Thom Spaces and the Oriented Cobordism Ring

Branko Juran

2020-5-20

Organization of the talk

- Definition of the oriented cobordism groups Ω_n

Organization of the talk

- Definition of the oriented cobordism groups Ω_n
- Isomorphism between Ω_n and a certain homotopy group $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Organization of the talk

- Definition of the oriented cobordism groups Ω_n
- Isomorphism between Ω_n and a certain homotopy group $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$
- Isomorphism $\pi_{n+k}(T(\tilde{\gamma}^k), t_0) \otimes \mathbb{Q} \cong H_n(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)) \otimes \mathbb{Q}$.

Oriented Cobordism

Convention

We assume all manifolds to be smooth, compact and oriented.

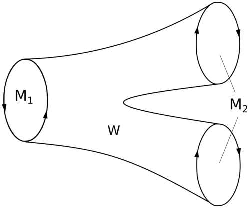
Oriented Cobordism

Convention

We assume all manifolds to be smooth, compact and oriented.

Definition

A *cobordism* between two n-dim. manifolds M_1 and M_2 is an $(n+1)$-dim. manifold with boundary W together with an orientation preserving diffeomorphism $\partial W \cong M_1 \sqcup (-M_2)$.
Two manifolds are said to be *cobordant* if there is a coboridsm between them.



Oriented Cobordism

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

□

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

- Reflexive: $\partial(M \times [0,1]) \cong M \sqcup (-M)$

$\square$

Oriented Cobordism

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

- Reflexive: $\partial(M \times [0,1]) \cong M \sqcup (-M)$
- Symmetric: $\partial(-W) \cong -\partial W \cong (-M_1) \sqcup M_2$

□

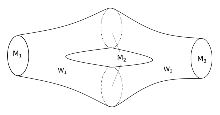
Oriented Cobordism

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

- Reflexive: $\partial(M \times [0,1]) \cong M \sqcup (-M)$
- Symmetric: $\partial(-W) \cong -\partial W \cong (-M_1) \sqcup M_2$
- Transitive: For W_1 cobordism between M_1 and M_2, W_2 cobordism between M_2 and M_3 use collar neighborhood theorem for gluing W_1 and W_2 along M_2



□

Oriented Cobordism

Definition

Let Ω_n be the set of cobordism classes of n-dimensional manifolds.

Oriented Cobordism

Definition

Let Ω_n be the set of cobordism classes of n-dimensional manifolds.

Lemma

The disjoint union induces a map $\Omega_n \times \Omega_n \to \Omega_n$ turning Ω_n into an abelian group. This group is called the n-th oriented cobordism group.

Oriented Cobordism

Definition

Let Ω_n be the set of cobordism classes of n-dimensional manifolds.

Lemma

The disjoint union induces a map $\Omega_n \times \Omega_n \to \Omega_n$ turning Ω_n into an abelian group. This group is called the n-th oriented cobordism group.

Proof.

For W cobordism between M_1, M_2 and N another n-dim. manifold, then $W \sqcup N \times [0,1]$ is cobordism between $M_1 \sqcup N$ and $M_2 \sqcup N$. $\square$

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ *turning* Ω_* *into a graded commutative ring. It is called the oriented cobordism ring.*

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ *turning* Ω_* *into a graded commutative ring. It is called the oriented cobordism ring.*

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2
- $\Omega_2 \cong 0$. S^2 and genus g surfaces bound

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2
- $\Omega_2 \cong 0$. S^2 and genus g surfaces bound
- $\Omega_3 \cong 0$. (Rohlin, 1951)

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2
- $\Omega_2 \cong 0$. S^2 and genus g surfaces bound
- $\Omega_3 \cong 0$. (Rohlin, 1951)
- $\Omega_4 \cong \mathbb{Z}$. Spanned by $\mathbb{C}P^2$

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ *ranges over all partitions of* r*, the manifolds*

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}*.*

Proof.

Follows immediatly from the facts that

□

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.

□

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.
- All Pontryagin numbers of the boundary of a $(4r+1)$-dimensional manifold are 0.

□

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.
- All Pontryagin numbers of the boundary of a $(4r+1)$-dimensional manifold are 0.
- Pontryagin numbers define a group homomorphism $\Omega_{4r} \to \mathbb{Z}^{p(r)}$

□

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \dots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.
- All Pontryagin numbers of the boundary of a $(4r+1)$-dimensional manifold are 0.
- Pontryagin numbers define a group homomorphism $\Omega_{4r} \to \mathbb{Z}^{p(r)}$
- The above manifolds have linearly independent Pontryagin numbers

□

The Thom Space of a Euclidean Vector Bundle

Definition

Let ξ be a k-dim. Euclidean vector bundle. Let $A \subset E(\xi)$ be the subset of all vectors v with $|v| \geq 1$. The *Thom space* $T(\xi)$ of ξ is defined as $E(\xi)/A$. Let t_0 denote the canonical base point.

