S4D4 - GRADUATE SEMINAR ON ADVANCED TOPOLOGY - CHARACTERISTIC CLASSES II

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Abstract. This document introduces multiplicative sequences and Hirzebruch's signature theorem. The main reference is [MS74].

Contents

1. Multiplicative Sequence

Multiplicative sequence is the algebraic preliminary to state and prove Hirzebruch's signature theorem. It somehow provides a unifying viewpoint to relate different characteristic classes.

We first fix some definitions and notations before introducing the concept of multiplicative sequence. Let A be a commutative ring with multiplicative identity. In fact, the ring of rationals is sufficient in our practice.

Definition 1.1. *A unitary unital commutative A-algebra A^* is* **non-negatively graded** *if there exist additive subgroups A_i of A^* for $i \geq 0$ such that $A^* = \bigoplus_{i\geq 0} A_i$ with $AA_i \subset A_i$ and $A_iA_j \subset A_{i+j}$ for all $i, j \geq 0$.*

Remark 1.2. It is obvious that $1 \in A_0$ and $A\cdot 1 \subset A_0$. Since $A^* = \bigoplus_{i\geq 0} A_i$ is an internal weak direct product decomposition, each element $a \in A^*$ can be uniquely expressed as the sum $\sum_{i\geq 0} a_i$ with $a_i \in A_i$ such that only a finitely many a_i's are nonzero. In the main application, A_n will usually be the cohomology group $H^{4n}(B; A)$. In this case, be careful that $A_n = H^{4n}(B; A)$ is of degree n in the graded algebra but is of degree $4n$ as a cohomology group.

Definition 1.3. *To each such A^*, we associate the ring A^Π consisting of all formal sums $\sum_{i\geq 0} a_i$ with $a_i \in A_i$, i.e. the internal direct product decomposition $A^\Pi = \sum_{i\geq 0} A_i$ holds such that $AA_i \subset A_i$ and $A_iA_j \subset A_{i+j}$ for all $i, j \geq 0$.*

Remark 1.4. Due to the same reason, for each $a \in A^\Pi$ we have a unique expression $a = \sum_{i\geq 0} a_i$ with $a_i \in A$. We will be particularly interested in elements of the form $a = 1 + \sum_{i\geq 1} a_i$ in A^Π which are invertible in A^Π by the theory of formal power series. The product of two such elements $a, b \in A^\Pi$ is

$$ab = (1 + a_1 + a_2 + \cdots)(1 + b_1 + b_2 + \cdots) = 1 + (a_1 + b_1) + (a_2 + a_1b_1 + b_2) + \cdots$$

where $\sum_{j=0}^{k} a_j b_{k-j} \in A_k$ for all $k \geq 0$ if we set $a_0, b_0 = 1$.

The following easy example reveals what the above two definitions actually mean.

Example 1.5. Let $A^* = A[X]$ and $A^\Pi = A[[X]]$. More concretely, set $A = \mathbb{Q}$, then we have $A^* = \mathbb{Q}[X]$ and $A^\Pi = \mathbb{Q}[[X]]$.

Now consider a sequence of polynomials

$$K_1(X_1),\ K_2(X_1, X_2),\ K_3(X_1, X_2, X_3),\ \ldots$$

with coefficients in A satisfying **homogeneity property**:

$$\text{each } K_n(X_1, X_2^2, X_3^3, \ldots, X_n^n) \text{ is homogeneous of degree } n.$$

Given an element $a \in A^\Pi$ with leading term 1, define a new element $K(a) \in A^\Pi$ also with leading term 1 by the formula

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots .$$

Definition 1.6. *The sequence $\{K_n\}_{n\geq 1}$ is a* **multiplicative sequence** *or briefly an* **m-sequence** *of polynomials if it satisfies* **multiplicative property***:*

$$K(ab) = K(a)K(b)$$

holds for all A-algebras A^* $\big($or $A^\Pi\big)$ and for all $a, b \in A^\Pi$ with leading term 1.

Example 1.7. Given any constant $\lambda \in A$, the polynomials

$$K_n(X_1, \ldots, X_n) = \lambda^n X_n$$

form an m-sequence with

$$K(1 + a_1 + a_2 + \cdots) = 1 + \lambda a_1 + \lambda^2 a_2 + \cdots .$$

The case $\lambda = 1$ (so that $K(a) = a$) and $\lambda = -1$ are of particular interest.

