Pontrjagin classes

[Milnor, Stasheff §15]

I Complexification of a real vector bundle

II Definition & Properties of Pontrjagin classes

III Application: Cohomology of the oriented Grassmannian

Convention:

- real vector bundles are denoted by ξ
- complex vector bundles are denoted by ζ

Reminder. For a complex v.b. ζ we have its underlying real v.b. $\zeta_\mathbb{R}$

(Note: $\dim(\zeta_\mathbb{R}) = 2\dim_\mathbb{C}(\zeta)$)

We defined

$$c_n(\zeta) := e(\zeta_\mathbb{R})$$

has a preferred orientation

I Complexification

Def. For a real v.b. ξ we define its complexification $\xi_\mathbb{C} = \xi \otimes_\mathbb{R} \mathbb{C}$ by applying the functor $- \otimes_\mathbb{R} \mathbb{C}$ to each fibre.

(Note: $\dim_\mathbb{C}(\xi_\mathbb{C}) = \dim(\xi)$)

Proposition.

(1) $(\xi_\mathbb{C})_\mathbb{R} \cong \xi \oplus \xi$

is oriented

(2) $(\zeta_\mathbb{R})_\mathbb{C} \cong \zeta \oplus \bar{\zeta}$

Proof. For (1): Let F be the fibre of ξ.

$$(F \otimes_\mathbb{R} \mathbb{C})_\mathbb{R} = F \otimes_\mathbb{R} \mathbb{C}_\mathbb{R} = \underbrace{(F\otimes 1) \oplus (F \otimes i)}_{\text{internal direct sum}} \cong F \oplus F$$

$\longrightarrow$ (1)

For (2): Let F be the fibre of ζ.

$$F_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\varphi} F \oplus \bar{F}, \quad a \otimes z \longmapsto (za, \bar{z}a)$$

which is a well-defined map of $\mathbb{R}$-v.s.

Claim: φ is an iso. of $\mathbb{C}$-v.s.

- φ is $\mathbb{C}$-linear

$$\begin{aligned}\varphi(w(a\otimes z)) &= \varphi(a \otimes wz)\\ &= (wza, \overline{wz}a)\\ &= w(za, \bar{z}a)\\ &= w\varphi(a\otimes z)\end{aligned}$$

- φ is bijective

$$\varphi\left(\tfrac{1}{2}(a+b)\otimes 1 - \tfrac{1}{2}i(a-b)\otimes i\right) = (a,b)$$

□

Proposition. If ξ is an oriented real v.b. of dim. n, then

$$(\xi_\mathbb{C})_\mathbb{R} \cong (-1)^{\frac{n(n-1)}{2}} \xi \oplus \xi$$

Proof. Let $a_1, \dots, a_n$ be an ordered basis of F (of ξ).

On the rhs:

$\rightsquigarrow (a_1, 0), \dots, (a_n, 0), (0, a_1), \dots, (0, a_n)$ (*)

On the lhs:

$\rightsquigarrow a_1 \otimes 1, \dots, a_n \otimes 1$ of $F \otimes_\mathbb{R} \mathbb{C}$

$\rightsquigarrow a_1 \otimes 1, a_1 \otimes i, \dots, a_n \otimes 1, a_n \otimes i$ of $(F \otimes_\mathbb{R} \mathbb{C})_\mathbb{R}$

$$(F \otimes_\mathbb{R} \mathbb{C})_\mathbb{R} = F \otimes_\mathbb{R} \mathbb{C}_\mathbb{R} = (F\otimes 1) \oplus (F \otimes i) \cong F \oplus F$$

$$a_k \otimes 1 \longmapsto (a_k, 0)$$
$$a_k \otimes i \longmapsto (0, a_k)$$

Have to compare (*) and

$(a_1, 0), (0, a_1), \dots, (a_n, 0), (0, a_n)$

By induction n you see that the sign of the permutation is

$$(-1)^{(n-1)+\dots+1} = (-1)^{\frac{n(n-1)}{2}}$$

□

II Def. & Properties of the Pontrjagin classes

Motivation. What is $w(\zeta_\mathbb{R})$ and $c(\xi_\mathbb{C})$?