The Thom Space of a Euclidean Vector Bundle

Definition

Let ξ be a k-dim. Euclidean vector bundle. Let $A \subset E(\xi)$ be the subset of all vectors v with $|v| \geq 1$. The *Thom space* $T(\xi)$ of ξ is defined as $E(\xi)/A$. Let t_0 denote the canonical base point.

Remark

If ξ has a compact base space, then $T(\xi)$ is homeomorphic to the one-point-compactification of $E(\xi)$.

The Thom Space of a Euclidean Vector Bundle

Definition

Let ξ be a k-dim. Euclidean vector bundle. Let $A \subset E(\xi)$ be the subset of all vectors v with $|v| \geq 1$. The *Thom space* $T(\xi)$ of ξ is defined as $E(\xi)/A$. Let t_0 denote the canonical base point.

Remark

If ξ has a compact base space, then $T(\xi)$ is homeomorphic to the one-point-compactification of $E(\xi)$.

Proof.

Extend the $E(\xi) - A \to E(\xi), v \mapsto v/(1 - |v|)$ to a map $T(\xi) \to E(\xi) \cup \{\infty\}$. □

Theorem of Thom

Definition

Let $\tilde{\gamma}^k$ denote the universal oriented k-bundle over $\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)$.

Definition

Let $\tilde{\gamma}^k$ denote the universal oriented k-bundle over $\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)$.

Theorem (Thom, 1954)

There is an isomorphism $\pi_{n+k}(T(\tilde{\gamma}^k), t_0) \cong \Omega_n$ *for* $k \geq n+2$.

The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.

The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i\colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$

The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

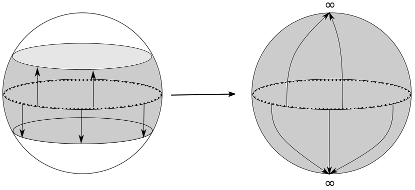
- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i\colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$

The Thom-Pontryagin Construction: $\alpha \colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i \colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$
  - $i^{-1} \colon i(E(\nu_M)) \to E(\nu_M) \subset T(\nu_M)$ on $i(E(\nu_M))$

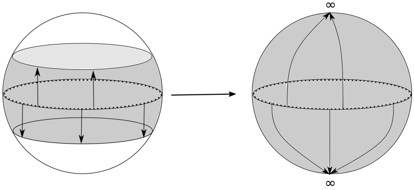
The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i\colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$
  - $i^{-1}\colon i(E(\nu_M)) \to E(\nu_M) \subset T(\nu_M)$ on $i(E(\nu_M))$
  - ∞ outside tubular neighborhood



The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i\colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$
  - $i^{-1}\colon i(E(\nu_M)) \to E(\nu_M) \subset T(\nu_M)$ on $i(E(\nu_M))$
  - ∞ outside tubular neighborhood



- Define $\alpha([M]) = [f]$ where $f\colon S^{n+k} \to T(\nu_M) \xrightarrow{\text{Gauss}} T(\tilde{\gamma}^k)$

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.

Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic

Well-definiteness of $\alpha \colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic
- Embedding $\partial W \to S^{n+k}$ extends to neat embedding $W \hookrightarrow D^{n+k+1}$

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic
- Embedding $\partial W \to S^{n+k}$ extends to neat embedding $W \hookrightarrow D^{n+k+1}$
- Intersection of S^{n+k} and a tubular neighborhood of W in D^{n+k+1} is a tubular neighborhood of ∂W in S^{n+k}

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic
- Embedding $\partial W \to S^{n+k}$ extends to neat embedding $W \hookrightarrow D^{n+k+1}$
- Intersection of S^{n+k} and a tubular neighborhood of W in D^{n+k+1} is a tubular neighborhood of ∂W in S^{n+k}
- Use Thom-Pontryagin construction for W:

$$\begin{array}{ccccc} S^{n+k} & \longrightarrow & T(\nu_{\partial W}) & & \\ \big\downarrow & & \big\downarrow & \searrow & \\ & & & & T(\tilde{\gamma}^k) \\ & & & \nearrow & \\ D^{n+k+1} & \longrightarrow & T(\nu_W) & & \end{array}$$

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).
- Idea: For $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$: $\beta([f]) = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$.

