness of

$$0 \to H^{4m+3}(\mathbb{P}^m(\mathbb{H})) \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{\cup e} H^{4m+4}(\mathbb{P}^m(\mathbb{H})) \to 0,$$

$H^{4m+3}(\mathbb{P}^m(\mathbb{H})) = 0$.
So $H^*(\mathbb{P}^m(\mathbb{H}))$ is a truncated polynomial ring, generated by $u := e(\gamma_R) \in H^4(\mathbb{P}^m(\mathbb{H}))$.

Preparations

Proof

Consider the section

$$[z_0 : z_1 : \ldots : z_m] \mapsto \frac{(\overline{z_0} z_0, \overline{z_0} z_1, \ldots, \overline{z_0} z_m)}{\sum_i |z_i|^2}$$

of γ_R. This is a well-defined, smooth section, which can only vanish when $z_0 = 0$.
Using this, we can construct a nowhere vanishing section of $\bigoplus_{i=0}^m \gamma_R$, and hence $e(\bigoplus_{i=0}^m \gamma_R) = 0$. Consequently, $e(\gamma_R)^{m+1} = 0$.
So $H^{4m+4}(\mathbb{P}^m(\mathbb{H})) = 0$, and by exactness of

$$0 \to H^{4m+3}(\mathbb{P}^m(\mathbb{H})) \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{\cup e} H^{4m+4}(\mathbb{P}^m(\mathbb{H})) \to 0,$$

$H^{4m+3}(\mathbb{P}^m(\mathbb{H})) = 0$.
So $H^*(\mathbb{P}^m(\mathbb{H}))$ is a truncated polynomial ring, generated by $u := e(\gamma_R) \in H^4(\mathbb{P}^m(\mathbb{H}))$.
As $H^2(\mathbb{P}^m(\mathbb{H})) = 0$, the total Chern class of γ_C is $c(\gamma_C) = 1 + u$.

Preparations

Lemma

For a complex vector bundle ω, *consider the Chern classes* $c_i := c_i(\omega)$ *and the Pontryagin classes* $p_i := p_i(\omega_R)$. *Then we have*

$$1 - p_1 + p_2 - \ldots \pm p_n = (1 - c_1 + c_2 - \ldots \pm c_n)(1 + c_1 + c_2 + \ldots + c_n).$$

Preparations

Lemma

For a complex vector bundle ω, consider the Chern classes $c_i := c_i(\omega)$ and the Pontryagin classes $p_i := p_i(\omega_R)$. Then we have

$$1 - p_1 + p_2 - \ldots \pm p_n = (1 - c_1 + c_2 - \ldots \pm c_n)(1 + c_1 + c_2 + \ldots + c_n).$$

Proof.

This follows from 3 facts we have seen before:

1. $(\omega_R)_C \cong \omega \oplus \overline{\omega}$,
2. $c(\xi_1 \oplus \xi_2) = c(\xi_1) \cup c(\xi_2)$ for any two complex vector bundles ξ_1, ξ_2,
3. $c_k(\overline{\omega}) = (-1)^k c_k(\omega)$. In particular, $c(\overline{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \ldots$

Preparations

Lemma

For a complex vector bundle ω, consider the Chern classes $c_i := c_i(\omega)$ and the Pontryagin classes $p_i := p_i(\omega_R)$. Then we have

$$1 - p_1 + p_2 - \ldots \pm p_n = (1 - c_1 + c_2 - \ldots \pm c_n)(1 + c_1 + c_2 + \ldots + c_n).$$

Proof.

This follows from 3 facts we have seen before:

1. $(\omega_R)_C \cong \omega \oplus \overline{\omega}$,
2. $c(\xi_1 \oplus \xi_2) = c(\xi_1) \cup c(\xi_2)$ for any two complex vector bundles ξ_1, ξ_2,
3. $c_k(\overline{\omega}) = (-1)^k c_k(\omega)$. In particular, $c(\overline{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \ldots$

Now use the definition of Pontryagin classes, and note that terms of odd degree of $(1 - c_1 + c_2 - \ldots \pm c_n)(1 + c_1 + c_2 + \ldots + c_n)$ vanish. □

Back to the proof of the proposition.

Applying this lemma to γ_C, we see that

$$1 - p_1(\gamma_R) + p_2(\gamma_R) = 1 + 2u + u^2,$$

so $p_1(\gamma_R) = -2u$.

Back to the proof of the proposition.

