Proof of the h-Cobordims Theorem

Dominik Kirstein

June 30, 2020

Theorem

Let M_0 be a closed simply connected manifold of dimension $n \geq 5$. Then every h-cobordism (W, M_0, M_1) over M_0 is trivial over M_0.

h-cobordism:
- W compact manifold,
- $\partial W = M_0 \amalg M_1$
- $M_0 \hookrightarrow W$, $M_1 \hookrightarrow W$ h-equivalences

trivial: $W \overset{\sim}{=} M_0 \times [0,1]$

diffeomorphism relative M_0

Short Reminder

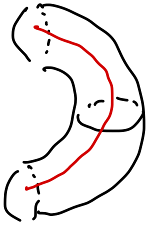
In the following, let W be a compact n-manifold with boundary $\partial W = \partial_0 W \amalg \partial_1 W$.
There exists a handlebody decomposition

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n) \tag{1}$$

- There exists a relative CW complex $(X, \partial_0 W) \simeq (W, \partial_0 W)$

  q-cells of $X \leftrightarrow q$-handles of (1)

  $H_q(W, \partial_0 W)$ can be computed from the handlebody chain complex $C_q(W, \partial_0 W) := H_q(W_q, W_{q-1})$



Strategy

We first want to simplify the handlebody decomposition in the following way:

Lemma (Normal form Lemma)

Let $(W, \partial_0 W, \partial_1 W)$ be a h-cobordism of dimension $n \geq 6$ and $2 \leq q \leq n-3$. Then there exists a handlebody decomposition of the form

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}). \tag{2}$$

Sketch of proof: • know: No 0, 1 - handles

• Replace each k-handle by (k+2) - handle

$\rightsquigarrow$ $W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} \phi_i^q + \dots + \sum_{i=1}^{p_n} \phi_i^n$

• Dual handlebody decomposition

Short Reminder

Lemma (Elimination Lemma)

Let $1 \leq q \leq n-3$ and suppose that $p_j = 0$ for $j < q$. Fix any $1 \leq i_0 \leq p_q$ and suppose that there is an embedding $\psi^{q+1}\colon S^q \times D^{n-1-q} \to \partial_1^\circ W_q$ with the following properties:

- *$\psi^{q+1}|_{S^q \times 0}$ is isotopic in $\partial_1 W_q$ to an embedding $\psi_1^{q+1}\colon S^q \times 0 \to \partial_1 W_q$ which meets the transverse sphere of $\phi_{i_0}^q$ transversally and in exactly one point and is disjoint from the transverse sphere of ϕ_i^q for all $i \neq i_0$;*
- *$\psi^{q+1}|_{S^q \times 0}$ is isotopic in $\partial_1 W_{q+1}$ to a trivial embedding $\psi_2^{q+1}\colon S^q \times 0 \to \partial_1^\circ W_{q+1}$.*

Then

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1, i \neq i_0}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi^{q+2}) + \sum_{i=1}^{p_{q+2}} (\overline{\phi}_i^{q+2}) + \sum_{i=1}^{p_n} (\overline{\phi}_i^n) \tag{3}$$

Homology Lemma

Lemma

Suppose $n \geq 6$, $2 \leq q \leq n-3$ *and* $i_0 \in \{1, \dots, p_q\}$ *and let* $f\colon S^q \to \partial_1 W_q$ *be an embedding. The following are equivalent:*

(i) f *is isotopic to an embedding* $g\colon S^q \to \partial_1 W_q$ *such that* g *meets the transverse sphere of* $(\phi^q_{i_0})$ *transversally and in exactly one point and is disjoint from the transverse sphere of* (ϕ^q_i) *for all* $i \neq i_0$.

(ii) *There exists* $\gamma \in \pi$ $= \pi_1(\partial_0 W)$ *such that*

$$[\widetilde{f}] = \pm\gamma[\phi^q_{i_0}] \tag{4}$$

in $C_q(\widetilde{W}, \widetilde{\partial_0 W})$*, where* $\widetilde{f}\colon S^q \to \widetilde{W}_q$ *is a lift of* f.

$$\begin{tikzcd} \widetilde{W}_q \arrow[r] & W_q \\ S^q \arrow[u, dashed, "\tilde f"] \arrow[ur, "f"'] \end{tikzcd}$$

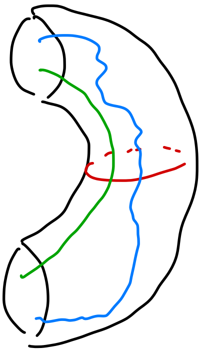
$$\pi_q(\widetilde{W}_q) \to \pi_q(\widetilde{W}_q, \widetilde{W}_{q-1}) \to H_q(\widetilde{W}_q, \widetilde{W}_{q-1}) = C_q(\widetilde{W}, \partial_0\widetilde{W})$$

$$[\tilde f] \in \pi_q(\widetilde{W}_q)$$

Proof: (c) $\Rightarrow$ (iii) isotope f s.t.

