Hirzebruch's Signature Theorem

Multiplicative Sequence Approach

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Overview

Outline of the talk:

- What is a Signature & the Signature Theorem?
- Why do we need multiplicative sequences?
- Classification of multiplicative sequences
- Proof of Signature Theorem
- Some applications of Signature Theorem

References

Friedrich Hirzebruch,
Topological methods in algebraic geometry,
Classics in Mathematics. Springer-Verlag, Berlin, 1995.

John W. Milnor and James D. Stasheff,
Characteristic classes,
Annals of Mathematics Studies, No. 76, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.

Signature

Definition

Let M^m be a connected compact oriented manifold.

The **signature** $\sigma(M)$ of M is defined to be

- zero if the dimension is not a multiple of 4 and
- as follows for $m = 4n$: Pick a basis $a_1, \ldots, a_r$ for $H^{2n}(M^{4n}; \mathbb{Q})$ so that the symmetric matrix $[\langle a_i \cup a_j, \mu_{4n} \rangle]$ is diagonal, then $\sigma(M^{4n})$ is the number of positive diagonal entries minus the number of negative ones.

The signature of a compact oriented but not connected manifold is the sum of the signatures of its connected components.

Signature

Remark

- Note that $H^{2n}(M^{4n};\mathbb{Q})$ is a unitary module over a division ring, i.e. a vector space, so it is valid to speak of basis.
- Although the cup product is non-strict commutative, the matrix $[\langle a_i \cup a_j, \mu_{4n}\rangle]$ is symmetric since the degree of the cohomology group is even.
- Then we derive a rational quadratic form $a \mapsto \langle a \cup a, \mu_{4n}\rangle$.
- So under a suitable (rational) change of basis, the matrix $[\langle a_i \cup a_j, \mu_{4n}\rangle]$ is diagonal. (Note that this is congruent diagonalization not the usual diagonalization.) Then $\sigma(M^{4n})$ is well-defined by Sylvester's law of inertia, which means that $\sigma(M^{4n})$ can also be equivalently defined as the difference $\#$ of the positive and negative eigenvalues of $[\langle a_i \cup a_j, \mu_{4n}\rangle]$.
- The manifolds are compact. So the number of connected components is finite.

Signature

Remark

The symmetric matrix $[\langle a_i \cup a_j, \mu_{4n} \rangle]$ is nonsingular!

We are doing (co)homology with coefficients in a field! There is no torsion in (co)homology. Consider

$$H^{4n-k}(M^{4n};\mathbb{Q}) \xrightarrow[\text{UCT}]{h} \mathrm{Hom}_{\mathbb{Q}}(H_{4n-k}(M^{4n};\mathbb{Q}),\mathbb{Q})$$
$$\xrightarrow[\text{Hom-dual}]{D^*} \mathrm{Hom}_{\mathbb{Q}}(H^k(M^{4n};\mathbb{Q}),\mathbb{Q}).$$

$D^* \circ h$ sends $\psi \in H^{4n-k}(M^{4n};\mathbb{Q})$ to the homomorphism

$$H^k(M^{4n};\mathbb{Q}) \ni \varphi \mapsto \psi(\mu_{4n} \cap \varphi) = (\varphi \cup \psi)(\mu_{4n}).$$

Nonsingularity in the other variable follows by (non-strict) commutativity of cup product.

Signature

Lemma (Thom)

The signature σ has the following three properties:

- $\sigma(M + M') = \sigma(M) + \sigma(M')$, $\sigma(-M) = -\sigma(M)$,
- $\sigma(M \times M') = \sigma(M)\sigma(M')$, and
- if M is an oriented boundary, then $\sigma(M) = 0$.

Proof: See [Hirzebruch, Theorem 8.2.1].

Corollary

The signature σ descends to a ring morphism from the cobordism ring Ω_* to the ring $\mathbb{Z}$ of integers, or equivalently it gives rise to an algebra morphism from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$ with $M \otimes 1 \mapsto$ an integer.

Signature Theorem

Consider the L-polynomials in Pontrjagin classes:

$$L_1 = \frac{1}{3}p_1,\ L_2 = \frac{1}{45}(7p_2 - p_1^2),\ L_3 = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3),$$
$$L_4 = \frac{1}{14175}(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4), \text{etc.}$$

The sequence $\{L_n(p_1, \ldots, p_n)\}_{n\geq 1}$ consists of polynomial in Pontrjagin classes such that $L_n(p_1, \ldots, p_n) \in H^{4n}(M; \mathbb{Q})$ for each $n \geq 1$.