Applying this lemma to γ_C, we see that

$$1 - p_1(\gamma_R) + p_2(\gamma_R) = 1 + 2u + u^2,$$

so $p_1(\gamma_R) = -2u$.
Specializing to the case $m = 1$, we get a real vector bundle over $\mathbb{P}^1(\mathbb{H}) \cong S^4$, satisfying the requirements. $\square$

Back to the proof of the proposition.

Applying this lemma to γ_C, we see that

$$1 - p_1(\gamma_R) + p_2(\gamma_R) = 1 + 2u + u^2,$$

so $p_1(\gamma_R) = -2u$.
Specializing to the case $m = 1$, we get a real vector bundle over $\mathbb{P}^1(\mathbb{H}) \cong S^4$, satisfying the requirements. □

Now, fix this $u \in H^4(S^4)$.

Preparations

Proposition

Let k, l be integers with $k \equiv 2l \mod 4$. Then there exists an oriented 4-plane bundle ξ over S^4 with

$$p_1(\xi) = ku \quad \text{and} \quad e(\xi) = lu.$$

Proposition

Let k, l be integers with $k \equiv 2l \mod 4$. Then there exists an oriented 4-plane bundle ξ over S^4 with

$$p_1(\xi) = ku \quad \text{and} \quad e(\xi) = lu.$$

Proof

Let $\widetilde{G}_4$ be the space of oriented 4-planes in $\mathbb{R}^\infty$, with universal bundle $\widetilde{\gamma}^4$. Consider the function

$$\varphi : \pi_4(\widetilde{G}_4) \to H^4(S^4) \cong \mathbb{Z} : [f] \mapsto p_1(f^*\widetilde{\gamma}^4) = f^* p_1(\widetilde{\gamma}^4).$$

Preparations

Proposition

Let k, l be integers with $k \equiv 2l \mod 4$. Then there exists an oriented 4-plane bundle ξ over S^4 with

$$p_1(\xi) = ku \quad \text{and} \quad e(\xi) = lu.$$

Proof

Let $\widetilde{G}_4$ be the space of oriented 4-planes in $\mathbb{R}^\infty$, with universal bundle $\widetilde{\gamma}^4$. Consider the function

$$\varphi : \pi_4(\widetilde{G}_4) \to H^4(S^4) \cong \mathbb{Z} : [f] \mapsto p_1(f^*\widetilde{\gamma}^4) = f^*p_1(\widetilde{\gamma}^4).$$

This is a homomorphism, as $\langle f^*p_1(\widetilde{\gamma}^4), [S^4]\rangle = \langle p_1(\widetilde{\gamma}^4), f_*[S^4]\rangle$, and the Hurewicz map $h : \pi_4(\widetilde{G}_4) \to H_4(\widetilde{G}_4) : [f] \mapsto f_*[S_4]$ is a homomorphism.

Preparations

Proposition

Let k, l be integers with $k \equiv 2l \mod 4$. Then there exists an oriented 4-plane bundle ξ over S^4 with

$$p_1(\xi) = ku \quad \text{and} \quad e(\xi) = lu.$$

Proof

Let $\widetilde{G}_4$ be the space of oriented 4-planes in $\mathbb{R}^\infty$, with universal bundle $\widetilde{\gamma}^4$. Consider the function

$$\varphi : \pi_4(\widetilde{G}_4) \to H^4(S^4) \cong \mathbb{Z} : [f] \mapsto p_1(f^*\widetilde{\gamma}^4) = f^*p_1(\widetilde{\gamma}^4).$$

This is a homomorphism, as $\langle f^*p_1(\widetilde{\gamma}^4), [S^4]\rangle = \langle p_1(\widetilde{\gamma}^4), f_*[S^4]\rangle$, and the Hurewicz map $h : \pi_4(\widetilde{G}_4) \to H_4(\widetilde{G}_4) : [f] \mapsto f_*[S_4]$ is a homomorphism.
Similarly, we have a homomorphism

$$\psi : \pi_4(\widetilde{G}_4) \to H^4(S^4) \cong \mathbb{Z} : [f] \mapsto e(f^*\widetilde{\gamma}^4).$$

Preparations

Proof.

Let $f_1 : S^4 \to \widetilde{G}_4$ classify the (oriented) tangent bundle of S^4. Then

$$\varphi(f_1) = 0 \quad \text{and} \quad \psi(f_1) = 2u.$$

Preparations

Proof.