- $f|_{S^q_+}$ looks like $S^q_+ \to D^q \times \{x\} \to \partial(\phi^q_{i_0})$
- $f|_{S^q_-}$ is disjoint from all ϕ^q_i

$\Rightarrow [\tilde{f}] = \pm \gamma [\phi^q_{i_0}]$



Homology Lemma

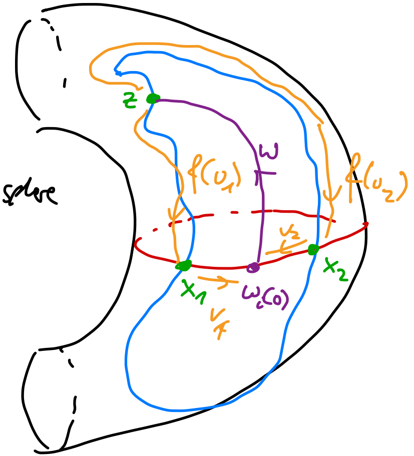
(ii) $\Rightarrow$ (i) Assume $[\tilde{f}] = \pm \gamma [\phi^q_{i_0}]$

Isotop f s.t. f meets all transverse spheres transversally

$\dim S^q + \dim(\text{transverse sphere}) = q + n - q - 1 = n - 1 = \dim \partial_1 W_q$

transversality $\Rightarrow$ f intersects each transverse sphere of ϕ^q_i in only finitely many points $x_{i1}, \dots, x_{ir_i}$

- Pick $y \in S^q$, $z := f(y)$
- $v_{ij} : I \to S^q$, $v_{ij}(0) = y$, $f(v_{ij}(1)) = x_{ij}$
- $w_i : I \to \partial_1 W_q$ starting in the transverse sphere of ϕ^q_{i}, $w_i(1) = z$
- $v_{ij} : I \to$ transverse sphere, $v_{ij}(0) = x_{ij}$, $v_{ij}(1) = w_i(0)$
- $\gamma_{ij} := f(v_{ij}) * v_{ij} * w_i$



As before: $\pm\gamma[\phi^q_{i_0}] = [\hat{f}] = \sum_{i=1}^{p_q}\sum_{j=1}^{r_i} \varepsilon_{ij}\,\gamma_{ij}\,[\phi^q_i]$

$\llcorner$ intersection number as ch Whitney trick ± 1

Assume that $\sum_i r_i > 1$

$\Rightarrow \exists\, 1 \leq i \leq p_q,\ \exists\, 1 \leq j_1, j_2 \leq r_i$ s.t. $\gamma_{ij_1} = \gamma_{ij_2}$ in π and $\varepsilon_{ij_1} = -\varepsilon_{ij_2}$

$\Rightarrow \gamma_{ij_1} * \gamma_{ij_2}^{-} = f(u_{ij_1}) * v_{ij_1} * v_{ij_2}^{-} * f(u_{ij_2})^{-}$ is nullhomotopic

Whitney trick $\Rightarrow$ isotope to remove x_{ij_1}, x_{ij_2}

$\rightsquigarrow \sum_i r_i = 1$ $\square$

Modification Lemma

Lemma

Let $f\colon S^q \to \partial_1^\circ W_q$ be an embedding and $x_j \in \mathbb{Z}\pi$ $(j = 1, \ldots, p_{q+1})$. Then there exists an embedding $g\colon S^q \to \partial_1^\circ W_q$ such that:

- *f and g are isotopic in $\partial_1 W_{q+1}$*
- *For a lift $\widetilde{f}\colon S^q \to \widetilde{W}_q$ one can find a lift $\widetilde{g}\colon S^q \to \widetilde{W}_q$ such that*

$$[\widetilde{g}] = [\widetilde{f}] + \underbrace{\sum_{j=1}^{p_{q+1}} x_j \cdot d_{q+1}[\phi_j^{q+1}]}_{\in \operatorname{im}(d_{q+1})} \tag{5}$$

in $C_q(\widetilde{W}, \widetilde{\partial_0 W})$.