Let $\lambda = -1$ and ω be a complex n-plane bundle. Consider the Chern class of the conjugate bundle $\bar{\omega}$, then we have $c_k(\bar{\omega}) = (-1)^k c_k(\omega)$ due to [MS74, Lemma 14.9]. Hence we derive

$$c(\bar{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \cdots + (-1)^k c_n(\omega) = K(c(\omega)).$$

Example 1.8. The formula

$$\begin{aligned} K(a) = a^{-1} &= \frac{1}{1 + (a_1 + a_2 + \cdots)} \\ &= 1 - (a_1 + a_2 + \cdots) + (a_1 + a_2 + \cdots)^2 - (a_1 + a_2 + \cdots)^3 + \cdots \end{aligned}$$

defines an m-sequence with

$$\begin{aligned} K_1(X_1) &= -X_1, \\ K_2(X_1, X_2) &= X_1^2 - X_2, \\ K_3(X_1, X_2, X_3) &= -X_1^3 + 2X_1X_2 - X_3, \\ K_4(X_1, X_2, X_3, X_4) &= X_1^4 - 3X_1^2X_2 + 2X_1X_3 + X_2^2 - X_4, \end{aligned}$$

and so on. In general, we have

$$K_n(X_1, \ldots, X_n) = \sum_{i_1 + 2i_2 + \cdots + ni_n = n} \frac{(i_1 + \cdots + i_n)!}{i_1! \cdots i_n!} (-X_1)^{i_1} \cdots (-X_n)^{i_n}.$$

These polynomials can be used to describe the relations between the Stiefel-Whitney classes (or the Chern classes, or the Pontrjagin classes) of two real vector bundles with trivial Whitney sum. Consider two real vector bundles ξ and η over the same base space. We have the equation $w(\xi \oplus \eta) = w(\xi)w(\eta)$ which can be uniquely solved as

$$w(\eta) = (w(\xi))^{-1} w(\xi \oplus \eta) = K(w(\xi)) w(\xi \oplus \eta)$$

(see [MS74, Lemma 4.1]). In particular, if $\xi \oplus \eta$ is trivial, then $w(\eta) = K(w(\xi))$. One important special case is Whitney duality theorem ([MS74, Lemma 4.2]): If τ_M is the tangent bundle of a manifold in Euclidean space and ν is the normal bundle, then $w(\nu) = K(w(\tau_M))$. Also, similar statements hold for the other two characteristic classes.

Example 1.9. The polynomials $K_{2n-1} = 0$ and

$$K_{2n}(X_1, \dots, X_{2n}) = X_n^2 - 2X_{n-1}X_{n+1} + \cdots + (-1)^{n-1}2X_1X_{2n-1} + (-1)^n 2X_{2n}$$

form an m-sequence which can be used to describe the relationship between the Chern classes of a complex vector bundle ω and the Pontrjagin classes of the underlying real bundle $\omega_{\mathbb{R}}$. Specifically, for any complex n-bundle, the Chern classes $c_k(\omega)$ determine the Pontrjagin classes $p_k(\omega_{\mathbb{R}})$ by the formula

$$1 - p_1 + p_2 - \cdots + (-1)^n p_n = \big(1 - c_1 + c_2 - \cdots + (-1)^n c_n\big)(1 + c_1 + c_2 + \cdots + c_n).$$

(see [MS74, Corollary 15.5]). Thus we have

$$\begin{aligned} p_k(\omega_{\mathbb{R}}) &= c_k(\omega)^2 - 2c_{k-1}(\omega)c_{k+1}(\omega) + \cdots + (-1)^{k-1}2c_1(\omega)c_{2k-1}(\omega) + (-1)^k 2c_{2k}(\omega) \\ &= K_{2n}\big(c_1(\omega), c_2(\omega), \dots, c_{2k}(\omega)\big) \end{aligned}$$

Then the total Pontrjagin class $p(\omega_{\mathbb{R}})$ can be written as $p(\omega_{\mathbb{R}}) = K(c(\omega))$.

Consider $A^* = A[t]$ where t can be seen as a generator of A_1 which is of degree 1. Then an element of $A^{\Pi} = A[[t]]$ with leading term 1 is the formal power series

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$$

with coefficients in A. In particular, $1+t$ is such a term which is obvious but important.

