Chern and Pontrjagin numbers

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Definition

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Juxtaposition

We define the juxtaposition of $I = (i_1, \ldots, i_r)$ (a partition of k) and $J = (j_1, \ldots, j_s)$ (a partition of l) to be

$$IJ = (i_1, \ldots, i_r, j_1, \ldots, j_s)$$

(a partition of $k + l$).

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Refinement

A refinement of $I = (i_1, \dots, i_r)$ is a partition of the form $I_1 \cdot I_2 \cdot \ldots \cdot I_r$, where I_j is a partition of i_j.
Example: $(1,1)(3)(1,1,2) = (1,1,1,1,2,3)$ is a refinement of $(2,3,4)$.

Denote the total number of partitions of n by $p(n)$.

Let K^n be a compact complex manifold of dimension n.

Definition

For $I = (i_1, \ldots, i_r)$ a partition of n, we define the I-th Chern number:

$$c_I[K^n] = \langle c_{i_1}(\tau^n) \cdots c_{i_r}(\tau^n), \mu_{2n} \rangle$$

where τ^n denotes the tangent bundle of K^n and μ_{2n} the fundamental homology class induced by the preferred orientation.
If I is not a partition of n, set $c_I[K^n] = 0$.

$\mathbb{C}P^n$

Reminder

In Chapter 14 we have seen that the *i*-th Chern class is:

$$c_i(\tau^n) = \binom{n+1}{i} a^i$$

and that $\langle a^n, \mu_{2n} \rangle = 1$.

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Reminder

In Chapter 14 we have seen that the i-th Chern class is:

$$c_i(\tau^n) = \binom{n+1}{i} a^i$$

and that $\langle a^n, \mu_{2n} \rangle = 1$.

Hence for any $I = (i_1, \ldots, i_r)$ a partition of n we get

$$c_I[K^n] = \binom{n+1}{i_1} \ldots \binom{n+1}{i_r}$$

Observations

$$c_n[K^n] = \langle c_n(\tau^n), \mu_{2n} \rangle = \xi(K^n)$$

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In general there are $p(n)$ different Chern numbers, which are linearly independent.
This is meant in the sense, that there is no linear relation between them, that is satisfied for all n-manifolds.

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Observation $\implies$ we can just compute all
$\langle c_{i_1}(\gamma^n)\dots c_{i_r}(\gamma^n), f_*(\mu_{2n})\rangle = \langle f^*(c_{i_1}(\gamma^n)\dots c_{i_r}(\gamma^n)), \mu_{2n}\rangle = c_I[K^n]$.

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Definition

The I-th Pontrjagin number is defined to be

$$p_I[M^{4n}] = \langle p_{i_1}(\tau^{4n}) \ldots p_{i_r}(\tau^{4n}), \mu_{4n} \rangle$$

where again τ^{4n} is the tangent bundle, μ_{4n} is the fundamental class.

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Results from chapter 15

$$p_I[\mathbb{C}\mathrm{P}^{2n}] = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_r}$$

Pontrjagin number vs Euler number

Reversing orientation on M^{4n} leaves the Pontrjagin classes stable, but changes the sign of the fundamental class, hence $p_I[-M^{4n}] = -p_I[M^{4n}]$. The Euler number stays the same.

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If any $p_I[M^{4n}] \neq 0$, then M is not a boundary of a smooth compact oriented $(4n+1)$-manifold.

See (4.9) - all Stiefel Whitney numbers vanish.

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Example

$t_1^2 + t_2^2 + t_1 t_2 \in \mathbb{Z}[t_1, t_2]$ is symmetric.
$t_1 + t_2^2$ is not.

We denote by $\mathcal{S}_n$ the subring of the symmetric polynomials.

Elementary symmetric polynomials

Theorem

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The elementary symmetric function σ_k can be characterized by being the homogeneous component of degree k in $\coprod_{i=1}^{n}(1 + t_i)$.
For example: in $n = 3$ we have $\sigma_2 = t_1 t_2 + t_2 t_3 + t_3 t_1$.

Grading

Assign each t_i in $\mathbb{Z}$ degree 1, then we can see that

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Definition

We write $\mathcal{S}^k$ for the subring of $\mathbb{Z}[t_1, \ldots, t_n]$ of symmetric polynomials of degree k.

