Complex Manifolds and Chern Classes

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Definition (Real Vector Bundles)

An n-dimensional real vector bundle ω over a topological space B consists of

- a topological space E
- a continuous map $\pi : E \to B$
- an n-dimensional real vector space structure on each fiber $F_b = \pi^{-1}(b)$

such that for all $b \in B$ there is an open subset $U \ni b$ and a homeomorphism $E|_U \xrightarrow{\cong} U \times \mathbb{R}^n$ that is fiberwise an $\mathbb{R}$-linear vector space isomorphism.

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Remark

We get a complex vector bundle with $(x + iy) \cdot v = x \cdot v + y \cdot J(v)$ *on each fiber.*

Example

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A manifold together with a given complex structure is called complex manifold.

Remark

Equivalently, a complex manifold is a manifold M such that there is a smooth atlas $\{h_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ (i.e. $U_\alpha \subseteq \mathbb{C}^n$ open and $\{V_\alpha\}_{\alpha \in A}$ an open covering of M) such that

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- For complex dimension ≥ 3, we do not know if there is an almost complex but not complex manifold.

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Remark

The total Chern class $c(\omega) = \sum_{i=0}^\infty c_i(\omega)$ fulfils $c(\omega \oplus \eta) = c(\omega) \smile c(\eta)$.

Construction

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Given a complex vector space

- *choose complex basis* $u_1, \ldots, u_n$
- *get real basis* $u_1, iu_1, \ldots, u_n, iu_n$
- *get orientation, independent of choice of complex basis*

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Similarly, for each complex vector bundle ω*, the underlying real vector bundle* ω_R *has a canonical orientation* $\Rightarrow$ *well defined Euler class* $e(\omega) := e(\omega_R)$.

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Remark

From this, we get a complex inner product
$\langle v, w\rangle = \frac{1}{2}(|v + w|^2 - |v|^2 - |w|^2) + \frac{1}{2}i(|v + iw|^2 - |v|^2 - |iw|^2)$.

- $\langle v, v\rangle = |v| > 0$
- *complex linear in* v*:* $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$, $\langle v + \tilde{v}, w\rangle = \langle v, w\rangle + \langle \tilde{v}, w\rangle$
- *conjugate linear in* w*:* $\langle v, \lambda w\rangle = \overline{\lambda}\langle v, w\rangle$, $\langle v, w + \tilde{w}\rangle = \langle v, w\rangle + \langle v, \tilde{w}\rangle$

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$$\cdots \longrightarrow H^{i-2n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \longrightarrow H^{i-2n+1}(B) \longrightarrow \cdots$$

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and $H^{i-2n}(B) = 0$, $H^{i-2n+1}(B) = 0$, we find that $\pi_0^* : H^i(B) \to H^i(E_0)$ is an isomorphism.

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- Using the previous construction and the isomorphism $\pi_0^* : H^i(B) \to H^i(E_0)$, we define recursively $c_i(\omega) = (\pi_0^*)^{-1} c_i(\omega_0)$.

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Then the conjugate bundle $\overline{\omega}$ is the complex vector bundle with the same underlying real bundle $\overline{\omega}_R = \omega_R$ and the conjugate complex structure $\overline{J} = -J$, i. e. for v in any fiber $\overline{F}_b = F_b$ and any $z \in \mathbb{C}$, we have $z \cdot_{\overline{\omega}} v = \overline{z} \cdot_{\omega} v$.

Example

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Proof: Assume there were an isomorphism. On each fiber, this has to be the reflection in some line. This yields a continuous field of tangent lines, hence a continuous nowhere vanishing vector field on $\mathbb{C}P^1 \simeq S^2$, in contradiction to the hairy ball theorem.

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  By choosing the opposite orientation, the Euler class changes its sign, so we get $c_n(\overline{\omega}) = e(\overline{\omega}) = (-1)^n e(\omega) = (-1)^n c_n(\omega)$.