The following nice lemma gives a simple but very sharp classification of all possible m-sequences.

Lemma 1.10 (Hirzebruch). *Given a formal power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$ with coefficients in A, there is one and only one m-sequence $\{K_n\}_{n\geq 1}$ with coefficients in A satisfying the condition*

$$K(1 + t) = f(t)$$

or equivalent satisfying the condition that

the coefficient of X_1^n in each polynomial $K_n(X_1, \dots, X_n)$ is equal to λ_n.

Definition 1.11. *The m-sequence $\{K_n\}_{n\geq 1}$ is called the m-sequence* **belonging to** *the formal power series $f(t)$.*

Remark 1.12. If the m-sequence $\{K_n\}_{n\geq 1}$ belongs to the power $f(t)$, then for any A^* and any $a_1 \in A_1$, the equation $K(1 + a_1) = f(a_1)$ is satisfied. Of course, this equation would most likely be false if something of degree $\neq 1$ were substituted in place of a_1. This trivial observation will be used in the proof.

Example 1.13. The three m-sequences mentioned above belong to the formal power series $1 + \lambda t$, $1 - t + t^2 - t^3 + \cdots$, and $1 + t^2$ respectively.

Proof of Lemma 1.10. Uniqueness: For any positive integer n, we set $A^* = A[t_1, \dots, t_n]$, then $t_1, \dots, t_n \in A_1$. Let $\sigma = (1 + t_1) \cdots (1 + t_n) =: 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n$ where the polynomials $\sigma_i \in A_i$ are elementary symmetric polynomials in $t_1, \dots, t_n$, then

$$\begin{aligned} K(\sigma) &= K(1 + t_1) \cdots K(1 + t_n) = f(t_1) \cdots f(t_n) \\ &= (1 + \lambda_1 t_1 + \lambda_2 t_1^2 + \cdots) \cdots (1 + \lambda_1 t_n + \lambda_2 t_n^2 + \cdots). \end{aligned}$$

Taking homogeneous part of degree n, it follows that $K_n(\sigma_1, \dots, \sigma_n)$ is completely determined by the formal power series $f(t)$. Furthermore, note that the elementary symmetric polynomials are algebraically independent, so each K_n is finally proven to be unique.

Existence: For any partition $I = (i_1, \ldots, i_r)$ of n with positive integers, let $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_r}$. Define the polynomial K_n by the formula

$$K_n(\sigma_1, \ldots, \sigma_n) = \sum \lambda_I s_I(\sigma_1, \ldots, \sigma_n)$$

summing over all partitions I of n. Recall that $s_I(\sigma_1, \ldots, \sigma_n)$, which is a homogeneous symmetric polynomial of degree n, is the unique polynomial in the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ equal to

$$\sum t_{\sigma(1)}^{i_1} \cdots t_{\sigma(r)}^{i_r}$$

summing over all permutations σ of $\{1, 2, \ldots, r\}$. Note that if we fix σ, then for each permutation σ' such that $t_{\sigma(1)}^{i_1} \cdots t_{\sigma(r)}^{i_r} = t_{\sigma'(1)}^{i_1} \cdots t_{\sigma'(r)}^{i_r}$, the monomial $t_{\sigma(1)}^{i_1} \cdots t_{\sigma(r)}^{i_r}$ will be recorded only once in the sum. By convention we have

$$s_I(a) = s_I(1 + a_1 + a_2 + \cdots) = s_I(a_1, \ldots, a_n)$$

for any partition I of n. Note that we have the identity

$$s_I(ab) = \sum_{HJ=I} s_H(a) s_J(b)$$

summing over all partitions H, J with juxtaposition $HJ = I$. Therefore, we obtain

$$\begin{aligned} K(ab) &= \sum_I \lambda_I s_I(ab) = \sum_I \lambda_I \sum_{HJ=I} s_H(a) s_J(b) \\ &= \sum_I \sum_{HJ=I} \big(\lambda_H s_H(a)\big)\big(\lambda_J s_J(b)\big) = \sum_{H,J} \big(\lambda_H s_H(a)\big)\big(\lambda_J s_J(b)\big) \\ &= \sum_H \lambda_H s_H(a) \sum_J \lambda_J s_J(b) = K(a)K(b), \end{aligned}$$

which holds for all $a, b \in A^\Pi$. If I is not a trivial partition of n, i.e. $I \neq (n)$, then $s_I(\sigma_1, 0, \ldots, 0) = 0$. Since $s_n(\sigma_1, 0, \ldots, 0) = \sigma_1^n$, we derive

$$K(1+t) = \sum_I \lambda_I s_I(t, 0, \ldots, 0) = \sum_{n \geq 0} \lambda_{(n)} s_{(n)}(t, 0, \ldots, 0) = \sum_{n \geq 0} \lambda_n t^n = f(t).$$