Let $f_1 : S^4 \to \widetilde{G}_4$ classify the (oriented) tangent bundle of S^4. Then

$$\varphi(f_1) = 0 \quad \text{and} \quad \psi(f_1) = 2u.$$

Similarly, we have $f_2 : S^4 \to \widetilde{G}_4$ classifying γ_R from the previous proposition, with

$$\varphi(f_2) = -2u \quad \text{and} \quad \psi(f_2) = u.$$

Preparations

Proof.

Let $f_1 : S^4 \to \widetilde{G}_4$ classify the (oriented) tangent bundle of S^4. Then

$$\varphi(f_1) = 0 \quad \text{and} \quad \psi(f_1) = 2u.$$

Similarly, we have $f_2 : S^4 \to \widetilde{G}_4$ classifying γ_R from the previous proposition, with

$$\varphi(f_2) = -2u \quad \text{and} \quad \psi(f_2) = u.$$

Then the vector bundle $\xi = [\frac{k+2l}{4} f_1 - \frac{k}{2} f_2]^* \widetilde{\gamma}^4$ is oriented and satisfies

$$p_1(\xi) = ku \quad \text{and } e(\xi) = lu.$$

□

Let $k \equiv 2 \mod 4$ be an integer. By the previous proposition, there exists an oriented 4-plane bundle ξ over S^4 with

$$p_1(\xi) = ku \quad \text{and} \quad e(\xi) = u.$$

Let $k \equiv 2 \mod 4$ be an integer. By the previous proposition, there exists an oriented 4-plane bundle ξ over S^4 with

$$p_1(\xi) = ku \quad \text{and} \quad e(\xi) = u.$$

Let $E = E(\xi)$ be the total space of this vector bundle, E' the space of vectors with length ≤ 1, and $\partial E'$ the space of unit vectors. We want to show $\partial E'$ is homotopy equivalent to S^7.

Existence of Exotic Spheres

Look at the Gysin sequence

$$\dots \to H^{i+3}(\partial E') \to H^{i}(S^4) \xrightarrow{\cup e} H^{i+4}(S^4) \to H^{i+4}(\partial E') \to H^{i+1}(S^4) \to \dots$$

Existence of Exotic Spheres

Look at the Gysin sequence

$$\ldots \to H^{i+3}(\partial E') \to H^i(S^4) \xrightarrow{\cup e} H^{i+4}(S^4) \to H^{i+4}(\partial E') \to H^{i+1}(S^4) \to \ldots$$

As $e(\xi) = u$ is a generator of $H^4(S^4)$, the map $\cup e : H^0(S^4) \to H^4(S^4)$ is an isomorphism. Consequently, $H^*(\partial E', \mathbb{Z}) \cong H^*(S^7, \mathbb{Z})$.

Look at the Gysin sequence

$$\ldots \to H^{i+3}(\partial E') \to H^i(S^4) \xrightarrow{\cup e} H^{i+4}(S^4) \to H^{i+4}(\partial E') \to H^{i+1}(S^4) \to \ldots$$

As $e(\xi) = u$ is a generator of $H^4(S^4)$, the map $\cup e : H^0(S^4) \to H^4(S^4)$ is an isomorphism. Consequently, $H^*(\partial E', \mathbb{Z}) \cong H^*(S^7, \mathbb{Z})$.

We also have a fibration $S^3 \to \partial E' \to S^4$. The associated long exact homotopy sequence shows that

$$\ldots \to \pi_1(S^3) \to \pi_1(\partial E') \to \pi_1(S^4) \to \ldots$$

is exact.

Look at the Gysin sequence

$$\ldots \to H^{i+3}(\partial E') \to H^i(S^4) \xrightarrow{\cup e} H^{i+4}(S^4) \to H^{i+4}(\partial E') \to H^{i+1}(S^4) \to \ldots$$

As $e(\xi) = u$ is a generator of $H^4(S^4)$, the map $\cup e : H^0(S^4) \to H^4(S^4)$ is an isomorphism. Consequently, $H^*(\partial E', \mathbb{Z}) \cong H^*(S^7, \mathbb{Z})$.

We also have a fibration $S^3 \to \partial E' \to S^4$. The associated long exact homotopy sequence shows that

$$\ldots \to \pi_1(S^3) \to \pi_1(\partial E') \to \pi_1(S^4) \to \ldots$$

is exact.

