Exercises for Algebraic Topology I – Sheet 8

Uni Bonn, WS 2018/19

Aufgabe 29. Let (X, A) be a relative CW-complex and Y be a space. Decide which of the following assertions are true in general:

(a) Every map $A \to Y$ extends to a map $X_0 \to Y$;

(b) Consider maps $f, g\colon X_0 \to Y$ with $f|_A = g|_A$. They are homotopic relative A if and only if there is a natural equivalence $\Pi(f) \xrightarrow{\cong} \Pi(g)$ of functors from $\Pi(X_0)$ to $\Pi(Y)$;

(c) Consider a map $f\colon X_0 \to Y$. It has an extension $F\colon X_1 \to Y$, if and only if there is a functor $\phi\colon \Pi(X_1) \to \Pi(Y)$ such that for the inclusion $i_0\colon X_0 \to X_1$ the functors $\phi \circ \Pi(i_0)$ and $\Pi(f)$ from $\Pi(X_0)$ to $\Pi(Y)$ are naturally equivalent;

(d) Let $f_i\colon X_1 \to Y$ be maps for $i = 0, 1$ and $h\colon X_0 \times [0, 1] \to Y$ be a homotopy $f_0|_{X_0} \simeq f_1|_{X_0}$. Let $F\colon X_0 \times [0, 1] \cup X_1 \times \{0, 1\} \to Y$ be the map given by f_0, f_1 and h. Then there exists a homotopy $H\colon f_0 \simeq f_1$ extending h if and only if there is a functor $\psi\colon \Pi(X_1 \times [0, 1]) \to \Pi(Y)$ such that $\psi \circ \Pi(j)$ and $\Pi(F)$ are naturally equivalent for the inclusion $j\colon X_0 \times [0, 1] \cup X_1 \times \{0, 1\} \to X_1 \times [0, 1]$.

Aufgabe 30. Let X be a CW-complex. Prove that the following assertions are equivalent:

(a) X is an Eilenberg Mac-Lane space of type 1, i.e., X is path connected and $\pi_n(X, x) = 0$ for some $x \in X$ and $n \geq 2$;

(b) X is path connected and the universal covering is contractible;

(c) X is homotopy equivalent to BG for some discrete group G;

(d) X is homotopy equivalent to $B\pi_1(X, x)$ for some $x \in X$;

(e) If Y is a simply connected CW-complex, then there is up to homotopy precisely one map from Y to X.

handover on Wednesday, 04.12 between 11 and 12:20 in room 4.020 (Florian's office)

Aufgabe 31. Let Y be a connected CW-complex. Prove or disprove that it is an Eilenberg-MacLane space of type 1 if and only if for every CW-complex X the map

$$[X, Y] \to [\Pi(X), \Pi(Y)], \quad [f] \mapsto [\Pi(f)]$$

is bijective, where $[\Pi(X), \Pi(Y)]$ denotes the natural equivalence classes of functors from $\Pi(X)$ to $\Pi(Y)$.

Aufgabe 32. Let G be an abelian group. Construct for every CW-complex X a bijection, natural in X

$$[X, BG] \xrightarrow{\cong} H^1(X; G).$$