Note that for partition I of 0 we have $\sum \lambda_I s_I(t, 0, \ldots, 0) = \lambda_I s_I(\,) = 1$ trivially. Now we have finished the proof of existence which is quite constructive. ☐

Example 1.14. Consider the m-sequence $\{K_n\}_{n \geq 1}$ in Example 1.9, which belongs to the formal power series $1 + t^2$. For $n \geq 1$, we have $K_{2n}(\sigma_1, \ldots, \sigma_n) = \sum \lambda_I s_I(\sigma_1, \ldots, \sigma_{2n}) = s_{(2,\ldots,2)}(\sigma_1, \ldots, \sigma_{2n})$, which implies

$$s_{(2,\ldots,2)}(\sigma_1, \ldots, \sigma_{2n}) = \sigma_n^2 - 2\sigma_{n-1}\sigma_{n+1} + \cdots + (-1)^{n-1} 2\sigma_1\sigma_{2n-1} + (-1)^n 2\sigma_{2n}.$$

2. Signature Theorem

Now consider some m-sequence $\{K_n(X_1, \ldots, X_n)\}_{n \geq 1}$ with rational coefficients. Let M^m be a compact oriented smooth m-dimensional manifold. We also put $A = \mathbb{Q}$ and $A_n = H^{4n}(M^m; \mathbb{Q})$.

Definition 2.1. *The* K-**genus** $K[M^m]$ *is zero if the dimension* m *is not divisible by* 4 *and is equal to the rational number*

$$K_n[M^{4n}] = \langle K_n(p_1, \ldots, p_n), \mu_{4n} \rangle$$

if $m = 4n$ *where* p_i *denotes the* i*-th Pontrjagin class of the tangent bundle and* μ_{4k} *denotes the fundamental homology class of* M^{4n}*. Thus,* $K[M^m]$ *is a certain rational linear combination of the Pontrjagin numbers of* M^m*.*

Lemma 2.2. *For any m-sequence $\{K_n\}_{n\geq 1}$ with rational coefficients, the correspondence $M \mapsto K[M]$ defines a ring morphism from the cobordism ring Ω_* to the ring $\mathbb{Q}$ of rational numbers, or equivalently this correspondence gives rise to an algebra morphism from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$.*

Proof. Since Pontrjagin numbers are cobordism invariants, so $M \mapsto K[M]$ descends to a well-defined map $\Omega_* \to \mathbb{Q}$. This map is additive since addition is given by disjoint union and Pontrjagin numbers are additive under such addition. Consider the product manifold $M \times M'$. Note that the tangent bundle of $M \times M'$ splits as a Whitney sum $TM \times TM' \cong \pi_1^* TM \oplus \pi_2^* TM'$ where π_1 and π_2 are the canonical projections of $M_1 \times M_2$ into the two factors. Modulo elements of order 2, we obtain

$$\begin{aligned} K_n(p_1, \ldots, p_n)(T(M \times M')) &= K_n(p_1, \ldots, p_n)(TM \times TM') \\ &= K_n(p_1, \ldots, p_n)(\pi_1^* TM \oplus \pi_2^* TM') \\ &= K_n(p_1, \ldots, p_n)(\pi_1^* TM) \cup K_n(p_1, \ldots, p_n)(\pi_2^* TM'). \end{aligned}$$