So $\partial E'$ is simply connected. As we also have $H_*(\partial E', \mathbb{Z}) \cong H_*(S^7, \mathbb{Z})$ by Poincaré duality, and a degree 1 map $S^7 \to \partial E'$ by the Hopf theorem, $\partial E'$ is homotopy equivalent to S^7.

Existence of Exotic Spheres

Now suppose there exists a diffeomorphism $\varphi : S^7 \to \partial E'$. Define

$$M := E' \cup_\varphi D^8.$$

Now suppose there exists a diffeomorphism $\varphi : S^7 \to \partial E'$. Define

$$M := E' \cup_\varphi D^8.$$

Then $M \simeq E' \cup_{\partial E'} C(\partial E')$. As this is defined by coning of $\partial E'$ in E', this is homotopy equivalent to $E'/\partial E' = Th(\xi)$.

Now suppose there exists a diffeomorphism $\varphi : S^7 \to \partial E'$. Define

$$M := E' \cup_\varphi D^8.$$

Then $M \simeq E' \cup_{\partial E'} C(\partial E')$. As this is defined by coning of $\partial E'$ in E', this is homotopy equivalent to $E'/\partial E' = Th(\xi)$.

Recall:

Lemma

If the base space B of k-plane bundle ξ is a CW-complex, then $Th(\xi)$ has the structure of a $(k-1)$-connected CW-complex having, in addition to the basepoint, one $(n+k)$-cell for each n-cell of B.

Using this lemma, we see that $H^n(Th(\xi), \mathbb{Z}) = \mathbb{Z}$ if $n = 0, 4, 8$, and 0 otherwise.

Using this lemma, we see that $H^n(Th(\xi), \mathbb{Z}) = \mathbb{Z}$ if $n = 0, 4, 8$, and 0 otherwise.

In particular, the signature $\sigma(M) = \pm 1$. By choosing the correct orientation of M, we can assume $\sigma(M) = 1$.

Existence of Exotic Spheres

Using this lemma, we see that $H^n(Th(\xi), \mathbb{Z}) = \mathbb{Z}$ if $n = 0, 4, 8$, and 0 otherwise.

In particular, the signature $\sigma(M) = \pm 1$. By choosing the correct orientation of M, we can assume $\sigma(M) = 1$.

We want to compute $p_1(TM)$. As

$$H^3(\partial E') \to H^4(M) \to H^4(E') \oplus H^4(C(\partial E')) \to H^4(\partial E')$$

is exact, we see that $i : E' \to M$ induces an isomorphism on $H^4(-, \mathbb{Z})$ (which is isomorphic to $\mathbb{Z}$). So we can compute $p_1(TE')$ instead.

Consider some Riemannian metric on E'. Then TE' splits as the Whitney sum of the bundle of vectors tangent to the fibers (of $\xi : E' \to S^4$), and the bundle of vectors normal to the fiber.

Existence of Exotic Spheres

Consider some Riemannian metric on E'. Then TE' splits as the Whitney sum of the bundle of vectors tangent to the fibers (of $\xi : E' \to S^4$), and the bundle of vectors normal to the fiber. So we get

$$TE' = \xi^*(\xi) \oplus \xi^*(TS^4).$$

Consider some Riemannian metric on E'. Then TE' splits as the Whitney sum of the bundle of vectors tangent to the fibers (of $\xi : E' \to S^4$), and the bundle of vectors normal to the fiber. So we get

$$TE' = \xi^*(\xi) \oplus \xi^*(TS^4).$$

This shows

$$2p(TE') = 2(p(\xi^*(\xi)) \cup p(\xi^*(TS^4))).$$

Consider some Riemannian metric on E'. Then TE' splits as the Whitney sum of the bundle of vectors tangent to the fibers (of $\xi : E' \to S^4$), and the bundle of vectors normal to the fiber. So we get

$$TE' = \xi^*(\xi) \oplus \xi^*(TS^4).$$

This shows

$$2p(TE') = 2(p(\xi^*(\xi)) \cup p(\xi^*(TS^4))).$$

Hence

$$2p_1(TE') = 2(p_1(\xi^*(\xi)) + p_1(\xi^*(TS^4))).$$

Consider some Riemannian metric on E'. Then TE' splits as the Whitney sum of the bundle of vectors tangent to the fibers (of $\xi : E' \to S^4$), and the bundle of vectors normal to the fiber. So we get