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  From this, we get our conclusion

$$c_k(\overline{\omega}) = \pi_0^{*-1}(c_k(\overline{\omega}_0)) = \pi_0^{*-1}((-1)^k c_k(\omega_0)) = (-1)^k c_k(\omega).$$

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Remark

If we have a Hermitian metric on ω, *we have an isomorphism*

$$\overline{\omega} \simeq \mathrm{Hom}_{\mathbb{C}}(\omega, \mathbb{C})$$
$$v \mapsto \langle -, v \rangle.$$

Theorem

Consider the tangent bundle τ^n of the projective space $\mathbb{C}P^n$. Its total Chern class is $c(\tau^n) = (1+a)^{n+1}$, with $a \in H^2(\mathbb{C}P^n, \mathbb{Z})$ being a Generator.

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- $\mathrm{Hom}_{\mathbb{C}}(L, L^\perp)$ is a vector space, hence the tangent space at L.
- get a map of vector bundles that is fiberwise an isomorphism and hence an isomorphism of vector bundles

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Proof (Theorem): We have $\tau^n \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \omega^n)$. Adding the one dimensional trivial bundle $\epsilon^1 \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \gamma^1)$, we get

$$\tau^n \oplus \epsilon^1 \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \omega^n \oplus \gamma^1) \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \epsilon^{n+1}) \simeq \bigoplus_{i=1}^{n+1} \mathrm{Hom}_{\mathbb{C}}(\gamma^1, \epsilon^1) \simeq \bigoplus_{i=1}^{n+1} \overline{\gamma}^1.$$

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Therefore, using $a = -c_1(\gamma^1)$, we get

$$c(\tau^n) = c(\tau^n \oplus \epsilon^1) = c(\overline{\gamma}^1)^{n+1} = (1 + c_1(\overline{\gamma}^1))^{n+1} = (1 - c_1(\gamma^1))^{n+1} = (1+a)^{n+1}.$$

- John W. Milnor, James D. Stasheff, *Characteristic Classes*
- Aleksandar Milivojevic, *Examples of almost complex four manifolds with no complex structure*

Thank You for Your attention!

Part of the proof of the linearity of the Hermitian product:

Claim

$\langle v + \tilde{v}, w \rangle = \langle v, w \rangle + \langle \tilde{v}, w \rangle$

Proof: From $|v| = |iv|$, we get $|v|^2 = \Re(v)^2 + \Im(v)^2$ with respect to some complex basis. In a real Euklidean vector space, we have

$$|v + \tilde{v} + w|^2 = v^2 + \tilde{v}^2 + w^2 + 2v\tilde{v} + 2vw + 2\tilde{v}w$$
$$= (v^2 + 2v\tilde{v} + \tilde{v}^2) + (v^2 + 2vw + w^2) + \tilde{v}^2 + 2\tilde{v}w + w^2) - v^2 - \tilde{v}^2 - w^2$$
$$= |v + \tilde{v}|^2 + |v + w|^2 + |\tilde{v} + w|^2 - |v|^2 - |\tilde{v}|^2 - |w|^2$$

In the complex case, we get
$\Re(v+\tilde{v}+w)^2 = \Re(v+\tilde{v})^2 + \Re(v+w)^2 + \Re(\tilde{v}+w)^2 + \Re(v)^2 + \Re(\tilde{v})^2 + \Re(w)^2$
and
$\Im(v+\tilde{v}+w)^2 = \Im(v+\tilde{v})^2 + \Im(v+w)^2 + \Im(\tilde{v}+w)^2 + \Im(v)^2 + \Im(\tilde{v})^2 + \Im(w)^2$
and therefore $|v+\tilde{v}+w|^2 = |v+\tilde{v}|^2 + |v+w|^2 + |\tilde{v}+w|^2 - |v|^2 - |\tilde{v}|^2 - |w|^2$.
From this, we get

$$\begin{aligned}\langle v+\tilde{v}, w\rangle &= \frac{1}{2}\left(|v+\tilde{v}+w|^2 - |v+\tilde{v}|^2 - |w|^2\right) + \frac{1}{2}i\,(\dots)\\ &= \frac{1}{2}\left(|v+w|^2 + |\tilde{v}+w|^2 - |v|^2 - |\tilde{v}|^2 - 2|w|^2\right) + \frac{1}{2}i\,(\dots)\\ &= \langle v, w\rangle + \langle \tilde{v}, w\rangle.\end{aligned}$$