Only construct g st. $[\tilde{g}] = [\tilde{f}] \pm x \, d_{q+1} [\phi_i^{q+1}]$

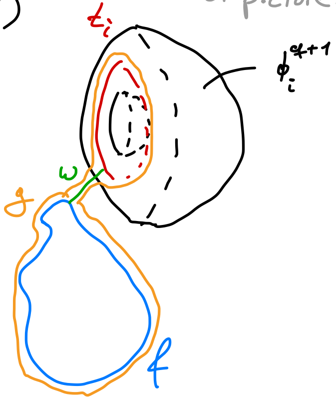
Modification Lemma

Consider $t_i : S^q \cong S^q \times z \subseteq S^q \times S^{n-q-2} \subseteq \partial(\phi_i^{q+1})$

$g = f \#_\omega t_i$

- g istopic to f in $\partial_1 \omega_{q+1}$
- $[\tilde{g}] = [\tilde{f}] + \tilde{\gamma}\, d_{q+1}([\phi_i^{q+1}])$

correct picture:



Dual Handelbody Decomposition

Suppose that

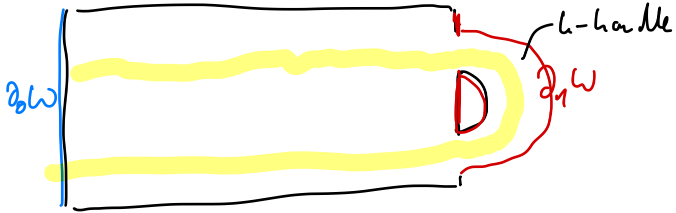
ϕ_i^k

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n). \tag{6}$$

Then there exists a handlebody decomposition relative $\partial_1 W$ of the form

ψ_i^{n-k}

$$W \cong \partial_1 W \times [0,1] + \sum_{i=1}^{p_n} (\psi_i^0) + \cdots + \sum_{i=1}^{p_0} (\psi_i^n). \tag{7}$$



$W \setminus \bullet \cong \partial_1 W \times I$

$\bullet = (n-k)$-handle

Normal Form Lemma

Lemma

Let $(W, \partial_0 W, \partial_1 W)$ be a h-cobordism of dimension $n \geq 6$ and $2 \leq q \leq n-3$. Then W has a handlebody decomposition with handles only of index q and $(q+1)$:

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) \tag{8}$$

<u>Proof</u> No 0,1 – handles, $p_0 = p_1 = 0$

Want $p_r = 0$ for $r < q$

Induction: Assume $p_r = 0$ $(r < k)$, $p_k > 1$

Normal Form Lemma

$W = \partial_0 W \times I + \sum_i \phi_i^k + \dots + \sum_i \phi_i^n$

Fix trivial $\overline{\varphi}^{k+1}: S^k \times D^{n-k-1} \to \partial_1^\circ W_k$

No $(k-1)$-handles $\Rightarrow$ $d_{k+1}: C_{k+1}(\tilde{W}, \partial_0\tilde{W}) \to C_k(\tilde{W}, \partial_0\tilde{W})$

surjective

$[\phi_1^k] = \sum_{i=1}^{p_{k+1}} x_i \, d_{k+1}[\phi_i^{k+1}]$ for $x_1, \dots, x_{p_{k+1}} \in \mathbb{Z}\pi$

Modification-Lemma: $\exists$ $\varphi^{k+1}: S^k \times D^{n-k-1} \to \partial_1^\circ W_k$ st.

- $\overline{\varphi}^{k+1}, \varphi^{k+1}$ are isotopic in $\partial_1 W_{k+1}$
- $[\widetilde{\varphi^{k+1}|_{S^k \times 0}}] = [\underbrace{\widetilde{\overline{\varphi}^{k+1}|_{S^k \times 0}}}_{=0}] + \sum_i x_i d_{k+1}[\phi_i^{k+1}] = [\phi_1^k]$

Normal Form Lemma

Homology + Elimination Lemma $\Rightarrow$ replace ϕ_1^{q} by some ψ^{q+2}

$\rightsquigarrow \quad W \cong \partial_0 W \times I + \sum_{i=1}^{p_q} \phi_i^{q} + \ldots + \sum_{i=1}^{p_n} \phi_i^{n}$

Dual $\rightsquigarrow \quad W = \partial_1 W \times I + \sum_i \psi_i^{0} + \ldots + \sum_i \psi_i^{n-q}$

$\rightsquigarrow \quad W = \partial_1 W \times I + \sum \overline{\psi}_i^{\,n-q-1} + \sum \overline{\psi}_i^{\,n-q}$

Dual $\rightsquigarrow \quad W = \partial_0 W \times I + \sum \overline{\phi}_i^{\,q} + \sum_i \overline{\phi}_i^{\,q+1}$

$\square$

The Normal Form Lemma gives us a decomposition

$$W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} \phi_i^q + \sum_{i=1}^{p_{q+1}} \phi_i^{q+1} \tag{9}$$

Then $d_{q+1}\colon C_{q+1}(\widetilde{W}, \widetilde{\partial_0 W}) \to C_q(\widetilde{W}, \widetilde{\partial_0 W})$ is an isomorphism. The invertible matrix representing this isomorphism with respect to the $\mathbb{Z}\pi$-bases $([\phi_i^{q+1}])_{1\le i\le p_{q+1}}$ and $([\phi_i^q])_{1\le i\le p_q}$ is called the **representative matrix**.