$$TE' = \xi^*(\xi) \oplus \xi^*(TS^4).$$

This shows

$$2p(TE') = 2(p(\xi^*(\xi)) \cup p(\xi^*(TS^4))).$$

Hence

$$p_1(TE') = p_1(\xi^*(\xi)) + p_1(\xi^*(TS^4)).$$

Consider some Riemannian metric on E'. Then TE' splits as the Whitney sum of the bundle of vectors tangent to the fibers (of $\xi : E' \to S^4$), and the bundle of vectors normal to the fiber. So we get

$$TE' = \xi^*(\xi) \oplus \xi^*(TS^4).$$

This shows

$$2p(TE') = 2(p(\xi^*(\xi)) \cup p(\xi^*(TS^4))).$$

Hence

$$p_1(TE') = p_1(\xi^*(\xi)) + p_1(\xi^*(TS^4)).$$

Consequently, $p_1(TE') = \xi^* p_1(\xi)$ is k times a generator of $H^4(E')$.

Existence of Exotic Spheres

So $p_1(TM) = ka$, for some generator $a \in H^4(M, \mathbb{Z})$. By the chosen orientation for M, it follows that $p_1^2[M] = k^2$. ($\langle a \cup a, [M] \rangle = 1$, as $\sigma(M) = 1$.)

So $p_1(TM) = ka$, for some generator $a \in H^4(M, \mathbb{Z})$. By the chosen orientation for M, it follows that $p_1^2[M] = k^2$. ($\langle a \cup a, [M] \rangle = 1$, as $\sigma(M) = 1$.)

By the signature theorem we have

$$\sigma(M) = \frac{7}{45} p_2[M] - \frac{1}{45} p_1^2[M],$$

So $p_1(TM) = ka$, for some generator $a \in H^4(M, \mathbb{Z})$. By the chosen orientation for M, it follows that $p_1^2[M] = k^2$. ($\langle a \cup a, [M] \rangle = 1$, as $\sigma(M) = 1$.)

By the signature theorem we have

$$\sigma(M) = \frac{7}{45} p_2[M] - \frac{1}{45} p_1^2[M],$$

and hence $p_2[M] = \frac{45+k^2}{7}$.

So $p_1(TM) = ka$, for some generator $a \in H^4(M, \mathbb{Z})$. By the chosen orientation for M, it follows that $p_1^2[M] = k^2$. ($\langle a \cup a, [M] \rangle = 1$, as $\sigma(M) = 1$.)

By the signature theorem we have

$$\sigma(M) = \frac{7}{45} p_2[M] - \frac{1}{45} p_1^2[M],$$

and hence $p_2[M] = \frac{45+k^2}{7}$.

For $k = 6$, we find $p_2[M] = 11 + \frac{4}{7}$, but the Pontryagin numbers must be integers.

Existence of Exotic Spheres

So $p_1(TM) = ka$, for some generator $a \in H^4(M, \mathbb{Z})$. By the chosen orientation for M, it follows that $p_1^2[M] = k^2$. ($\langle a \cup a, [M] \rangle = 1$, as $\sigma(M) = 1$.)

By the signature theorem we have

$$\sigma(M) = \frac{7}{45} p_2[M] - \frac{1}{45} p_1^2[M],$$

and hence $p_2[M] = \frac{45+k^2}{7}$.

For $k = 6$, we find $p_2[M] = 11 + \frac{4}{7}$, but the Pontryagin numbers must be integers.

This contradicts our assumption that there exists a diffeomorphism $S^7 \to \partial E'$.

Existence of Exotic Spheres

Conclusion

We have found a smooth 7-manifold which is homotopy equivalent, but not diffeomorphic to S^7.

Existence of Exotic Spheres

Conclusion

We have found a smooth 7-manifold which is homotopy equivalent, but not diffeomorphic to S^7.

Remark

By the generalized Poincaré conjecture, this manifold must even be homeomorphic to S^7.

Group structures on sets of homotopy spheres

From now on, all manifolds will assumed to be smooth and oriented.

Group structures on sets of homotopy spheres

From now on, all manifolds will assumed to be smooth and oriented.

Definition

A homotopy sphere is an n-manifold, which is homotopy equivalent to S^n.

Group structures on sets of homotopy spheres

From now on, all manifolds will assumed to be smooth and oriented.

Definition

A homotopy sphere is an n-manifold, which is homotopy equivalent to S^n.

Proposition (Mazur)

For a closed manifold M, the following are equivalent:

1. *There exists a manifold N with* $M \# N \cong S^n$,
2. $M \# \mathbb{R}^n \cong \mathbb{R}^n$,
3. *M can be covered by two open subsets, both diffeomorphic to* $\mathbb{R}^n$.