We can also see S^k as a free $\mathbb{Z}$-module. An obvious basis are the monomials $\sigma_{i_1} \ldots \sigma_{i_r}$ with $(i_1, \ldots, i_r)$ a partition of k.

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Notation

We write

$$\sum t_1^{a_1} \dots t_n^{a_n} \in \mathcal{S}_n$$

for the sum of all monomials equivalent to $t_1^{a_1} \dots t_n^{a_n}$.

Example: $\sigma_k = \sum t_1 t_2 \dots t_k$

Lemma

$$\left\{ \sum t_1^{a_1} \dots t_r^{a_r} \;\middle|\; r \leq n,\ (a_1, \dots, a_r) \text{ is a partition of } k \right\}$$

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We want to assign to a partition I of k a polynomial s_I in k variables. For $n \geq k$, the $\sigma_1, \dots, \sigma_k$ are algebraically independent in $\mathbb{Z}[t_1, \dots, t_n]$ and we can say s_I is specified by the equation

$$s_I(\sigma_1, \dots, \sigma_k) = \sum t_1^{i_1} \dots t_r^{i_r}$$

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This does not depend on n, we can set $t_{k+1} = \dots = t_n = 0$ to recover the equation for $n = k$.

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This does not depend on n, we can set $t_{k+1} = \cdots = t_n = 0$ to recover the equation for $n = k$.
From the definition follows that the $\{s_I | I$ is a partition of $k\}$ are linearly independent. Last Lemma $\implies$ this is a basis.
This is the one we wanted to construct!

If a complex n-plane bundle ω splits as $\nu_1 \oplus \cdots \oplus \nu_n$ a Whitney sum of line bundles, the formula
$1 + c_1(\omega) + \cdots + c_n(\omega) = (1 + c_1(\nu_1)) \ldots (1 + c_1(\nu_n))$ shows that

$$c_k(\omega) = \sigma_k(c_1(\nu_1), \ldots, c_1(\nu_n))$$

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Example: $\gamma^1 \times \cdots \times \gamma^1$ the n-fold cartesian product over $\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty$.
Note that $H^*(\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[a_1, \dots, a_n]$ with $\deg a_i = 2$ and

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$$c(\gamma^1 \times \cdots \times \gamma^1) = (1 + a_1) \dots (1 + a_n)$$

Hence $H^*(Gr_n(\mathbb{C}^\infty); \mathbb{Z}) \cong \mathcal{S}^n$ and our new basis of $\mathcal{S}_k$ gives us a basis of $H^{2k}(Gr_n(\mathbb{C}^\infty); \mathbb{Z})$

Let ω be a complex n-plane bundle with paracompact base space B and total Chern class $c = 1 + c_1 + \cdots + c_n$. For $k > 0$ and I a partition of k we write

$$s_I(c) = s_I(c_1, \ldots, c_k) \in H^{2k}(B; \mathbb{Z})$$

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Lemma - Thom

$$s_I(c(\omega \oplus \omega')) = \sum_{JK=I} s_J(c(\omega)) s_I(c(\omega'))$$

where we sum over all partitions J, K with $JK = I$.

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Taking the trivial partition $I = (k)$, we see that

$$s_k(c(\omega \oplus \omega')) = s_k(c(\omega)) + s_k(c(\omega'))$$

Proof: $s_I(c(\omega \oplus \omega')) = \sum_{JK=I} s_J(c(\omega))s_I(c(\omega'))$

Let σ_k be the elementary symmetric polynomial in $t_1, \ldots, t_n$ and σ'_k be the one in $t_{n+1}, \ldots, t_{2n}$. Define $\sigma'' = \sum_{i=0}^{k} \sigma_i \sigma'_{k-i}$, which is just the k-th elementary symmetric polynomial in $t_1, \ldots, t_n$

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Claim

$$s_I(\sigma''_1, \dots, \sigma''_k) = \sum_{JK=I} s_J(\sigma_1, \dots)s_K(\sigma'_1, \dots)$$

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Claim

$$s_I(\sigma''_1, \dots, \sigma''_k) = \sum_{JK=I} s_J(\sigma_1, \dots)s_K(\sigma'_1, \dots)$$

Once we have proven this, we can use that $\sigma_1, \dots, \sigma_n, \sigma'_1, \dots, \sigma'_n$ are algebraically independent to set $\sigma_i = c_i(\omega)$ and $\sigma'_i = c_i(\omega')$. The product formula for Chern classes (14.7) yields $\sigma''_i = c_i(\omega \oplus \omega')$.