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).
- Idea: For $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$: $\beta([f]) = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$.
- Problem: $f^{-1}(\mathrm{Gr}_k(\mathbb{R}^\infty))$ does not need to be a manifold (even if f is smooth!)

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).
- Idea: For $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$: $\beta([f]) = f^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$.
- Problem: $f^{-1}(\mathrm{Gr}_k(\mathbb{R}^\infty))$ does not need to be a manifold (even if f is smooth!)
- Need *transversality*.

Sard's Theorem

Definition

Let $f\colon M \to N$ be a smooth map. A point $y \in N$ is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $T_x f\colon T_x M \to T_y N$ is surjective.

Sard's Theorem

Definition

Let $f\colon M \to N$ be a smooth map. A point $y \in N$ is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $T_x f\colon T_x M \to T_y N$ is surjective.

Theorem (Sard)

Let $f\colon M \to N$ be a smooth map. The set of regular values of f is dense in N.

Transversality

Definition

Let M, N be manifolds, X a subset of M and Y a submanifold of N. A smooth function $f : M \to N$ is *transverse* to Y throughout X if $T_x M \xrightarrow{T_x f} T_{f(x)} N \to T_{f(x)} N / T_{f(x)} Y$ is surjective for all $x \in f^{-1}(Y) \cap X$.

Transversality

Definition

Let M, N be manifolds, X a subset of M and Y a submanifold of N. A smooth function $f: M \to N$ is *transverse* to Y throughout X if $T_xM \xrightarrow{T_xf} T_{f(x)}N \to T_{f(x)}N/T_{f(x)}Y$ is surjective for all $x \in f^{-1}(Y) \cap X$.

Lemma

If $f: M \to N$ is transverse to $Y \subset N$, then $f^{-1}(Y)$ is a smooth manifold. The normal bundle of Y in N pulls back to the normal bundle of $f^{-1}(Y)$ in M. In particular, $f^{-1}(Y)$ inherits an orientation from an orientation on M and an orientation of the normal bundle of Y in N.

Definition

Let M, N be manifolds, X a subset of M and Y a submanifold of N. A smooth function $f\colon M \to N$ is *transverse* to Y throughout X if $T_x M \xrightarrow{T_x f} T_{f(x)} N \to T_{f(x)} N / T_{f(x)} Y$ is surjective for all $x \in f^{-1}(Y) \cap X$.

Lemma

If $f\colon M \to N$ is transverse to $Y \subset N$, then $f^{-1}(Y)$ is a smooth manifold. The normal bundle of Y in N pulls back to the normal bundle of $f^{-1}(Y)$ in M. In particular, $f^{-1}(Y)$ inherits an orientation from an orientation on M and an orientation of the normal bundle of Y in N.

Proof.

If φ is a local defining function for Y in N, then $\varphi \circ f$ is one for $f^{-1}(Y)$ in M. □

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ *open subset,* $f\colon W \to \mathbb{R}^k$ *smooth, origin regular value throughout closed subset* $X \subset W$, K *a compact subset of* W *and* $\varepsilon > 0$. *There exists smooth* $g\colon W \to \mathbb{R}^k$ *such that*

- $f = g$ *outside compact subset*

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f: W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g: W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0,1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0,1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.
- Set $g(x) = f(x) - \lambda(x)y$ for y regular value of f and $|y| < \varepsilon$

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0,1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.
- Set $g(x) = f(x) - \lambda(x)y$ for y regular value of f and $|y| < \varepsilon$
- If $g(x) = 0$ for $x \in K$, then $f(x) = y \implies T_x f = T_x g$ full rank at x

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0, 1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.
- Set $g(x) = f(x) - \lambda(x)y$ for y regular value of f and $|y| < \varepsilon$
- If $g(x) = 0$ for $x \in K$, then $f(x) = y \implies T_x f = T_x g$ full rank at x
- We can choose partial derivatives of f and g uniformly close to each other $\implies$ origin regular value throughout X □

Thom's Transversality Theorem

Theorem

Every map $S^m \to T(\xi)$ is homotopic to a map $\hat{f}$ which is smooth throughout $\hat{f}^{-1}(T(\xi) - t_0)$ and transverse to the zero-section. The map $\pi_{n+k}(T(\xi), t_0) \to \Omega_n$, $f \mapsto [\hat{f}^{-1}(B(\xi))]$ is well-defined.