Signature Theorem says "Signatures are just Pontrjagin numbers"!!!

Theorem (Signature Theorem)

Let M be any compact oriented smooth $4n$-manifold, then we have $\sigma(M) = L_n(p_1, \ldots, p_n)[M]$.

You Could Have Invented Multiplicative Sequences!!!

You Could Have Invented Spectral Sequences

Timothy Y. Chow

You Could Have Invented Multiplicative Sequences!!!

Consider two real vector bundles ξ and η over the same base space with trivial Whitney sum. We have the equation $w(\xi \oplus \eta) = w(\xi)w(\eta)$ which can be uniquely solved as

$$w(\eta) = (w(\xi))^{-1} w(\xi \oplus \eta) = (w(\xi))^{-1} =: K(w(\xi))$$

(see [Milnor, Lemma 4.1]). One important special case is Whitney duality theorem ([Milnor, Lemma 4.2]): $w(\nu) = K(w(\tau))$ where τ is the tangent bundle of a manifold in Euclidean space and ν is its normal bundle. Now we "expand" $K(w(\xi))$:

$$\begin{aligned} K(w(\xi)) = w(\xi)^{-1} &= \frac{1}{1 + (w_1(\xi) + w_2(\xi) + \cdots)} \\ &= 1 - (w_1(\xi) + w_2(\xi) + \cdots) + (w_1(\xi) + w_2(\xi) + \cdots)^2 - \cdots \\ &= 1 + K_1(w_1(\xi)) + K_2(w_1(\xi), w_2(\xi)) + \cdots . \end{aligned}$$

You Could Have Invented Multiplicative Sequences!!!

Consider two vector bundles ξ and η over the same base space. We have the equation $w(\xi \oplus \eta) = w(\xi)w(\eta)$ which can be uniquely solved as

$$w(\eta) = (w(\xi))^{-1} w(\xi \oplus \eta) = K(w(\xi))$$

(see [Milnor, Lemma 4.1]) where

$$\begin{aligned}
K(w(\xi)) = w(\xi)^{-1} &= 1 + K_1(w_1(\xi)) + K_2(w_1(\xi), w_2(\xi)) + \cdots \\
K_1(X_1) &= -X_1, \\
K_2(X_1, X_2) &= X_1^2 - X_2, \\
K_3(X_1, X_2, X_3) &= -X_1^3 + 2X_1X_2 - X_3, \\
K_4(X_1, X_2, X_3, X_4) &= X_1^4 - 3X_1^2X_2 + 2X_1X_3 + X_2^2 - X_4, \ \ldots, \\
K_n(X_1, \ldots, X_n) &= \sum_{i_1 + 2i_2 + \cdots + ni_n = n} \frac{(i_1 + \cdots + i_n)!}{i_1! \cdots i_n!} (-X_1)^{i_1} \cdots (-X_n)^{i_n}.
\end{aligned}$$

You Could Have Invented Multiplicative Sequences!!!

Consider the general term

$$K_n(X_1, \ldots, X_n) = \sum_{i_1+2i_2+\cdots+ni_n=n} \frac{(i_1 + \cdots + i_n)!}{i_1! \cdots i_n!}(-X_1)^{i_1} \cdots (-X_n)^{i_n}.$$

We easily find that

each $K_n(X_1, X_2^2, X_3^3, \ldots, X_n^n)$ is homogeneous of degree n.

You Could Have Invented Multiplicative Sequences!!!

Consider the formal sum

$$K(1 + X_1 + X_2 + \cdots) = 1 + K_1(X_1) + K_2(X_1, X_2) + \cdots .$$

Let $X = 1 + X_1 + X_2 + \cdots$ and $Y = 1 + Y_1 + Y_2 + \cdots$ formally, then $K(X) = X^{-1}$ and $K(Y) = Y^{-1}$. Assume $XY = YX$. We have

$$\begin{aligned} K \text{ is multiplicative}: \quad K(XY) &= (XY)^{-1} = Y^{-1}X^{-1} \\ &= X^{-1}Y^{-1} = K(X)K(Y). \end{aligned}$$

You Could Have Invented Multiplicative Sequences!!!