- $w_{2k+1}(\zeta_\mathbb{R}) = 0$ and $w_{2k}(\zeta_\mathbb{R})$ is the mod 2 reduction of $c_k(\zeta)$
- $c_{2k+1}(\xi_\mathbb{C}) = \boxed{\beta(w_{2k}(\xi) \cup w_{2k+1}(\xi))}$

Definition. For a real v.b. ξ we define its k-th Pontrjagin class

$$p_k(\xi) := (-1)^k c_{2k}(\xi_\mathbb{C}) \in H^{4k}(B(\xi); \mathbb{Z})$$

The total Pontrjagin class of ξ is

$$p(\xi) := p_0(\xi) + p_1(\xi) + \dots$$

Proposition. If ξ is an oriented real v.b. of dim $2n$, then $p_n(\xi) = e(\xi)^2$.

Proof.

$$\begin{aligned}p_n(\xi) &= (-1)^n c_{2n}(\xi_\mathbb{C})\\ &= (-1)^n e((\xi_\mathbb{C})_\mathbb{R})\\ &= (-1)^n e\left((-1)^{\frac{2n(2n-1)}{2}} \xi \oplus \xi\right)\\ &= \underbrace{(-1)^n (-1)^{n(2n-1)}}_{=+1} e(\xi \oplus \xi)\\ &= e(\xi)^2\end{aligned}$$

□

Theorem.

(1) $p_0(\xi) = 1$ and $p_k(\xi) = 0$ if $k > \frac{\dim(\xi)}{2}$

(2) $p(f^*\xi) = f^* p(\xi)$

(3) $2p(\xi \oplus \xi') = 2p(\xi) \cup p(\xi')$

Part (1) and (2) are following from the properties of the Chern classes. For (3) we need the following

Lemma. For real v.b. ξ we have

$$2c_{2k+1}(\xi_\mathbb{C}) = 0$$

Proof. Note $\xi_\mathbb{C} \cong \overline{\xi_\mathbb{C}}$, because

$$F \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\cong} \overline{F \otimes_\mathbb{R} \mathbb{C}} = F \otimes_\mathbb{R} \overline{\mathbb{C}}, \quad a \otimes z \longmapsto a \otimes \bar{z}.$$

Now, we compute

$$\begin{aligned}c_{2k+1}(\xi_\mathbb{C}) &= c_{2k+1}(\overline{\xi_\mathbb{C}})\\ &= \underbrace{(-1)^{2k+1}}_{=-1} c_{2k+1}(\xi_\mathbb{C})\end{aligned}$$

$$\longrightarrow 2c_{2k+1}(\xi_\mathbb{C}) = 0$$

□

Proof of Thm (3):

$$2p(\xi \oplus \xi') = 2p(\xi) \cup p(\xi')$$

We compute

$$2p_k(\xi \oplus \xi') = 2(-1)^k c_{2k}(\underbrace{(\xi \oplus \xi')_\mathbb{C}}_{= \xi_\mathbb{C} \oplus \xi'_\mathbb{C}})$$

$$= 2(-1)^k \sum_{j=0}^{2k} c_{2k-j}(\xi_\mathbb{C}) \cup c_j(\xi'_\mathbb{C})$$

$$\overset{\text{Lemma}}{=} 2(-1)^k \sum_{j=0}^{k} c_{2k-2j}(\xi_\mathbb{C}) \cup c_{2j}(\xi'_\mathbb{C})$$

$$= 2\sum_{j=0}^{k} \underbrace{(-1)^{k-j} c_{2k-2j}(\xi_\mathbb{C})}_{= p_{k-j}(\xi)} \cup \underbrace{(-1)^j c_{2j}(\xi'_\mathbb{C})}_{= p_j(\xi')}$$