Proof of Claim

$$s_I(\sigma_1'', \ldots, \sigma_k'') = \sum_{JK=I} s_J(\sigma_1, \ldots) s_K(\sigma_1', \ldots)$$

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Fixing $J = (j_1, \ldots, j_s), K = (k_1, \ldots, k_{r-s})$ partitions with $JK = I$, and taking the sum over all α that induce them, yields

$$\underbrace{\sum t_1^{j_1} \ldots t_s^{j_s}}_{\text{taken in } t_1, \ldots, t_n} \cdot \underbrace{\sum t_{n+1}^{k_1} \ldots t_{n+r-s}^{k_{r-s}}}_{\text{taken in } t_{n+1}, \ldots, t_{2n}} = s_J(\sigma_1, \ldots, \sigma_s) s_K(\sigma_1', \ldots, \sigma_{r-s}')$$

Consider K^n a complex manifold of complex dimension n. For a partition I of n define $s_I(c)[K^n] = \langle s_I(c(\tau^n)), \mu_{2n} \rangle \in \mathbb{Z}$.

Corollary

$$s_I[K^m \times L^n] = \sum_{JK=I} s_J[K^m] s_K[L^n]$$

where J is a partition of m and K one of n.

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$$s_I[K^m \times L^n] = \langle s_I(\tau \times \tau'), \mu_{2m} \times \mu_{2n'}\rangle = \sum_{JK=I} \langle s_J(\tau), \mu_{2m}\rangle \langle s_K(\tau'), \mu'_{2n}\rangle$$

The signs die since all degrees are even.

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The signs die since all degrees are even.
As a corollary, $s_{n+m}[K^m \times L^n] = 0$ if $m, n \neq 0$.

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and taking $n = k$ yields $s_n[\mathbb{C}\mathrm{P}^n] = n+1 \neq 0$
This means $\mathbb{C}\mathrm{P}^n$ cannot be expressed as a (non-trivial) product of complex manifolds.

Pontrjagin numbers

... have analogous results. Let ξ be a real vector bundle over the base space B and I a partition of n. Define

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Then we get another product formula, but only modulo 2 (see 16.2):

$$s_I(p(\xi \oplus \xi')) = \sum_{JK=I} s_J(p(\xi)) s_K(p(\xi'))$$

which implies $s_I(p)[M \times N] = \sum_{JK=I} s_J(p)[M] s_K(p)[N]$

Main Result

Theorem - Thom

For $K_1, \dots, K^n$ complex manifolds with $s_k(c)[K^k] \neq 0$, we have that the matrix

$$[c_{i_1} \dots c_{i_r}[K^{j_1} \times \dots \times K^{j_s}]]_{I,J \text{ are partitions of } n}$$

is non-singular.

For example, $K^r = \mathbb{C}\mathrm{P}^r$ has this property

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Theorem

If $M^4, \ldots, M^{4n}$ are oriented manifolds and $s_k(p)[M^{4k}] \neq 0$ then

$$[p_{i_1} \ldots p_{i_r}[M^{4j_1} \times \cdots \times M^{4j_s}]]_{I,J \text{ are partitions of } n}$$

is non-singular.

Proof

We can easily generalize our product formula to

$$s_I[K^{j_1} \times \cdots \times K^{j_q}] = \sum_{I_1 \ldots I_q = I} s_{I_1}[K^{j_1}] \ldots s_{I_q}[K^{j_q}]$$

where we sum over I_l partitions of j_l.

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We see that $s_I[K^{j_1} \times \dots \times k^{j_q}] = 0$, unless $I = (i_1, \dots, i_r)$ is a refinement of $(j_1, \dots, j_q)$.

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But that means we can arrange the partitions so that

$$[c_{i_1} \dots c_{i_r}[K^{j_1} \times \cdots \times K^{j_s}]]_{I,J \text{ are partitions of } n}$$

Is a triangular matrix with zeros above the diagonal.

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Is a triangular matrix with zeros above the diagonal.
The diagonal entries are $s_{(i_1,\ldots,i_r)}[K^{i_1} \times \cdots \times K^{i_r}] = \coprod_{l=1}^{r} s_{i_l}[K^{i_l}] \neq 0$, so the determinant of the matrix is non-zero.