We actually obtain a sequence $\{K_n(X_1, \ldots, X_n)\}_{n\geq 1}$ with $K(X) = 1 + K_1(X_1) + K_2(X_1, X_2) + \cdots$ where $X = 1 + X_1 + X_2 + \cdots$ satisfying

- **homogeneity property**:

  each $K_n(X_1, X_2^2, X_3^3, \ldots, X_n^n)$ is homogeneous of degree n;

- **multiplicative property**: $K(XY) = K(X)K(Y)$.

Now we have invented a multiplicative sequence!!!

Multiplicative Sequence

Let A be a commutative ring with multiplicative identity.

Definition

- A unitary unital commutative A-algebra A^* is **non-negatively graded** if there exist additive subgroups A_i of A^* for $i \geq 0$ such that $A^* = \bigoplus_{i\geq 0} A_i$ with $AA_i \subset A_i$ and $A_iA_j \subset A_{i+j}$ for all $i, j \geq 0$.
- To each such A^*, we associate the ring A^Π consisting of all formal sums $\sum_{i\geq 0} a_i$ with $a_i \in A_i$, i.e. the internal direct product decomposition $A^\Pi = \sum_{i\geq 0} A_i$ holds such that $AA_i \subset A_i$ and $A_iA_j \subset A_{i+j}$ for all $i, j \geq 0$.

Example

Let $A^* = A[X]$ and $A^\Pi = A[[X]]$. More concretely, set $A = \mathbb{Q}$, then we have $A^* = \mathbb{Q}[X]$ and $A^\Pi = \mathbb{Q}[[X]]$.

Multiplicative Sequence

Remark

- In the main application, we put $A_n = H^{4n}(B; A)$.
- Note that $A_n = H^{4n}(B; A)$ is of degree n in the graded algebra but is of degree $4n$ as a cohomology group.
- $1 \in A_0$ and $A \cdot 1 \subset A_0$.
- $A^* = \bigoplus_{i \geq 0} A_i$ is an internal weak direct product decomposition $\Rightarrow$ each element $a \in A^*$ can be uniquely expressed as the sum $\sum_{i \geq 0} a_i$ with $a_i \in A_i$ such that only a finitely many a_i's are nonzero.

Multiplicative Sequence

Remark

- Due to the same reason, for each $a \in A^{\Pi}$ we have a unique expression $a = \sum_{i \geq 0} a_i$ with $a_i \in A$.
- We will be particularly interested in elements of the form $a = 1 + \sum_{i \geq 1} a_i$ in A^{Π} which are invertible in A^{Π} by the theory of formal power series.

Multiplicative Sequence

Now consider a sequence of polynomials

$$K_1(X_1),\ K_2(X_1, X_2),\ K_3(X_1, X_2, X_3),\ \dots$$

with coefficients in A satisfying **homogeneity property**:

each $K_n(X_1, X_2^2, X_3^3, \dots, X_n^n)$ is homogeneous of degree n.

Given an element $a = 1 + a_1 + a_2 + \cdots \in A^\Pi$ with leading term 1 which is invertible in A^Π, define a new element $K(a) \in A^\Pi$ also with leading term 1 by the formula

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots .$$

Multiplicative Sequence

Definition (Multiplicative Sequence)

The sequence $\{K_n\}_{n\geq 1}$ is a **multiplicative sequence** or briefly an **m-sequence** of polynomials if it satisfies **multiplicative property**:

$$K(ab) = K(a)K(b)$$

holds for all A-algebras A^* $\left(\text{or } A^\Pi\right)$ and for all $a, b \in A^\Pi$ with leading term 1.

Multiplicative Sequence

Example

Given any constant $\lambda \in A$, the polynomials

$$K_n(X_1, \ldots, X_n) = \lambda^n X_n$$

form an m-sequence with

$$K(1 + a_1 + a_2 + \cdots) = 1 + \lambda a_1 + \lambda^2 a_2 + \cdots .$$

The case $\lambda = 1$ (so that $K(a) = a$) and $\lambda = -1$ are of particular interest:

Let $\lambda = -1$ and ω be a complex n-plane bundle. Consider the Chern class of the conjugate bundle $\bar{\omega}$, then we have $c_k(\bar{\omega}) = (-1)^k c_k(\omega)$ due to [Milnor, Lemma 14.9]. Hence we derive

$$c(\bar{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \cdots + (-1)^k c_n(\omega) = K(c(\omega)).$$