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies$ $|f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
  - $f_i = f_{i-1}$ outside compact subset of W_i

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies$ $|f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
    - $f_i = f_{i-1}$ outside compact subset of W_i
    - f_i transverse to B throughout $K_1 \cup \cdots \cup K_i$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \dots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \dots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \dots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
  - $f_i = f_{i-1}$ outside compact subset of W_i
  - f_i transverse to B throughout $K_1 \cup \dots \cup K_i$
  - $\xi \circ f_0 = \xi \circ f_i$ on $f_0^{-1}(T - t_0)$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \dots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
    - $f_i = f_{i-1}$ outside compact subset of W_i
    - f_i transverse to B throughout $K_1 \cup \cdots \cup K_i$
    - $\xi \circ f_0 = \xi \circ f_i$ on $f_0^{-1}(T - t_0)$
    - $|f_i(x) - f_{i-1}(x)| < c/k$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
  - $f_i = f_{i-1}$ outside compact subset of W_i
  - f_i transverse to B throughout $K_1 \cup \cdots \cup K_i$
  - $\xi \circ f_0 = \xi \circ f_i$ on $f_0^{-1}(T - t_0)$
  - $|f_i(x) - f_{i-1}(x)| < c/k$
- Use coordinates $U_i \times \mathbb{R}^k \cong \xi^{-1}(U_i) \supset f_0(W_i)$: Need to construct map $f_i|_{W_i} \colon W_i \to U_i \times \mathbb{R}^k$ transversal to U_i throughout $(K_1 \cup \cdots \cup K_{i-1}) \cup K_i$. First coordinate given by third condition. Second coordinate given by lemma.

Independence of $\hat{f}^{-1}(B)$ representative.

$\square$

Thom's Transversality Theorem

Independence of $\hat{f}^{-1}(B)$ representative.

- Given two homotopic maps $\hat{f}_1$ and $\hat{f}_2$, choose homotopy $h_0 \colon S^m \times [0,3] \to T(\xi)$, smooth throughout $h_0^{-1}(T - t_0)$, $h_0(x,t) = \hat{f}_1(x)$ for $t \leq 1$ and $h_0(x,t) = \hat{f}_2(x)$ for $t \geq 2$.

□

Thom's Transversality Theorem

Independence of $\hat{f}^{-1}(B)$ representative.

- Given two homotopic maps $\hat{f}_1$ and $\hat{f}_2$, choose homotopy $h_0 \colon S^m \times [0,3] \to T(\xi)$, smooth throughout $h_0^{-1}(T - t_0)$, $h_0(x,t) = \hat{f}_1(x)$ for $t \leq 1$ and $h_0(x,t) = \hat{f}_2(x)$ for $t \geq 2$.
- Construct new homotopy $h \colon S^m \times [0,3] \to T(\xi)$ which coincides with h_0 outside compact subset of $S^m \times (0,3)$, transverse to B.

$\square$

Thom's Transversality Theorem

Independence of $\hat{f}^{-1}(B)$ representative.

- Given two homotopic maps $\hat{f}_1$ and $\hat{f}_2$, choose homotopy $h_0 \colon S^m \times [0,3] \to T(\xi)$, smooth throughout $h_0^{-1}(T - t_0)$, $h_0(x,t) = \hat{f}_1(x)$ for $t \leq 1$ and $h_0(x,t) = \hat{f}_2(x)$ for $t \geq 2$.
- Construct new homotopy $h \colon S^m \times [0,3] \to T(\xi)$ which coincides with h_0 outside compact subset of $S^m \times (0,3)$, transverse to B.
- $h^{-1}(B)$ is cobordism between $\hat{f}_1^{-1}(B)$ and $\hat{f}_2^{-1}(B)$

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.
- **Input 1:** f transverse to $B \implies f$ homotopic to a map $\Phi\colon S^{n+k} \to T(\tilde{\gamma}^k)$ which restricts to a bundle map on D

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.
- **Input 1:** f transverse to $B \implies f$ homotopic to a map $\Phi\colon S^{n+k} \to T(\tilde{\gamma}^k)$ which restricts to a bundle map on D
- $\Phi|_D\colon D \to E(\tilde{\gamma}^k)$ is differential of f at M. Homotopic to $f|_D$ via homotopy $h_t(x) = f(tx)/t$.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.
- **Input 1:** f transverse to $B \implies f$ homotopic to a map $\Phi\colon S^{n+k} \to T(\tilde{\gamma}^k)$ which restricts to a bundle map on D
- $\Phi|_D\colon D \to E(\tilde{\gamma}^k)$ is differential of f at M. Homotopic to $f|_D$ via homotopy $h_t(x) = f(tx)/t$.
- **Input 2:** The Thom-Pontryagin collapse map and Φ agree on D and they map $S^{n+k} - \mathrm{int}\,(D)$ to the contractible space $T(\tilde{\gamma}^k) - \tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty) \implies$ they are homotopic □

Topology of the Thom space

Lemma

If the base space B of ξ admits a CW-structure, then $T(\xi)$ admits a $(k-1)$-connected CW-structure where the $(n+k)$-cells correspond one-to-one to n-cells of B (and one additional base point).

