

Complex Manifolds and Chern Classes

Jonathan Pampel

29. April 2020

Definition (Real Vector Bundles)

An n -dimensional real vector bundle ω over a topological space B consists of

- a topological space E
- a continuous map $\pi : E \rightarrow B$
- an n -dimensional real vector space structure on each fiber $F_b = \pi^{-1}(b)$

such that for all $b \in B$ there is an open subset $U \ni b$ and a homeomorphism

$$\begin{array}{ccc} E|_U & \xrightarrow{\cong} & U \times \mathbb{R}^n \\ \pi \searrow & & \swarrow pr_1 \\ & U & \end{array}$$

that is fiberwise an \mathbb{R} -linear vector space isomorphism.

Definition (Complex Vector Bundles)

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Let ω be an $2n$ -dimensional real vector bundle over a space B .

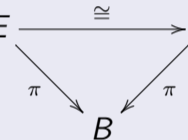
Definition (Complex Structure)

Let ω be an $2n$ -dimensional real vector bundle over a space B . A complex structure on ω is a homomorphism of vector bundles $J : E \xrightarrow{\cong} E$ such that

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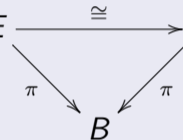
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Remark

We get a complex vector bundle with $(x + iy) \cdot v = x \cdot v + y \cdot J(v)$ on each fiber.

Example

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A manifold together with a given complex structure is called complex manifold.

Remark

Equivalently, a complex manifold is a manifold M such that there is a smooth atlas $\{h_\alpha : U_\alpha \rightarrow V_\alpha\}_{\alpha \in A}$ (i.e. $U_\alpha \subseteq \mathbb{C}^n$ open and $\{V_\alpha\}_{\alpha \in A}$ an open covering of M) such that

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- For complex dimension ≥ 3 , we do not know if there is an almost complex but not complex manifold.

Chern Classes

Axiomatic Definition

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Remark

The total Chern class $c(\omega) = \sum_{i=0}^{\infty} c_i(\omega)$ fulfils $c(\omega \oplus \eta) = c(\omega) \smile c(\eta)$.

Construction

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Given a complex vector space

- *choose complex basis u_1, \dots, u_n*
- *get real basis $u_1, iu_1, \dots, u_n, iu_n$*
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Similarly, for each complex vector bundle ω , the underlying real vector bundle ω_R has a canonical orientation \Rightarrow well defined Euler class $e(\omega) := e(\omega_R)$.

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Remark

From this, we get a complex inner product

$$\langle v, w \rangle = \frac{1}{2}(|v + w|^2 - |v|^2 - |w|^2) + \frac{1}{2}i(|v + iw|^2 - |v|^2 - |iw|^2).$$

- $\langle v, v \rangle = |v| > 0$
- *complex linear in v* : $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$, $\langle v + \tilde{v}, w \rangle = \langle v, w \rangle + \langle \tilde{v}, w \rangle$
- *conjugate linear in w* : $\langle v, \lambda w \rangle = \bar{\lambda} \langle v, w \rangle$, $\langle v, w + \tilde{w} \rangle = \langle v, w \rangle + \langle v, \tilde{w} \rangle$

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$$\dots \longrightarrow H^{i-2n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \longrightarrow H^{i-2n+1}(B) \longrightarrow \dots$$

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and $H^{i-2n}(B) = 0$, $H^{i-2n+1}(B) = 0$, we find that $\pi_0^* : H^i(B) \rightarrow H^i(E_0)$ is an isomorphism.

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- Using the previous construction and the isomorphism $\pi_0^* : H^i(B) \rightarrow H^i(E_0)$, we define recursively $c_i(\omega) = (\pi_0^*)^{-1} c_i(\omega_0)$.

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Then the conjugate bundle $\bar{\omega}$ is the complex vector bundle with the same underlying real bundle $\bar{\omega}_R = \omega_R$ and the conjugate complex structure $\bar{J} = -J$, i. e. for v in any fiber $\bar{F}_b = F_b$ and any $z \in \mathbb{C}$, we have $z \cdot_{\bar{\omega}} v = \bar{z} \cdot_{\omega} v$.

Example

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Proof: Assume there were an isomorphism. On each fiber, this has to be the reflection in some line. This yields a continuous field of tangent lines, hence a continuous nowhere vanishing vector field on $\mathbb{C}P^1 \simeq S^2$, in contradiction to the hairy ball theorem.

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By choosing the opposite orientation, the Euler class changes its sign, so we get $c_n(\bar{\omega}) = e(\bar{\omega}) = (-1)^n e(\omega) = (-1)^n c_n(\omega)$.