Multiplicative Sequence

Example

$$K_n(X_1, \ldots, X_n) = \sum_{i_1+2i_2+\cdots+ni_n=n} \frac{(i_1+\cdots+i_n)!}{i_1!\cdots i_n!}(-X_1)^{i_1}\cdots(-X_n)^{i_n},$$

$$\begin{aligned} K(a) = a^{-1} &= \frac{1}{1+(a_1+a_2+\cdots)} \\ &= 1-(a_1+a_2+\cdots)+(a_1+a_2+\cdots)^2 \\ &\quad -(a_1+a_2+\cdots)^3+\cdots . \end{aligned}$$

Multiplicative Sequence

Example

The polynomials $K_{2n-1} = 0$ and

$$K_{2n}(X_1, \ldots, X_{2n}) = X_n^2 - 2X_{n-1}X_{n+1} + \cdots + (-1)^{n-1}2X_1X_{2n-1} + (-1)^n 2X_{2n}$$

form an m-sequence. For any complex n-bundle ω, the Chern classes $c_k(\omega)$ determine the Pontrjagin classes $p_k(\omega_{\mathbb{R}})$ by the formula

$$1-p_1+p_2-\cdots+(-1)^n p_n = (1-c_1+c_2-\cdots+(-1)^n c_n)(1+c_1+c_2+\cdots+c_n)$$

(see [Milnor, Corollary 15.5]). Thus we have

$$p_k(\omega_{\mathbb{R}}) = c_k(\omega)^2 - 2c_{k-1}(\omega)c_{k+1}(\omega) + \cdots + (-1)^{k-1}2c_1(\omega)c_{2k-1}(\omega) + (-1)^k 2c_{2k}(\omega) = K_{2n}(c_1(\omega), c_2(\omega), \ldots, c_{2k}(\omega))$$

The total Pontrjagin class $p(\omega_{\mathbb{R}})$ is just $p(\omega_{\mathbb{R}}) = K(c(\omega))$.

Multiplicative Sequence

Consider $A^* = A[t]$ where t can be seen as a generator of A_1 which is of degree 1.

Then an element of $A^\Pi = A[[t]]$ with leading term 1 is the formal power series

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$$

with coefficients in A.

In particular, $1 + t$ is such a term which is obvious but important.

Multiplicative Sequence

The following nice lemma gives a simple but very sharp classification of all possible m-sequences:

Lemma (Classification of m-Sequences)

Given a formal power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$ with coefficients in A, there is one and only one m-sequence $\{K_n\}_{n\geq 1}$ with coefficients in A satisfying the condition

$$K(1+t) = f(t)$$

or equivalent satisfying the condition that

the coefficient of X_1^n in each polynomial $K_n(X_1, \ldots, X_n)$ is equal to λ_n.

Multiplicative Sequence

Definition

The m-sequence $\{K_n\}_{n\geq 1}$ is called the m-sequence **belonging to** the formal power series $f(t)$.

Remark

If the m-sequence $\{K_n\}_{n\geq 1}$ belongs to the power $f(t)$, then for any A^* and any $a_1 \in A_1$, the equation $K(1+a_1) = f(a_1)$ is satisfied. Of course, this equation would most likely be false if something of degree $\neq 1$ were substituted in place of a_1. This trivial observation will be used in the proof.

Example

The three m-sequences mentioned above belong to the formal power series $1+\lambda t$, $1-t+t^2-t^3+\cdots$, and $1+t^2$ respectively.

Multiplicative Sequence

Uniqueness: For any positive integer n, we set $A^* = A[t_1, \dots, t_n]$, then $t_1, \dots, t_n \in A_1$. Let $\sigma = (1+t_1)\cdots(1+t_n) =: 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n$ where the polynomials $\sigma_i \in A_i$ are elementary symmetric polynomials in $t_1, \dots, t_n$, then

$$\begin{aligned} K(\sigma) &= K(1+t_1)\cdots K(1+t_n) = f(t_1)\cdots f(t_n) \\ &= (1 + \lambda_1 t_1 + \lambda_2 t_1^2 + \cdots)\cdots(1 + \lambda_1 t_n + \lambda_2 t_n^2 + \cdots). \end{aligned}$$

Taking homogeneous part of degree n, it follows that $K_n(\sigma_1, \dots, \sigma_n)$ is completely determined by the formal power series $f(t)$. Furthermore, note that the elementary symmetric polynomials are algebraically independent, so each K_n is finally proven to be unique.