Group structures on sets of homotopy spheres

From now on, all manifolds will assumed to be smooth and oriented.

Definition

A homotopy sphere is an n-manifold, which is homotopy equivalent to S^n.

Proposition (Mazur)

For a closed manifold M, the following are equivalent:

1. *There exists a manifold N with $M \# N \cong S^n$,*
2. $M \# \mathbb{R}^n \cong \mathbb{R}^n$,
3. *M can be covered by two open subsets, both diffeomorphic to $\mathbb{R}^n$.*

Denote by A^n the set of manifolds satisfying these conditions, up to orientation-preserving diffeomorphism.

Group structures on sets of homotopy spheres

By the condition $M \# \mathbb{R}^n \cong \mathbb{R}^n$, we have

Lemma

Any $M \in A^n$ is homeomorphic to S^n.

Group structures on sets of homotopy spheres

By the condition $M \# \mathbb{R}^n \cong \mathbb{R}^n$, we have

Lemma

Any $M \in A^n$ is homeomorphic to S^n.

By the first condition, we get

Lemma

The set A^n, together with the connected sum operation, is an abelian group.

Group structures on sets of homotopy spheres

By the condition $M\#\mathbb{R}^n \cong \mathbb{R}^n$, we have

Lemma

Any $M \in A^n$ is homeomorphic to S^n.

By the first condition, we get

Lemma

The set A^n, together with the connected sum operation, is an abelian group.

Remark

- When this group was first constructed, it was not yet known that any homotopy sphere was homeomorphic to S^n.
- Even now, there might be homotopy spheres not in A^n.
- While it can be shown that these groups are countable, and some lower bounds are known as well, it is very difficult to determine upper bounds for the orders of these groups.

Group structures on sets of homotopy spheres

To address these problems, it is useful to replace the "diffeomorphism" by a different concept of equivalence.

Group structures on sets of homotopy spheres

To address these problems, it is useful to replace the "diffeomorphism" by a different concept of equivalence.

Definition

Two closed manifolds M, N are h-cobordant (or J-equivalent), if there exists a manifold W with boundary such that

1. $\partial W \cong M \sqcup -N$
2. $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

To address these problems, it is useful to replace the "diffeomorphism" by a different concept of equivalence.

Definition

Two closed manifolds M, N are h-cobordant (or J-equivalent), if there exists a manifold W with boundary such that

1. $\partial W \cong M \sqcup -N$
2. $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

In particular, h-cobordant manifolds are cobordant, and homotopy equivalent.

Group structures on sets of homotopy spheres

Theorem

For a closed manifold M, there exists a closed manifold N with $M \# N$ h-cobordant to S^n, if and only if M is homotopy equivalent to S^n.

Group structures on sets of homotopy spheres

Theorem

For a closed manifold M, there exists a closed manifold N with $M \# N$ h-cobordant to S^n, if and only if M is homotopy equivalent to S^n.
In particular, the set of h-cobordism classes of homotopy spheres forms an abelian group under the connected sum operation. We denote this group by Θ^n.
It can be shown that the inverse of some homotopy sphere is given by the same homotopy sphere, but with the reversed orientation.

Group structures on sets of homotopy spheres

Theorem

For a closed manifold M, there exists a closed manifold N with $M \# N$ h-cobordant to S^n, if and only if M is homotopy equivalent to S^n.
In particular, the set of h-cobordism classes of homotopy spheres forms an abelian group under the connected sum operation. We denote this group by Θ^n.
It can be shown that the inverse of some homotopy sphere is given by the same homotopy sphere, but with the reversed orientation.

Using this notion of h-cobordism instead of diffeomorphism, Milnor and Kervaire showed that Θ^n is finite for $n \neq 3$. They were also able to determine some of these groups completely.

Group structures on sets of homotopy spheres

Let $bP_{n+1} \subseteq \Theta^n$ be the set of elements which are the boundary of some parallellizable manifold. This is well-defined (i.e. does not depend of the choice of representative), and a subgroup.

Let $bP_{n+1} \subseteq \Theta^n$ be the set of elements which are the boundary of some parallellizable manifold. This is well-defined (i.e. does not depend of the choice of representative), and a subgroup.

Theorem

The quotient Θ^n / bP_{n+1} is finite.