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From this, we get our conclusion

$$c_k(\bar{\omega}) = \pi_0^{*-1}(c_k(\bar{\omega}_0)) = \pi_0^{*-1}((-1)^k c_k(\omega_0)) = (-1)^k c_k(\omega).$$

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Remark

If we have a Hermitian metric on ω , we have an isomorphism

$$\bar{\omega} \simeq \text{Hom}_{\mathbb{C}}(\omega, \mathbb{C})$$

$$v \mapsto \langle -, v \rangle.$$

Theorem

Consider the tangent bundle τ^n of the projective space $\mathbb{C}P^n$. Its total Chern class is $c(\tau^n) = (1 + a)^{n+1}$, with $a \in H^2(\mathbb{C}P^n, \mathbb{Z})$ being a Generator.

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- $\text{Hom}_{\mathbb{C}}(L, L^\perp)$ is a vector space, hence the tangent space at L .
- get a map of vector bundles that is fiberwise an isomorphism and hence an isomorphism of vector bundles

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Proof (Theorem): We have $\tau^n \simeq \text{Hom}_{\mathbb{C}}(\gamma^1, \omega^n)$. Adding the one dimensional trivial bundle $\epsilon^1 \simeq \text{Hom}_{\mathbb{C}}(\gamma^1, \gamma^1)$, we get

$$\tau^n \oplus \epsilon^1 \simeq \text{Hom}_{\mathbb{C}}(\gamma^1, \omega^n \oplus \gamma^1) \simeq \text{Hom}_{\mathbb{C}}(\gamma^1, \epsilon^{n+1}) \simeq \bigoplus_{i=1}^{n+1} \text{Hom}_{\mathbb{C}}(\gamma^1, \epsilon^1) \simeq \bigoplus_{i=1}^{n+1} \bar{\gamma}^1.$$

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Therefore, using $a = -c_1(\gamma^1)$, we get

$$c(\tau^n) = c(\tau^n \oplus \epsilon^1) = c(\bar{\gamma}^1)^{n+1} = (1 + c_1(\bar{\gamma}^1))^{n+1} = (1 - c_1(\gamma^1))^{n+1} = (1 + a)^{n+1}.$$

-  John W. Milnor, James D. Stasheff, *Characteristic Classes*
-  Aleksandar Milivojevic, *Examples of almost complex four manifolds with no complex structure*

Thank You for Your attention!

Part of the proof of the linearity of the Hermitian product:

Claim

$$\langle v + \tilde{v}, w \rangle = \langle v, w \rangle + \langle \tilde{v}, w \rangle$$

Proof: From $|v| = |iv|$, we get $|v|^2 = \Re(v)^2 + \Im(v)^2$ with respect to some complex basis. In a real Euklidean vector space, we have

$$\begin{aligned} |v + \tilde{v} + w|^2 &= v^2 + \tilde{v}^2 + w^2 + 2v\tilde{v} + 2vw + 2\tilde{v}w \\ &= (v^2 + 2v\tilde{v} + \tilde{v}^2) + (v^2 + 2vw + w^2) + (\tilde{v}^2 + 2\tilde{v}w + w^2) - v^2 - \tilde{v}^2 - w^2 \\ &= |v + \tilde{v}|^2 + |v + w|^2 + |\tilde{v} + w|^2 - |v|^2 - |\tilde{v}|^2 - |w|^2 \end{aligned}$$

In the complex case, we get

$$\Re(v + \tilde{v} + w)^2 = \Re(v + \tilde{v})^2 + \Re(v + w)^2 + \Re(\tilde{v} + w)^2 + \Re(v)^2 + \Re(\tilde{v})^2 + \Re(w)^2$$

and

$$\Im(v + \tilde{v} + w)^2 = \Im(v + \tilde{v})^2 + \Im(v + w)^2 + \Im(\tilde{v} + w)^2 + \Im(v)^2 + \Im(\tilde{v})^2 + \Im(w)^2$$

$$\text{and therefore } |v + \tilde{v} + w|^2 = |v + \tilde{v}|^2 + |v + w|^2 + |\tilde{v} + w|^2 - |v|^2 - |\tilde{v}|^2 - |w|^2.$$

From this, we get

$$\begin{aligned}\langle v + \tilde{v}, w \rangle &= \frac{1}{2} \left(|v + \tilde{v} + w|^2 - |v + \tilde{v}|^2 - |w|^2 \right) + \frac{1}{2}i(\dots) \\ &= \frac{1}{2} \left(|v + w|^2 + |\tilde{v} + w|^2 - |v|^2 - |\tilde{v}|^2 - 2|w|^2 \right) + \frac{1}{2}i(\dots) \\ &= \langle v, w \rangle + \langle \tilde{v}, w \rangle.\end{aligned}$$