Multiplicative Sequence

Existence: For any partition $I = (i_1, \ldots, i_r)$ of n with positive integers, let $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_r}$. Define the polynomial K_n by the formula

$$K_n(\sigma_1, \ldots, \sigma_n) = \sum \lambda_I s_I(\sigma_1, \ldots, \sigma_n)$$

summing over all partitions I of n. Recall that $s_I(\sigma_1, \ldots, \sigma_n)$, which is a homogeneous symmetric polynomial of degree n, is the unique polynomial in the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ equal to

$$\sum t_{\sigma(1)}^{i_1} \cdots t_{\sigma(r)}^{i_r}$$

summing over all permutations σ of $\{1, 2, \ldots, r\}$. Note that if we fix σ, then for each permutation σ' such that $t_{\sigma(1)}^{i_1} \cdots t_{\sigma(r)}^{i_r} = t_{\sigma'(1)}^{i_1} \cdots t_{\sigma'(r)}^{i_r}$, the monomial $t_{\sigma(1)}^{i_1} \cdots t_{\sigma(r)}^{i_r}$ will be recorded only once in the sum.

Multiplicative Sequence

By convention we have

$$s_I(a) = s_I(1 + a_1 + a_2 + \cdots) = s_I(a_1, \ldots, a_n)$$

for any partition I of n. Note that we have the identity

$$s_I(ab) = \sum_{HJ=I} s_H(a)s_J(b)$$

summing over all partitions H, J with juxtaposition $HJ = I$. Therefore, we obtain

$$\begin{aligned} K(ab) &= \sum_I \lambda_I s_I(ab) = \sum_I \lambda_I \sum_{HJ=I} s_H(a)s_J(b) \\ &= \sum_I \sum_{HJ=I} (\lambda_H s_H(a))(\lambda_J s_J(b)) = \sum_{H,J} (\lambda_H s_H(a))(\lambda_J s_J(b)) \\ &= \sum_H \lambda_H s_H(a) \sum_J \lambda_J s_J(b) = K(a)K(b), \end{aligned}$$

which holds for all $a, b \in A^\Pi$.

Multiplicative Sequence

If I is not a trivial partition of n, i.e. $I \neq (n)$, then $s_I(\sigma_1, 0, \ldots, 0) = 0$. Since $s_n(\sigma_1, 0, \ldots, 0) = \sigma_1^n$, we derive

$$K(1+t) = \sum_I \lambda_I s_I(t, 0, \ldots, 0) = \sum_{n \geq 0} \lambda_{(n)} s_{(n)}(t, 0, \ldots, 0)$$
$$= \sum_{n \geq 0} \lambda_n t^n = f(t).$$

Note that for partition I of 0 we have $\sum \lambda_I s_I(t, 0, \ldots, 0) = \lambda_I s_I(\,) = 1$ trivially.

Now we have finished the proof of existence which is constructive!!!

Multiplicative Sequence

Example

Consider the m-sequence $\{K_n\}_{n\geq 1}$ belonging to $1+t^2$, which belongs to the formal power series $1+t^2$.

For $n \geq 1$, we have

$$K_{2n}(\sigma_1, \dots, \sigma_n) = \sum \lambda_I s_I(\sigma_1, \dots, \sigma_{2n}) = s_{\underbrace{(2, \dots, 2)}_{n \text{ terms of } 2}}(\sigma_1, \dots, \sigma_{2n}),$$

which implies

$$s_{\underbrace{(2, \dots, 2)}_{n \text{ terms of } 2}}(\sigma_1, \dots, \sigma_{2n}) = \sigma_n^2 - 2\sigma_{n-1}\sigma_{n+1} + \cdots$$

$$+ (-1)^{n-1} 2\sigma_1\sigma_{2n-1} + (-1)^n 2\sigma_{2n}.$$

Signature Theorem

Now consider some m-sequence $\{K_n(X_1, \dots, X_n)\}_{n \geq 1}$ with rational coefficients. Let M^m be a compact oriented smooth m-dimensional manifold. We also put $A = \mathbb{Q}$ and $A_n = H^{4n}(M^m; \mathbb{Q})$.

Definition

The K-**genus** $K[M^m]$ is zero if the dimension m is not divisible by 4 and is equal to the rational number

$$K_n[M^{4n}] = \langle K_n(p_1, \dots, p_n), \mu_{4n} \rangle$$

if $m = 4n$ where p_i denotes the i-th Pontrjagin class of the tangent bundle and μ_{4k} denotes the fundamental homology class of M^{4n}. Thus, $K[M^m]$ is a certain rational linear combination of the Pontrjagin numbers of M^m.