Group structures on sets of homotopy spheres

Let $bP_{n+1} \subseteq \Theta^n$ be the set of elements which are the boundary of some parallellizable manifold. This is well-defined (i.e. does not depend of the choice of representative), and a subgroup.

Theorem

The quotient Θ^n / bP_{n+1} is finite.

These bP_{n+1}'s have a big influence in how big the Θ^n's are:

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\|\Theta^n\|$	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264
$\|bP_{n+1}\|$	1	1	1	28	1	2	1	992	1	1	1	8128	1	2	1	261632
$\|\Theta^n / bP_{n+1}\|$	1	1	1	1	2	4	6	1	1	3	2	2	2	8	16	2

Group structures on sets of homotopy spheres

This is not an accident:

Theorem

- *If n is even, then bP_{n+1} is trivial.*
- *If $n \equiv 1 \mod 4$, then bP_{n+1} has order at most 2.*

Group structures on sets of homotopy spheres

This is not an accident:

Theorem

- *If n is even, then bP_{n+1} is trivial.*
- *If $n \equiv 1 \mod 4$, then bP_{n+1} has order at most 2.*

In contrast to this, we have:

Theorem

For $k > 1$, we have that bP_{4k} is cyclic of order

$$2^{k-2}(2^{k-1} - 1) \cdot \text{numerator}(\frac{4B_k}{k}),$$

where B_k is the k'th Bernoulli number.

bP_{4k} is cyclic: idea of proof.

- Consider all the set of all parallellizable $4k$-manifolds M_0, which have boundary S^{4k-1}.
- There is a notion of *connected sum along the boundary*, and the signature is additive with respect to this operation.

bP_{4k} is cyclic: idea of proof.

- Consider all the set of all parallellizable $4k$-manifolds M_0, which have boundary S^{4k-1}.
- There is a notion of *connected sum along the boundary*, and the signature is additive with respect to this operation.
- As reversing the orientation changes the sign of the signature, and $\sigma(D^{4k}) = 0$, the set of signatures $\sigma(M_0)$ forms a subgroup of $\mathbb{Z}$.
- This subgroup is not trivial, and hence has a positive generator σ_k.

Group structures on sets of homotopy spheres

Theorem

Let Σ_1, Σ_2 be homotopy spheres of dimension $4k-1$, bounding parallellizable manifolds M_1 and M_2. Then Σ_1 and Σ_2 are h-cobordant if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k.$$

Group structures on sets of homotopy spheres

Theorem

Let Σ_1, Σ_2 be homotopy spheres of dimension $4k-1$, bounding parallellizable manifolds M_1 and M_2. Then Σ_1 and Σ_2 are h-cobordant if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k.$$

Proof sketch.

First, assume $\sigma(M_1) = \sigma(M_2) + \sigma(M_0)$, with M_0 parrallellizable and with boundary S^{4k-1}. Take the connected sum along the boundary:

$$(M, \partial M) = (-M_1, -\partial M_1)\#(M_2, \partial M_2)\#(M_0, \partial M_0).$$

Group structures on sets of homotopy spheres

Theorem

Let Σ_1, Σ_2 be homotopy spheres of dimension $4k-1$, bounding parallellizable manifolds M_1 and M_2. Then Σ_1 and Σ_2 are h-cobordant if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k.$$

Proof sketch.

First, assume $\sigma(M_1) = \sigma(M_2) + \sigma(M_0)$, with M_0 parrallellizable and with boundary S^{4k-1}.
Take the connected sum along the boundary:

$$(M, \partial M) = (-M_1, -\partial M_1)\#(M_2, \partial M_2)\#(M_0, \partial M_0).$$

Then $\partial M = -\partial M_1 \# \partial M_2 \# S^{4k-1} \cong -\Sigma_1 \# \Sigma_2$.
We have $\sigma(M) = -\sigma(M_1) + \sigma(M_2) + \sigma(M_0) = 0$.
It can then be shown, that ∂M must be h-cobordant to S^{4k-1}. (This fails if $k = 1$.)

Theorem

Let Σ_1, Σ_2 be homotopy spheres of dimension $4k-1$, bounding parallellizable manifoldsf M_1 and M_2. Then Σ_1 and Σ_2 are h-cobordant if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k.$$

Proof sketch.

Conversely, suppose W is an h-cobordism between $-\Sigma_1\#\Sigma_2$ and S^{4k-1}.
Glue W and $(-M_1, -\partial M_1)\#(M_2, \partial M_2)$ along their common boundary $-\Sigma_1\#\Sigma_2$ to get a manifold M. This has boundary S^{4k-1}, and is parallellizable.