Thus, we have

$$\begin{aligned} K[M \times M'] &= \langle K_n(p_1, \ldots, p_n)(\pi_1^* TM) \cup K_n(p_1, \ldots, p_n)(\pi_2^* TM'), \mu_{4n} \times \mu'_{4n'} \rangle \\ &= (-1)^{mm'} \langle K_n(p_1, \ldots, p_n)(\pi_1^* TM), \mu_{4n} \rangle \langle K_n(p_1, \ldots, p_n)(\pi_2^* TM'), \mu'_{4n'} \rangle \\ &= \langle \pi_1^* K_n(p_1, \ldots, p_n)(TM), \mu_{4n} \rangle \langle \pi_2^* K_n(p_1, \ldots, p_n)(TM'), \mu'_{4n'} \rangle \\ &= \langle K_n(p_1, \ldots, p_n)(TM), \mu_{4n} \rangle \langle K_n(p_1, \ldots, p_n)(TM'), \mu'_{4n'} \rangle \\ &= K[M]K[M']. \end{aligned}$$

There is no sign here since the K-genera is nonzero only when m, m' are divisible by 4. So the proof is finished. ☐

Remark 2.3. Note that the ring morphism here does preserve multiplicative identities. Note that $\Omega_0 \cong \mathbb{Z}$ is spanned by a singleton of positive orientation. Since singletons are of dimension 0, the K-genus is just $K_0 := 1$.

We are going to use this construction to compute an important homotopy type invariant of M.

Definition 2.4. *The* **signature** *$\sigma(M)$ of a connected compact oriented manifold M^m is defined to be zero if the dimension is not a multiple of 4 and as follows for $m = 4n$: Choose a basis $a_1, \ldots, a_r$ for $H^{2n}(M^{4n}; \mathbb{Q})$ so that the symmetric matrix $[\langle a_i \cup a_j, \mu_{4n} \rangle]$ is diagonal, then $\sigma(M^{4n})$ is the number of positive diagonal entries minus the number of negative ones. The signature of a compact oriented but not connected manifold is the sum of the signatures of its connected components.*

Remark 2.5. Note that $H^{2n}(M^{4n}; \mathbb{Q})$ is a unitary module over a division ring, i.e. a vector space, so it is valid to speak of basis. As the manifolds are compact, the number of connected components is finite. The definition of $\sigma(M^{4n})$ is then well-defined by Sylvester's law of inertia.

The signature σ is in other words the signature of the rational quadratic form $a \mapsto \langle a \cup a, \mu \rangle$. The number σ is often called the **index** of M (see for example [Hir95, 8.2]).

Since we are doing (co)homology with coefficients in a field, there is no torsion in (co)homology. The rational quadratic form $a \mapsto \langle a \cup a, \mu_{4n} \rangle$ is nonsingular, or equivalently the symmetric matrix $[\langle a_i \cup a_j, \mu_{4n} \rangle]$ is nonsingular.

Lemma 2.6 (Thom). *The signature σ has the following three properties:*

(i) $\sigma(M + M') = \sigma(M) + \sigma(M')$, $\sigma(-M) = -\sigma(M)$,
(ii) $\sigma(M \times M') = \sigma(M)\sigma(M')$, *and*
(iii) *if M is an oriented boundary, then $\sigma(M) = 0$.*

Proof. See [Hir95, Theorem 8.2.1]. □

The following theorem reveals that, using these properties, one can show that the signature of a manifold can be expressed as a linear function of its Pontrjagin numbers.

Theorem 2.7 (Signature Theorem). *Let $\{L_n(X_1,\dots,X_n)\}_{n\geq 1}$ be the m-sequence of polynomials belonging to the formal power series*

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} := 1+\frac{1}{3}t-\frac{1}{45}t^2+\cdots+(-1)^{k-1}\frac{2^{2k}B_k}{(2k)!}t^k+\cdots.$$

Then the signature $\sigma(M)$ of any compact oriented smooth manifold M is equal to the L-genus $L[M]$.

Here B_k denotes the k-th Bernoulli number which can be defined as the coefficients occur in the power series expansion

$$\frac{x}{\tanh x}=\frac{x\cosh x}{\sinh x}=1+\frac{B_1}{2!}(2x)^2-\frac{B_2}{4!}(2x)^4+\frac{B_3}{6!}(2x)^6-\cdots$$

convergent for $|x|<\pi$, or equivalently in the Laurent expansion

$$\frac{z}{e^z-1}=1-\frac{z}{2}+\frac{B_1}{2!}z^2-\frac{B_2}{4!}z^4+\frac{B_3}{6!}z^6-\cdots.$$

