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Summer term 2019

Exam - Topology II

September 27th 2019

Name: ____________________

Immatriculation no.: ____________________

Points:

1	2	3	4	5	Σ
					/50

Grade: ____________________

Problem 1 (10 points)**.** *Let M and N be connected, closed, d-dimensional, oriented manifolds, and $f\colon M \to N$ a map of degree 1.*

Show that f induces a surjection on fundamental groups (for all basepoints).

Hint: Consider covering spaces.

Proof. Let $p : \hat{N} \to N$ be the covering space corresponding to the image of $f_*\colon \pi_1(M, m) \to \pi_1(N, f(m))$ for some $m \in M$. The number of sheets of this covering space is equal to the index of $\mathrm{im}(f_*) \subseteq \pi_1(N, f(m))$, so the surjectivity of f_* is equivalent to p being single sheeted. Now, f tautologically lifts to a map $\hat{f} : M \to \hat{N}$ over p by the lifting criterion for maps. If the covering had an infinite number of sheets then $\hat{N}$ is a connected, non-compact manifold and thus $H_d(\hat{N}) = 0$. In particular, $f_*[M] = p_*\hat{f}_*[M] = 0$, forcing $\deg(f) = 0$ in contradiction to the assumption. Thus the covering has finitely many sheets and by the local formula for mapping degrees, the number of sheets is $\deg(p)$, upon giving $\hat{N}$ the orientation induced from N along p. But then

$$1 = \deg(f) = \deg(p) \cdot \deg(\hat{f})$$

forcing $1 = \deg(p)$ as desired. $\square$

Problem 2 (10 points)**.** *Let M be a connected, closed, non-orientable, 3-manifold. Show that $H_1(M;\mathbb{Q}) \neq 0$.*

Proof no. 1. Since M has odd dimension its Euler characteristic vanishes by Poincaré duality for (co)homology with $\mathbb{Z}/2$-coefficients; recall that the Euler characteristic can be computed with any field coefficients. Furthermore, as M is non-orientable $H_3(M;\mathbb{Q}) = 0$. But then if $H_1(M;\mathbb{Q}) = 0$, we would find $\chi(M) = 1 + \dim_{\mathbb{Q}} H_2(M;\mathbb{Q}) > 0$. □

A little more brute force is:

Proof no. 2. The integral homology of M is finitely generated by a result of the lecture. From the universal coefficient theorem with base $\mathbb{Z}$ we then find

$$\begin{aligned}
\dim_{\mathbb{Z}/2} H_1(M;\mathbb{Z}/2) &= \dim_{\mathbb{Q}} H_1(M;\mathbb{Q}) + \dim_{\mathbb{Z}/2} H_1(M;\mathbb{Z})/2 \\
\dim_{\mathbb{Z}/2} H_2(M;\mathbb{Z}/2) &= \dim_{\mathbb{Q}} H_2(M;\mathbb{Q}) + \dim_{\mathbb{Z}/2} H_2(M;\mathbb{Z})/2 + \dim_{\mathbb{Z}/2} H_1(M;\mathbb{Z})/2 \\
\dim_{\mathbb{Z}/2} H_3(M;\mathbb{Z}/2) &= \dim_{\mathbb{Q}} H_3(M;\mathbb{Q}) + \dim_{\mathbb{Z}/2} H_2(M;\mathbb{Z})/2
\end{aligned}$$

But Poincaré duality, the universal coefficient theorem with base $\mathbb{Z}/2$ and non-orientability imply

$$\dim_{\mathbb{Z}/2} H_3(M;\mathbb{Z}/2) = 1, \quad \dim_{\mathbb{Q}} H_3(M;\mathbb{Q}) = 0$$

$$\text{and} \quad \dim_{\mathbb{Z}/2} H_1(M;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^2(M;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H_2(M;\mathbb{Z}/2),$$

so

$$\dim_{\mathbb{Z}/2} H_2(M;\mathbb{Z})/2 = 1$$

and finally

$$\dim_{\mathbb{Q}} H_1(M;\mathbb{Q}) = \dim_{\mathbb{Q}} H_2(M;\mathbb{Q}) + 1 > 0$$

as desired. □

Problem 3 (10 points)**.** *Consider the counit* $\mathrm{ev}_X \colon |\mathrm{Sing}X| \to X$ *of the adjunction*

$$|-|\colon \mathrm{Top} \longleftrightarrow \mathrm{sSet}\colon \mathrm{Sing}.$$

Show that ev_X *is a homotopy equivalence if and only if the functor*

$$[X,-]\colon \mathrm{Top} \to \mathrm{Set}$$

sends weak homotopy equivalences to bijections.