Topology of the Thom space

Lemma

If the base space B of ξ admits a CW-structure, then $T(\xi)$ admits a $(k-1)$-connected CW-structure where the $(n+k)$-cells correspond one-to-one to n-cells of B (and one additional base point).

Proof.

Preimage of open n-cells in B under ξ are open $(n+k)$-cells in E. □

Homotopy and Homology groups modulo $\mathcal{C}$

Definition

Let $\mathcal{C} \subset \mathrm{Ab}$ denote the class of all finite abelian groups. A map $f : A \to B$ of abelian groups is a *$\mathcal{C}$-isomorphism* if $\ker(f) \in \mathcal{C}$ and $\mathrm{coker}(f) \in \mathcal{C}$.

Homotopy and Homology groups modulo $\mathcal{C}$

Definition

Let $\mathcal{C} \subset \mathrm{Ab}$ denote the class of all finite abelian groups. A map $f : A \to B$ of abelian groups is a *$\mathcal{C}$-isomorphism* if $\ker(f) \in \mathcal{C}$ and $\operatorname{coker}(f) \in \mathcal{C}$.

Theorem

Let X be finite $(k-1)$-connected CW-complex for an integer $k \geq 2$. The Hurewicz morphism $\pi_n(X, x_0) \to H_n(X)$ is a $\mathcal{C}$-isomorphism for $n < 2k - 1$.

$\mathcal{C}$-isomorphism $\pi_n(T(\xi), t_0) \to H_{n-k}(B(\xi))$

Corollary

There is a $\mathcal{C}$-isomorphism: $\pi_{n+k}(T(\xi), t_0) \to H_n(B(\xi))$ in degree $n < k - 1$.

$\mathcal{C}$-isomorphism $\pi_n(T(\xi), t_0) \to H_{n-k}(B(\xi))$

Corollary

There is a $\mathcal{C}$-isomorphism: $\pi_{n+k}(T(\xi), t_0) \to H_n(B(\xi))$ in degree $n < k - 1$.

Proof.

- Generalized Hurewicz: There is $\mathcal{C}$-isomorphism $\pi_{n+k}(T(\xi), t_0) \to H_{n+k}(T(\xi))$
- Let T_0 denote the complement of the zero-section in $T(\xi)$. Since T_0 is contractible: $H_{n+k}(T(\xi)) \cong H_{n+k}(T(\xi), T_0)$. By Excision: $H_{n+k}(T(\xi), T_0) \cong H_{n+k}(E(\xi), E_0)$. Thom isomorphism: $H_{n+k}(E(\xi), E_0) \cong H_n(B(\xi))$.

□

Description of Ω_n

Theorem (Thom, 1954)

The oriented cobordism group Ω_n is finite for $4 \nmid n$ and finitely generated of rank $p(r)$ (numbers of partitions of r) if $n = 4r$.

Proof.

- We know that $\Omega_n \cong \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ for $k \gg 0$
- There is a $\mathcal{C}$-isomorphism $\pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to H_n(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$.
- This group is finite for $4 \nmid n$ and finitely generated of rank $p(r)$ (number of partitions) if $n = 4r$.

$\square$

Corollary

The graded ring $\Omega_* \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$ with linearly independent generators $\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \ldots$.

Classification of oriented boundaries

Corollary

The multiple of an n-dimensional manifold M is diffeomorphic to an oriented boundary if and only if all Pontrjagin numbers vanish.

Theorem (Wall, 1960)

An n-dimensional manifold M is an oriented boundary if and only if all Pontrjagin numbers and all Stiefel-Whitney classes vanish.

References

Morris W. Hirsch. *Differential topology*. Vol. 33. Graduate Texts in Mathematics. Corrected reprint of the 1976 original. Springer-Verlag, New York, 1994, pp. x+222.

John W. Milnor and James D. Stasheff. *Characteristic classes*. Vol. 76. Annals of Mathematics Studies. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, pp. vii+331.

Edwin H. Spanier. *Algebraic topology*. Corrected reprint. Springer-Verlag, New York-Berlin, 1981, pp. xvi+528.

René Thom. “Quelques propriétés globales des variétés différentiables”. In: *Comment. Math. Helv.* 28 (1954), pp. 17–86.