- $C_{q+2}(\tilde{W}, \widetilde{\partial_0 W}) = C_{q-1}(\tilde{W}, \widetilde{\partial_0 W}) = 0$
- $p_q = p_{q+1}$

Whitehead Group

Define the **Whitehead group** $Wh(\pi)$ to be set of all invertible $\mathbb{Z}\pi$-matrices (with arbitrary dimension) modulo the following relations: A and B are equivalent if B is obtained from A by a sequence of the following operations:

1. Add x times the k-th row of A to the m-th row of A where $x \in \mathbb{Z}\pi$ and $k \neq m$
2. $B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$
3. $A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$
4. Multiply the m-th row of A with $\pm\gamma$ for $\gamma \in \pi$
5. Interchange two rows or columns of A.

- $Wh(0) = 0$
- $Wh(\pi)$ is abelian
- $[A] \cdot [B] := \left[\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \right]$

Modifying the Handlebody Decomposition

in normal form

Lemma

Suppose A is the representative matrix of a handlebody decomposition of W and $B \in GL(k, \mathbb{Z}\pi)$ such that $[A] = [B]$ in $Wh(\pi)$. Then W has a handlebody decomposition with representative matrix B.

Only need to show:

For each operation 1 - 5 we can modify the handlebody decomposition

Modifying the Handlebody Decomposition

1. Add x times the k-th row of A to the m-th row of A where $x \in \mathbb{Z}\pi$ and $k \neq m$

   Change ϕ_m^{q+1} s.t. $[\phi_m^{q+1}|_{S^k \times 0}] = [\phi_m^{q+1}|_{S^k \times 0}] + x\, d_{q+1} [\phi_k^{q+1}]$

2. $B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ Attach a trivial handle + cancel it

3. $A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ Homology lemma + Cancelation Lemma

4. Multiply the m-th row of A with $\pm\gamma$ for $\gamma \in \pi$

   Choose a different basepoint of Cft

   Change orientation of handle

5. Interchange two rows or columns of A.

   Change order of handles

Theorem

1. *Let $(W, \partial_0 W, \partial_1 W)$ be a h-cobordism of dimension $n \geq 6$. If A is the representative matrix of a handlebody decomposition of W such that $[A] = 0$ in $Wh(\pi)$, then W is trivial relative $\partial_0 W$.*
2. *Let M_0 be a closed manifold of dimension $n - 1 \geq 5$ and $u \in Wh(\pi)$. Then there exists a h-cobordism (W, M_0, M_1) with representative matrix A such that $[A] = u$.*

Proof: ① By the previous lemma, W has a handlebody decomposition with no handles,

② Let $A \in SL(k, \mathbb{Z}\pi)$ s.t. $[A] = 0$.

Start with $W' = W_0 \times I + \sum_{i=1}^{k} \phi_i^2$ for disjoint trivial ϕ_i^2's

Then construct ϕ_i^3's s.t.

$$[\phi_i^3|_{S^2\times 0}] = \sum_i a_{ij} [\phi_j^2]$$

$$\Rightarrow W = W' + \sum_{j=1}^{h} \phi_j^3 \quad \text{has representative matrix } A \qquad \square$$

The s-Cobordism Theorem

Theorem

Let M_0 be a closed connected manifold of dimension $n-1 \geq 5$ and $\pi = \pi_1(M_0)$.

1. *A h-cobordism (W, M_0, M_1) is trivial relative M_0 if the Whitehead torsion $\tau(W, M_0) \in Wh(\pi)$ is zero.*
2. *For any $u \in Wh(\pi)$ there exists a h-cobordism (W, M_0, M_1) such that $\tau(W, M_0) = u$.*
3. *The function*

$$(\{h\text{-cobordisms over } M_0\}/\cong) \to Wh(\pi), (W, M_0, M_1) \mapsto \tau(W, M_0) \tag{10}$$

is a bijection.

- $\tau(W, M_0) = [A]$: • doesn't depend on handlebody decomposition
  • doesn't depend on choice of lifts, ...
- $(W, M_0, M_1), (W', M_1, M_2)$ h-cobordisms $\Rightarrow \tau(W \cup_{M_1} W', M_0) = \tau(W, M_0) + \tau(W', M_1)$

References

- Diarmuid Crowley, Wolfgang Lueck, and Tibor Macko, *Surgery theory: Foundations*, Springer, 2019.
- Morris Hirsch, *Differential topology*, Springer-Verlag, 1976.
- John Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, 1965.
- Andrew Mackie-Mason, *The h-cobordism theorem*.