Recall that $[X,Y]$ denotes the set of homotopy classes of maps $X \to Y$. Furthermore, recall that ev_X always is a weak homotopy equivalence.

Proof. ($\Rightarrow$) If ev_X is a homotopy equivalence, then it induces a natural isomorphism $[X,-] \Rightarrow [|\mathrm{Sing}X|,-]$. But $|\mathrm{Sing}X|$ admits a canonical cell structure, so by Whitehead's theorem the right hand functor sends weak homotopy equivalences to bijections.

($\Leftarrow$) Since ev_X is a weak homotopy equivalence, so the map $(\mathrm{ev}_X)_* : [X, |\mathrm{Sing}X|] \to [X,X]$ is bijective by assumption. Therefore, there is a unique homotopy class of maps $f : X \to Y$ such that $\mathrm{ev}_X \circ f \simeq \mathrm{id}_X$. To see that also $f \circ \mathrm{ev}_X$ is homotopic to the identity of $|\mathrm{Sing}X|$, consider

$$(\mathrm{ev}_X)_* : [|\mathrm{Sing}X|, |\mathrm{Sing}X|] \to [|\mathrm{Sing}X|, X].$$

(note the different meaning of $(\mathrm{ev}_X)_*$ compared to the first line). As $|\mathrm{Sing}X|$ is a cell complex $[|\mathrm{Sing}X|,-]$ sends weak homotopy equivalences to bijections (again by Whitehead's theorem), so the above map is also bijective, in particular injective. Moreover, we see that

$$(\mathrm{ev}_X)_*[f \circ \mathrm{ev}_X] = [\mathrm{ev}_X \circ f \circ \mathrm{ev}_X] = [\mathrm{ev}_X] = (\mathrm{ev}_X)_*[\mathrm{id}_{|\mathrm{Sing}X|}]$$

yielding the claim.

□

Problem 4 (10 points)**.** *Let W, V be connected, compact manifolds of the same dimension with partitions $\partial W = \partial_0 W \sqcup \partial_1 W$ and $\partial V = \partial_0 V \sqcup \partial_1 V$ into connected components. Let $\phi\colon \partial_1 W \to \partial_0 V$ be a homeomorphism. Assume that $(W, \partial_0 W)$ and $(V, \partial_0 V)$ are k-connected.*

Show that $(W \cup_\phi V, \partial_0 W)$ is k-connected.

You may use that $(W \cup_\phi V, W, V)$ is an excisive triad.

Proof. By assumption the pair $(V, \partial_0 V)$ is k-connected and the pair $(W, \partial 1 W)$ is 0-connected. Thus by the Blakers-Massey theorem the map

$$\pi_i(V, \partial_0 V) \longrightarrow \pi_i(W \cup_\phi V, W)$$

is a surjection for $i \leq k$, so also $(W \cup_\phi V, W)$ is k-connected. But then the triple sequence of $(W \cup_\phi V, W, \partial_0 W)$ implies that the map

$$\pi_i(W, \partial_0 W) \longrightarrow \pi_i(W \cup_\phi V, \partial_0 W)$$

is a surjection for $i \leq k$ as well. Since the source vanishes in that range of degrees we are done. $\square$

Problem 5 (10 points). *Let* $\mathrm{Gr}_2(\mathbb{R}^4)$ *denote the Grassmannian of* 2*-planes in* $\mathbb{R}^4$*.*

Decompose $\pi_2(\mathrm{Gr}_2(\mathbb{R}^4), g)$ *into cyclic groups for* $g \in \mathrm{Gr}_2(\mathbb{R}^4)$*.*

Recall $\mathrm{Gr}_k(\mathbb{R}^n) \cong \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$ and how we computed $\pi_2(\mathrm{Gr}_2(\mathbb{R}^n)) \cong \mathbb{Z}$ for all other $n \geq 3$.