□

Example

$$p(\mathbb{CP}^n) := p(T_\mathbb{R}) = (1 + a^2)^{n+1}$$

III Application

The oriented Grassmannian $\tilde{G}_n(\mathbb{R}^{n+k})$ is the space of all oriented n-dim linear subspaces in $\mathbb{R}^{n+k}$ with quotient topo.:

$$\begin{tikzcd} V_n(\mathbb{R}^{n+k}) \arrow[r] \arrow[dr] & G_n(\mathbb{R}^{n+k}) \\ & \tilde{G}_n(\mathbb{R}^{n+k}) \arrow[u, "\pi"'] \end{tikzcd}$$

is a 2-fold covering

$$\begin{tikzcd} \tilde{G}_n := \operatorname{colim}_k \tilde{G}_n(\mathbb{R}^{n+k}) \arrow[d, "\pi"] \\ G_n \end{tikzcd}$$

and $\tilde{\gamma}^n := \pi^* \gamma^n$ is an oriented n-dim real v.b. $\tilde{G}_n$.

Remark.

- G_n is model for $BO(n)$
- $\tilde{G}_n$ is model for $BSO(n)$

Theorem. Let R be an integral domain with $\frac{1}{2} \in R$. Then

$$H^*(\tilde{G}_n; R) \cong \begin{cases} R[p_1, \dots, p_m] & \text{if } n = 2m+1 \\ R[p_1, \dots, p_{m-1}, e] & \text{if } n = 2m \end{cases}$$

where $p_k := p_k(\tilde{\gamma}^n)$.

Proof. By induction on n. For $n = 1$, we have $\tilde{G}_1 \cong S^\infty$ (Note $G_1 = \mathbb{RP}^\infty$). So for $n=1$ the statement holds.

Suppose $n > 1$. Consider the Gysin-sequence

$$\begin{tikzcd} \dots \arrow[r] & H^{i-n}(\tilde{G}_n) \arrow[r, "\cdot \cup e"] & H^i(\tilde{G}_n) \arrow[r] \arrow[dr, "\lambda"'] & H^i(E_0) \arrow[r] \arrow[d, "\cong"] & H^{i+1-n}(\tilde{G}_n) \arrow[r] & \dots \\ & & & H^i(\tilde{G}_{n-1}) \arrow[ur] & & \end{tikzcd}$$

Claim: there exist iso.s $H^i(E_0) \xrightarrow{\cong} H^i(\tilde{G}_{n-1})$ such that

- $\lambda: H^*(\tilde{G}_n) \longrightarrow H^*(\tilde{G}_{n-1})$ is a ring-hom
- λ maps $p_k(\tilde{\gamma}^n)$ to $p_k(\tilde{\gamma}^{n-1})$

Case 1: $n = 2m$. By IH, $H^*(\tilde{G}_{2m-1})$ is generated by $p_1(\tilde{\gamma}^{2m-1}), \dots, p_{m-1}(\tilde{\gamma}^{2m-1})$.

So, λ is surjective $\Longrightarrow$ The Gysin-sequence splits into s.e.s.s:

$$0 \longrightarrow H^{i-2m}(\tilde{G}_{2m}) \xrightarrow{\cdot \cup e} H^i(\tilde{G}_{2m}) \xrightarrow{\lambda} H^{i+1-2m}(\tilde{G}_{2m-1}) \longrightarrow 0$$

Inductively check that every element in $H^i(\tilde{G}_{2m})$ is (uniquely) a polynomial

$$p_1(\tilde{\gamma}^{2m}), \dots, p_{m-1}(\tilde{\gamma}^{2m}), e := e(\tilde{\gamma}^{2m})$$

Case 2: $n = 2m+1$.

Claim: $e(\tilde{\gamma}^{2m+1}) = 0$

- In general: $2e(\xi) = 0$ if $\dim(\xi)$ odd
- We're working over R

Again, we get s.e.s.s:

$$0 \longrightarrow H^i(\tilde{G}_{2m+1}) \xrightarrow{\lambda} H^i(\tilde{G}_{2m}) \longrightarrow H^{i+2m}(\tilde{G}_{2m+1}) \longrightarrow 0 \quad (+)$$