These two series are related by the easily verified identity

$$\frac{x}{\tanh x}=\frac{2x}{e^{2x}-1}+x.$$

With this notion one has:

$$B_1=\frac{1}{6},\ B_2=\frac{1}{30},\ B_3=\frac{1}{42},\ B_4=\frac{1}{30},\ B_5=\frac{5}{66},\ B_6=\frac{691}{2730},\ B_7=\frac{7}{6},\ B_8=\frac{3617}{510},$$

and so on. These numbers were first introduced by Jakob Bernoulli. The first four L-polynomials are

$$L_1=\frac{1}{3}p_1,\ L_2=\frac{1}{45}(7p_2-p_1^2),\ L_3=\frac{1}{945}(62p_3-13p_2p_1+2p_1^3),$$
$$L_4=\frac{1}{14175}(381p_4-71p_3p_1-19p_2^2+22p_2p_1^2-3p_1^4).$$

Proposition 2.8 (Cauchy's Integral Formula). *Let U be an open subset of the complex plane $\mathbb{C}$, and suppose that the closed disk D defined as $\{z\in\mathbb{C}:|z-z_0|\leq r\}$ is completely contained in U. Let $f:U\to\mathbb{C}$ be a holomorphic function, and let C be the circle, oriented counterclockwise and forming the boundary of D. Then for every ξ in the interior of D, we have*

$$f(\xi)=\oint_C\frac{f(z)}{z-\xi}dz.$$

Moreover, since holomorphic functions are analytic, i.e. they can be expanded as convergent Laurent power series, we have $f(z)=\sum_{-\infty}^{+\infty}a_n(z-\xi)^n$ for every z in the interior of D where

$$a_n=\frac{f^{(n)}(\xi)}{n!}=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-\xi)^{n+1}}dz.$$

Proof of Theorem 2.7. Note that the signature descends to a ring morphism $\Omega_*\to\mathbb{Q}$ (preserving multiplicative identities since the signature of a point is just 1). Since the correspondences $M\mapsto\sigma(M)$ and $M\mapsto L[M]$ both give rise to algebra morphisms from $\Omega_*\otimes\mathbb{Q}$ to $\mathbb{Q}$, it suffices to check this theorem on a set of generators for the algebra $\Omega_*\otimes\mathbb{Q}$, i.e. it suffices to prove the equality on each $\mathbb{CP}^{2k}$ since they generate the oriented cobordism ring.

Let τ be the tangent bundle of $\mathbb{C}\mathrm{P}^{2k}$. Let $\gamma^1 := \gamma^1(\mathbb{C}^{2k+1})$ be the canonical line bundle over $\mathbb{C}\mathrm{P}^{2k}$, then $a := -c_1(\gamma^1)$ is a generator of $H^2(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q}) \cong \mathbb{Q}$ such that the total Chern class of τ is $c(\tau^n) = (1+a)^{2k+1}$ and the total Pontrjagin class of $\tau_{\mathbb{R}}$ is $p := p(\tau_{\mathbb{R}}) = (1+a^2)^{2k+1}$.

It follows that the top Chern class $c_{2k}(\tau) = (2k+1)a^{2k}$. Therefore, the Euler number $e[\mathbb{C}\mathrm{P}^{2k}] = \langle e(\tau_{\mathbb{R}}), \mu_{4k}\rangle = \langle c_{2k}(\tau), \mu_{4k}\rangle = (2k+1)\langle c_{2k}(\tau), \mu_{4k}\rangle = (2k+1)\langle a^{2k}, \mu_n\rangle$. On the other hand, we have $\langle e(\tau_{\mathbb{R}}), \mu_{4k}\rangle$ (using integer or rational coefficients) is equal to the Euler characteristic $\chi(\mathbb{C}\mathrm{P}^{2k}) = \sum(-1)^i \dim H^i(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q}) = 2k+1$ (see [MS74, Corollary 11.12]). Thus, $\langle a^{2k}, \mu_{4k}\rangle = 1$ which means a^{2k} is precisely the generator of $H^{4k}(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q}) \cong \mathbb{Q}$ (which is compatible with the preferred orientation).

Note that the generator $a^k \in H^{2k}(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q}) \cong \mathbb{Q}$ actually forms a basis, so $\langle a^k \cup a^k, \mu_{4k}\rangle = \langle a^{2k}, \mu_{4k}\rangle = 1$ (for the preferred orientation of $\mathbb{C}\mathrm{P}^{2k}$). Hence we obtain the signature $\sigma(\mathbb{C}\mathrm{P}^{2k}) = 1$.