Group structures on sets of homotopy spheres

Theorem

Let Σ_1, Σ_2 be homotopy spheres of dimension $4k-1$, bounding parallellizable manifoldsf M_1 and M_2. Then Σ_1 and Σ_2 are h-cobordant if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k.$$

Proof sketch.

Conversely, suppose W is an h-cobordism between $-\Sigma_1 \# \Sigma_2$ and S^{4k-1}.
Glue W and $(-M_1, -\partial M_1)\#(M_2, \partial M_2)$ along their common boundary $-\Sigma_1 \# \Sigma_2$ to get a manifold M. This has boundary S^{4k-1}, and is parallellizable.
Hence we have $\sigma(M) \equiv 0 \mod \sigma_k$.
But $\sigma(M) = -\sigma(M_1) + \sigma(M_2)$, so $\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k$.

Group structures on sets of homotopy spheres

Theorem

Let Σ_1, Σ_2 be homotopy spheres of dimension $4k-1$, bounding parallellizable manifoldsf M_1 and M_2. Then Σ_1 and Σ_2 are h-cobordant if and only if

$$\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k.$$

Proof sketch.

Conversely, suppose W is an h-cobordism between $-\Sigma_1 \# \Sigma_2$ and S^{4k-1}.
Glue W and $(-M_1, -\partial M_1) \# (M_2, \partial M_2)$ along their common boundary $-\Sigma_1 \# \Sigma_2$ to get a manifold M. This has boundary S^{4k-1}, and is parallellizable.
Hence we have $\sigma(M) \equiv 0 \mod \sigma_k$.
But $\sigma(M) = -\sigma(M_1) + \sigma(M_2)$, so $\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_k$.

It follows that bP_{4k} is isomorphic to a subgroup of the cyclic group of order σ_k, and hence must be cyclic.

Group structures on sets of homotopy spheres

For $n \leq 3$, every topological n-manifold has a smooth structure, unique up to diffeomorphism. Hence, $\Theta^n = 0$ for $n \leq 3$. (Using the Poincaré conjecture for $n = 3$.)

Group structures on sets of homotopy spheres

For $n \leq 3$, every topological n-manifold has a smooth structure, unique up to diffeomorphism. Hence, $\Theta^n = 0$ for $n \leq 3$. (Using the Poincaré conjecture for $n = 3$.)

For $n = 4$, it is known that $\Theta^4 = 0$. However, this is the only case for which it is not known if there exist exotic spheres.

Group structures on sets of homotopy spheres

For $n \leq 3$, every topological n-manifold has a smooth structure, unique up to diffeomorphism. Hence, $\Theta^n = 0$ for $n \leq 3$. (Using the Poincaré conjecture for $n = 3$.)

For $n = 4$, it is known that $\Theta^4 = 0$. However, this is the only case for which it is not known if there exist exotic spheres.

Theorem (h-cobordism theorem)

If W is an h-cobordism between M and N, with M, N n-dimensional closed simply connected manifolds and $n \geq 5$, then W is trivial (i.e. diffeomorphic to $M \times [0, 1]$).

Group structures on sets of homotopy spheres

For $n \leq 3$, every topological n-manifold has a smooth structure, unique up to diffeomorphism. Hence, $\Theta^n = 0$ for $n \leq 3$. (Using the Poincaré conjecture for $n = 3$.)

For $n = 4$, it is known that $\Theta^4 = 0$. However, this is the only case for which it is not known if there exist exotic spheres.

Theorem (h-cobordism theorem)

If W is an h-cobordism between M and N, with M, N n-dimensional closed simply connected manifolds and $n \geq 5$, then W is trivial (i.e. diffeomorphic to $M \times [0,1]$).

So for $n \geq 5$, we see that each element of Θ^n is represented by exactly one homotopy sphere (up to oriented diffeomorphism).

References

- Kervaire, Michel A. and Milnor, John W. *Groups of homotopy spheres: I*. Annals of Mathematics, 77:504–537, 1963. JSTOR.
- Milnor, John and Stasheff, James D. *Characteristic Classes*. No 76. Princeton University Press, 1974.
- Milnor, John W. *Sommes de variétés différentiables et structures différentiables des spheres*. Bulletin de la Société Mathématique de France, 87:439–444, 1959.