Now, consider the diagram

$$\begin{tikzcd} R[p_1, \dots, p_m] \arrow[r, hook] \arrow[d, "p_m \mapsto e^2"'] & H^*(\tilde{G}_{2m+1}) \arrow[d, "\lambda"] \\ R[p_1, \dots, p_{m-1}, e] \arrow[r] & H^*(\tilde{G}_{2m}) \end{tikzcd}
\qquad
\begin{tikzcd} p_k \arrow[r, mapsto] \arrow[d, mapsto] & p_k(\tilde{\gamma}^{2m+1}) \arrow[d, mapsto] \\ p_k \arrow[r, mapsto] & p_k(\tilde{\gamma}^{2m}) \end{tikzcd}$$

the diagram commutes

$$\begin{tikzcd} p_m \arrow[r, mapsto] \arrow[d, mapsto] & p_m(\tilde{\gamma}^{2m+1}) \arrow[d, mapsto] \\ e^2 \arrow[r, mapsto] & e^2 = p_m(\tilde{\gamma}^{2m}) \end{tikzcd}$$

It follows that $R[p_1, \dots, p_m] \longrightarrow H^*(\tilde{G}_{2m+1})$ is injective

$$\left(\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\ A' \arrow[r, "\cong"] & B' \end{tikzcd}\right)$$

By abuse of notation

$$A^* := R[p_1, \dots, p_m] \subseteq H^*(\tilde{G}_{2m+1}) \subseteq \underbrace{H^*(\tilde{G}_{2m})}_{= R[p_1, \dots, p_{m-1}, e]}$$

$$p_m \longmapsto e^2$$

Want to show: $A^* = H^*(\tilde{G}_{2m+1})$

We compare ranks. We know

$$H^i(\tilde{G}_{2m}) = A^i \oplus e A^{i-2m}$$

so

(*) $\operatorname{rank}(H^i(\tilde{G}_{2m})) = \operatorname{rank}(A^i) + \operatorname{rank}(A^{i-2m})$

By (+) we get

(**) $\operatorname{rank}(H^i(\tilde{G}_{2m})) = \operatorname{rank}(H^i(\tilde{G}_{2m+1})) + \operatorname{rank}(H^{i-2m}(\tilde{G}_{2m+1}))$

By (*) & (**) and induction on i, we see

$$\operatorname{rank}(A^i) = \operatorname{rank}(H^i(\tilde{G}_{2m+1}))$$

Suppose $A^i \subsetneq H^i(\tilde{G}_{2m+1})$, i.e. there is an element $x + ey \in H^i(\tilde{G}_{2m+1}) \setminus A^i$ for $x \in A^i$ and non-zero $y \in A^{i-2m}$

$$\Longrightarrow \operatorname{rank}(A^i) < \operatorname{rank}(H^i(\tilde{G}_{2m+1})) \quad ↯$$

$$\Longrightarrow A^i = H^i(\tilde{G}_{2m+1})$$

□

Example.

$$p(\mathbb{CP}^n) = (1 + a^2)^{n+1}$$

First, we use $c(\mathbb{CP}^n) = (1+a)^{n+1}$.

Moreover: For a complex v.b. ζ we have

$$\begin{aligned}(-1)^k p_k(\zeta_\mathbb{R}) &= c_{2k}((\zeta_\mathbb{R})_\mathbb{C})\\ &= c_{2k}(\zeta \oplus \bar{\zeta})\\ &= \sum_{j=0}^{2k} c_{2k-j}(\zeta) \cup \underbrace{c_j(\bar{\zeta})}_{= (-1)^j c_j(\zeta)}\end{aligned}$$

$$\longrightarrow \left(\sum_{k \geq 0} (-1)^k p_k(\zeta_\mathbb{R})\right) = c(\zeta) \cup \left(\sum_{k \geq 0} (-1)^k c_k(\zeta)\right)$$

So in our example

$$\begin{aligned}\left(\sum_{k\geq 0} (-1)^k p_k(\mathbb{CP}^n)\right) &= (1+a)^{n+1} \cup (1-a)^{n+1}\\ &= (1-a^2)^{n+1}\end{aligned}$$

$$\longrightarrow p(\mathbb{CP}^n) = (1 + a^2)^{n+1}$$