Signature Theorem

Lemma

For any m-sequence $\{K_n\}_{n\geq 1}$ with rational coefficients, the correspondence $M \mapsto K[M]$ defines a ring morphism from the cobordism ring Ω_* to the ring $\mathbb{Q}$ of rational numbers, and this correspondence gives rise to an algebra morphism from $\Omega_* \otimes \mathbb{Q}$.

Remark

We will see that, using Signature Theorem, the ring morphism is actually $\Omega_* \to \mathbb{Z}$. So the algebra morphism $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$ satisfies $M \otimes 1 \to$ an integer, which means that the ring morphism $\Omega_* \to \mathbb{Q}$ can be recovered.

Signature Theorem

Proof: Since Pontrjagin numbers are cobordism invariants, so $M \mapsto K[M]$ descends to a well-defined map $\Omega_* \to \mathbb{Q}$.

This map is additive since addition is given by disjoint union and Pontrjagin numbers are additive under such addition.

Consider the product manifold $M \times M'$. Note that the tangent bundle of $M \times M'$ splits as a Whitney sum $TM \times TM' \cong \pi_1^* TM \oplus \pi_2^* TM'$ where π_1 and π_2 are the canonical projections of $M_1 \times M_2$ into the two factors. Modulo elements of order 2, we obtain

$$\begin{aligned} K_n(p_1, \ldots, p_n)(T(M \times M')) &= K_n(p_1, \ldots, p_n)(TM \times TM') \\ &= K_n(p_1, \ldots, p_n)(\pi_1^* TM \oplus \pi_2^* TM') \\ &= K_n(p_1, \ldots, p_n)(\pi_1^* TM) \cup K_n(p_1, \ldots, p_n)(\pi_2^* TM'). \end{aligned}$$

Signature Theorem

Thus, we have

$$
\begin{aligned}
&K[M \times M'] \\
&= \langle K_n(p_1, \ldots, p_n)(\pi_1^* TM) \cup K_n(p_1, \ldots, p_n)(\pi_2^* TM'), \mu_{4n} \times \mu'_{4n'} \rangle \\
&= (-1)^{mm'} \langle K_n(p_1, \ldots, p_n)(\pi_1^* TM), \mu_{4n} \rangle \langle K_n(p_1, \ldots, p_n)(\pi_2^* TM'), \mu'_{4n'} \rangle \\
&= \langle \pi_1^* K_n(p_1, \ldots, p_n)(TM), \mu_{4n} \rangle \langle \pi_2^* K_n(p_1, \ldots, p_n)(TM'), \mu'_{4n'} \rangle \\
&= \langle K_n(p_1, \ldots, p_n)(TM), \mu_{4n} \rangle \langle K_n(p_1, \ldots, p_n)(TM'), \mu'_{4n'} \rangle \\
&= K[M]K[M'].
\end{aligned}
$$

There is no sign here since the K-genera is nonzero only when m, m' are divisible by 4. So the proof is finished.

Signature Theorem

The following theorem reveals that, using these properties, one can show that the signature of a manifold can be expressed as a linear function of its Pontrjagin numbers.

Signature Theorem

Let $\{L_n(X_1, \ldots, X_n)\}_{n\geq 1}$ be the m-sequence of polynomials belonging to the formal power series

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} := 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{k-1}\frac{2^{2k}B_k}{(2k)!}t^k + \cdots .$$

Then the signature $\sigma(M)$ of any compact oriented smooth manifold M is equal to the L-genus $L[M]$.

Signature Theorem

Here B_k denotes the k-th Bernoulli number which can be defined as the coefficients occur in the power series expansion

$$\frac{x}{\tanh x} = \frac{x \cosh x}{\sinh x} = 1 + \frac{B_1}{2!}(2x)^2 - \frac{B_2}{4!}(2x)^4 + \frac{B_3}{6!}(2x)^6 - \cdots$$

convergent for $|x| < \pi$, or equivalently in the Laurent expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \frac{B_3}{6!}z^6 - \cdots .$$

These two series are related by the easily verified identity

$$\frac{x}{\tanh x} = \frac{2x}{e^{2x} - 1} + x.$$

Signature Theorem

With this notion one has:

$$B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30},$$
$$B_5 = \frac{5}{66},\ B_6 = \frac{691}{2730},\ B_7 = \frac{7}{6},\ B_8 = \frac{3617}{510},$$

and so on. These numbers were first introduced by Jakob Bernoulli. The first four L-polynomials are

$$L_1 = \frac{1}{3}p_1,\ L_2 = \frac{1}{45}\left(7p_2 - p_1^2\right),\ L_3 = \frac{1}{945}\left(62p_3 - 13p_2p_1 + 2p_1^3\right),$$
$$L_4 = \frac{1}{14175}\left(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4\right),$$

and so on.