Proof. Since Grassmannians are path connected, it suffices to treat $g = \mathbb{R}^2 \times 0$. Consider the bundle $\mathrm{O}(4) \to \mathrm{Gr}_2(\mathbb{R}^4)$ with typical fibre $\mathrm{O}(2) \times \mathrm{O}(2)$. For any lift $M \in \mathrm{O}(4)$ of g, its long exact sequence in homotopy groups reads

$$\pi_2(\mathrm{O}(4), e) \longrightarrow \pi_2(\mathrm{Gr}_2(\mathbb{R}^4), \mathbb{R}^2 \times 0) \longrightarrow \pi_1(\mathrm{O}(2) \times \mathrm{O}(2), e) \longrightarrow \pi_1(\mathrm{O}(4), e).$$

In the lecture we showed that the first term vanishes (for example since $\mathrm{SO}(4)$ has universal cover $S^3 \times S^3$), the third term is $\mathbb{Z} \times \mathbb{Z}$, since $\mathrm{O}(2) \cong S^1 + S^1$, and the final term is $\mathbb{Z}/2$. This already forces $\pi_2 \mathrm{Gr}_2(\mathbb{R}^4) = \mathbb{Z} \times \mathbb{Z}$ without determining the map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2$ more explicitely. But we also showed in the lecture that the inclusions $\mathrm{O}(n) \to \mathrm{O}(n+1)$ are surjections on π_1 for $n \geq 1$ (and isomorphisms for $n \geq 2$), so the map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2$ identifies with the addition on $\mathbb{Z}$ reduced modulo 2. $\square$

Bonus problem (3 points)**.** *Determine the action of* $\pi_1(\mathrm{Gr}_2(\mathbb{R}^4), g)$ *on* $\pi_2(\mathrm{Gr}_2(\mathbb{R}^4), g)$.

Hint: Recall that a generator for $\pi_1(\mathrm{Gr}_2(\mathbb{R}^n), \mathbb{R}^2 \times 0) = \mathbb{Z}/2$ is given by rotating $\mathbb{R}^2 \times 0$ about an angle of π around the first axis in $\mathbb{R}^3 \times 0$.

Proof. It is readily checked straight from the definitions that for any fibration $E \to B$, with simple fibre F over $b \in B$, the boundary map

$$\pi_n(B, b) \longrightarrow \pi_{n-1}(F)$$

is equivariant, if we let $[w] \in \pi_1(B, b)$ act on $\pi_{n-1}(F)$ by the fibre transport along w; recall that this the homotopy class of maps $F \to F$, determined by taking the end of any lift in

$$\begin{tikzcd} F \arrow[r, "\mathrm{incl}"] \arrow[d] & E \arrow[d] \\ F \times I \arrow[r, "w \circ \mathrm{pr}_2"] \arrow[ur, dashed] & B. \end{tikzcd}$$

(Note that the fibre transport is only well-defined up to unbased homotopy, so in order to obtain a well-defined map one needs the simplicity of F).

In the present situation, the fibre is $\mathrm{O}(2) \times \mathrm{O}(2)$, certainly simple as a topological group. Furthermore, using the description of the generator of $\pi_1(\mathrm{Gr}_2(\mathbb{R}^4), \mathbb{R}^2 \times 0)$, the fibre transport is given by the multiplication on $\mathrm{O}(2) \times \mathrm{O}(2)$ with the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in $\mathrm{O}(4)$ describing the rotation. As this matrix has determinant 1, it preserves $\mathrm{SO}(2) \times \mathrm{SO}(2)$ and under the diffeomorphism with $S^1 \times S^1$ corresponds to complex conjugation in both factors. Thus it induces the negative of the identity on π_1. Since the map

$$\pi_2(\mathrm{Gr}_2(\mathbb{R}^4), g) \longrightarrow \pi_1(\mathrm{O}(2) \times \mathrm{O}(2))$$

is injective, the same is true in the source. $\square$