Since the m-sequence $\{L_k\}_{k\geq 1}$ belongs to the power series $f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}}$, we derive

$$L(1+a^2) = \frac{\sqrt{a^2}}{\tanh\sqrt{a^2}} \xlongequal[\text{formal power series}]{\text{in the sense of}} \frac{a}{\tanh a}.$$

Note that $a^2 \in H^4(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q})$ is of degree 1 in the graded algebra since we take A_n to be $H^{4n}(\mathbb{C}\mathrm{P}^{2k};\mathbb{Q})$ and we have

$$L(p) = L\big((1+a^2)^{2k+1}\big) = \big(L(1+a^2)\big)^{2k+1} = \Big(\frac{a}{\tanh a}\Big)^{2k+1}.$$

Thus the L-genus $\langle L_k(p_1,\dots,p_k), \mu_{4k}\rangle = \langle L_k(\alpha_1 a^2, \dots, \alpha_k a^{2k}), \mu_{4k}\rangle$ is equal to the coefficient of a^{2k} in this power series where $\alpha_1,\dots,\alpha_k$ are binomial coefficients determined by $p(\tau_{\mathbb{R}}) = (1+a^2)^{2k+1}$.

Replace a by the complex variable z, the coefficient of z^{2k} in the Laurent expansion of $\big(\frac{z}{\tanh z}\big)^{2k+1}$ can be computed by dividing by $2\pi i z^{2k+1}$ and then integrating around the origin. In fact, the substitution $u = \tanh z$ with $dz = \frac{du}{1-u^2} = (1+u^2+u^4+\cdots)du$ shows that

$$\begin{aligned} L[\mathbb{C}\mathrm{P}^{2k}] &= \frac{1}{2\pi i}\oint_C \frac{dz}{(\tanh z)^{2k+1}} = \frac{1}{2\pi i}\oint_C \frac{1+u^2+u^4+\cdots}{u^{2k+1}}du \\ &= d^{2k}(1+u^2+u^4+\cdots+u^{2k}+\cdots)/d^{2k}u|_{u=0}/(2k)! \\ &= ((2k)!+\cdots)|_{u=0}/(2k)! = 1. \end{aligned}$$

Hence, we always have $L[\mathbb{C}\mathrm{P}^{2k}] = 1 = \sigma(\mathbb{C}\mathrm{P}^{2k})$, and it follows that $L[M] = \sigma(M)$ for all M. □

Since the signature of any manifold is an integer and only depends on the oriented homotopy type, we immediately obtain the following corollary:

Corollary 2.9. *The L-genus $L[M]$ of any smooth compact oriented M is an integer and only depends on the oriented homotopy type of M.*

Example 2.10. There exists no compact oriented smooth 4-connected 12-manifold M with $\dim H^6(M;\mathbb{Q})$ equal to an odd number.

In fact, note that the matrix used to define the signature of M is nonsingular, so $\sigma(M)$ is an odd number since $\dim H^6(M;\mathbb{Q})$ is odd.

It suffices to show that the signature is also even. Since M is 4-connected, we have $\pi_i(M) = 0$ for $i = 3, 4$. Hurewicz theorem implies $H_i(M;\mathbb{Q}) = 0$ for $i = 3, 4$. Then Universal coefficient theorem implies

$$0 \to 0 = \mathrm{Ext}^1_{\mathbb{Q}}(H_3(M;\mathbb{Q});\mathbb{Q}) \to H^4(M;\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(H_4(M;\mathbb{Q});\mathbb{Q}) = 0 \to 0.$$

So $H^4(M;\mathbb{Q}) = 0$ and then $p_1[M] = 0$. Hence, $L[M] = \langle L_3(p_1, p_2, p_3), \mu_{12}\rangle = \frac{62}{945} p_3[M]$, which is an integer. As 62 and 945 are coprime, we see that 945 divides $p_3[M]$ and $L[M]$ is even. Signature theorem says that $\sigma(M) = L[M]$ is even, which is a contradiction.

Example 2.11. The Pontrjagin number $p_1[M^4]$ is divisible by 3, and the Pontrjagin number $7p_2[M^8] - p_1[M^8]$ is divisible by 45. If M^8 is 4-connected, then $p_2[M^8]$ is divisible by 45.

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