Signature Theorem

Proof of Signature Theorem:

Since the correspondences $M \mapsto \sigma(M)$ and $M \mapsto L[M]$ both give rise to algebra morphisms from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$, it suffices to check this theorem on a set of generators for the algebra $\Omega_* \otimes \mathbb{Q}$, i.e. it suffices to prove the equality on each $\mathbb{C}\mathrm{P}^{2k}$ since they generate the oriented cobordism ring.

Let τ be the tangent bundle of $\mathbb{C}\mathrm{P}^{2k}$. Let $\gamma^1 := \gamma^1(\mathbb{C}^{2k+1})$ be the canonical line bundle over $\mathbb{C}\mathrm{P}^{2k}$, then $a := -c_1(\gamma^1)$ is a generator of $H^2(\mathbb{C}\mathrm{P}^{2k}; \mathbb{Q}) \cong \mathbb{Q}$ such that the total Chern class of τ is $c(\tau^n) = (1+a)^{2k+1}$ and the total Pontrjagin class of $\tau_{\mathbb{R}}$ is $p := p(\tau_{\mathbb{R}}) = (1+a^2)^{2k+1}$.

Signature Theorem

It follows that the top Chern class $c_{2k}(\tau) = (2k+1)a^{2k}$. Therefore, the Euler number $e[\mathbb{C}\mathrm{P}^{2k}] = \langle e(\tau_{\mathbb{R}}), \mu_{4k}\rangle = \langle c_{2k}(\tau), \mu_{4k}\rangle = (2k+1)\langle c_{2k}(\tau), \mu_{4k}\rangle = (2k+1)\langle a^{2k}, \mu_n\rangle$.

On the other hand, we have $\langle e(\tau_{\mathbb{R}}), \mu_{4k}\rangle$ (using integer or rational coefficients) is equal to the Euler characteristic $\chi(\mathbb{C}\mathrm{P}^{2k}) = \sum(-1)^i \dim H^i(\mathbb{C}\mathrm{P}^{2k}; \mathbb{Q}) = 2k+1$ (see [Milnor, Corollary 11.12]).

Thus, $\langle a^{2k}, \mu_{4k}\rangle = 1$ which means a^{2k} is precisely the generator of $H^{4k}(\mathbb{C}\mathrm{P}^{2k}; \mathbb{Q}) \cong \mathbb{Q}$ (which is compatible with the preferred orientation).

Note that the generator $a^k \in H^{2k}(\mathbb{C}\mathrm{P}^{2k}; \mathbb{Q}) \cong \mathbb{Q}$ actually forms a basis, so $\langle a^k \cup a^k, \mu_{4k}\rangle = \langle a^{2k}, \mu_{4k}\rangle = 1$ (for the preferred orientation of $\mathbb{C}\mathrm{P}^{2k}$). Hence we obtain the signature $\sigma(\mathbb{C}\mathrm{P}^{2k}) = 1$.

Signature Theorem

Since the m-sequence $\{L_k\}_{k\geq 1}$ belongs to the power series $f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}}$, we derive

$$L(1+a^2) = \frac{\sqrt{a^2}}{\tanh\sqrt{a^2}} \xlongequal[\text{formal power series}]{\text{in the sense of}} \frac{a}{\tanh a}.$$

Note that $a^2 \in H^4(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q})$ is of degree 1 in the graded algebra since we take A_n to be $H^{4n}(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q})$ and we have

$$L(p) = L\big((1+a^2)^{2k+1}\big) = \big(L(1+a^2)\big)^{2k+1} = \left(\frac{a}{\tanh a}\right)^{2k+1}.$$

Thus the L-genus

$$\langle L_k(p_1,\ldots,p_k),\mu_{4k}\rangle = \langle L_k(\alpha_1 a^2,\ldots,\alpha_k a^{2k}),\mu_{4k}\rangle$$

is equal to the coefficient of a^{2k} in this power series where $\alpha_1,\ldots,\alpha_k$ are binomial coefficients determined by $p(\tau_{\mathbb{R}}) = (1+a^2)^{2k+1}$.

Signature Theorem

Cauchy Integral Formula

Let U be an open subset of the complex plane $\mathbb{C}$, and suppose that the closed disk D defined as $\{z \in \mathbb{C} : |z - z_0| \leq r\}$ is completely contained in U. Let $f : U \to \mathbb{C}$ be a holomorphic function, and let C be the circle, oriented counterclockwise and forming the boundary of D. Then for every ξ in the interior of D, we have

$$f(\xi) = \oint_C \frac{f(z)}{z - \xi} dz.$$

Moreover, since holomorphic functions are analytic, i.e. they can be expanded as convergent Laurent power series, we have $f(z) = \sum_{-\infty}^{+\infty} a_n (z - \xi)^n$ for every z in the interior of D where

$$a_n = \frac{f^{(n)}(\xi)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - \xi)^{n+1}} dz.$$

Signature Theorem

Replace a by the complex variable z, the coefficient of z^{2k} in the Laurent expansion of $\left(\frac{z}{\tanh z}\right)^{2k+1}$ can be computed by dividing by $2\pi i z^{2k+1}$ and then integrating around the origin. In fact, the substitution $u = \tanh z$ with $dz = \frac{du}{1-u^2} = (1+u^2+u^4+\cdots)du$ shows that

$$
\begin{aligned}
L[\mathbb{C}\mathrm{P}^{2k}] &= \frac{1}{2\pi i}\oint_C \frac{dz}{(\tanh z)^{2k+1}} = \frac{1}{2\pi i}\oint_C \frac{1+u^2+u^4+\cdots}{u^{2k+1}}du \\
&= d^{2k}(1+u^2+u^4+\cdots+u^{2k}+\cdots)/d^{2k}u|_{u=0}/(2k)! \\
&= ((2k)!+\cdots)|_{u=0}/(2k)! = 1.
\end{aligned}
$$

Hence, we always have $L[\mathbb{C}\mathrm{P}^{2k}] = 1 = \sigma(\mathbb{C}\mathrm{P}^{2k})$, and it follows that $L[M] = \sigma(M)$ for all M.

Now we have finished the proof!

Signature Theorem

Signature Theorem

Let $\{L_n(X_1, \ldots, X_n)\}_{n\geq 1}$ be the m-sequence of polynomials belonging to the formal power series

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} := 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{k-1}\frac{2^{2k}B_k}{(2k)!}t^k + \cdots .$$

Then the signature $\sigma(M)$ of any compact oriented smooth manifold M is equal to the L-genus $L[M]$.

Since the signature of any manifold is an integer and only depends on the oriented homotopy type, we immediately obtain the following corollary:

Corollary

The L-genus $L[M]$ of any smooth compact oriented M is an integer and only depends on the oriented homotopy type of M.

Signature Theorem

Example

There exists no compact oriented smooth 4-connected 12-manifold M with $\dim H^6(M;\mathbb{Q})$ equal to an odd number.

Proof: The matrix used to define the signature of M is nonsingular, so $\sigma(M)$ is an odd number since $\dim H^6(M;\mathbb{Q})$ is odd. Since M is 4-connected, Hurewicz theorem implies $H_i(M;\mathbb{Q}) = 0$ for $i = 3, 4$. UCT implies

$$0 \to 0 = \mathrm{Ext}^1_{\mathbb{Q}}(H_3(M;\mathbb{Q});\mathbb{Q}) \to H^4(M;\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(H_4(M;\mathbb{Q});\mathbb{Q}) = 0 \to 0.$$

So $H^4(M;\mathbb{Q}) = 0 \Rightarrow p_1[M] = 0 \Rightarrow L[M] = \langle L_3(p_1, p_2, p_3), \mu_{12}\rangle = \frac{62}{945}p_3[M]$ is an integer. As 62 and 945 are coprime, we see that 945 divides $p_3[M]$ and $L[M]$ is even. Signature Theorem says that $\sigma(M) = L[M]$ is even, which is a contradiction.

Signature Theorem

Example

The Pontrjagin number $p_1[M^4]$ is divisible by 3, and the Pontrjagin number $7p_2[M^8] - p_1[M^8]$ is divisible by 45. If M^8 is 4-connected, then $p_2[M^8]$ is divisible by 45.

Acknowledgments

Thanks for your attention!