1. Groups as metric spaces

Definition 1.1. Let G be a finitely generated group with generating system S. The length function l_S associated to S is defined as follows:

$$l_S(g) := \min\{n \in \mathbb{N} \mid \exists s_1, .., s_n \in S \cup S^{-1} : g = s_1 \cdot \ldots \cdot s_n\}$$

The word metric d_S is defined by $d(g, h) := l_S(g^{-1}h)$.

Lemma 1.2. *Let G be a group with finite generating system S. Then the word metric d_S is left-invariant, i.e. $d(hg_1, hg_2) = d(g_1, g_2)$, and proper, i.e. balls are proper (and thus finite since the space is discrete).*

Proof. Left-invariance is obvious from the definition. For the second it suffices to show that for each n there are only finitely many g with $l_S(g) = d(e, g) \leq n$. This follows from the fact that S is finite and hence there are only finitely many words of length at most n in $S \cup S^{-1}$. □

Definition 1.3. Two maps $f, f' : X \to Y$ are close if there exists $c > 0$ such that for all $x \in X$ we have $d(f(x), f'(x)) \leq c$.

Definition 1.4. Let X, Y be metric spaces, a map $f : X \to Y$ is a quasi-isometric embedding if there exist constants $A \geq 1 B \geq 0$ such that for all $x, y \in X$

$$\tfrac{1}{A} d_X(x, y) - B \leq d_Y(f(x), f(y)) < A d_X(x, y) + B.$$

It is a quasi-isometry if there exists a quasi-isometric embedding $g : Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y.

X, Y are quasi-isometric if there exists a quasi-isometry $f : X \to Y$.

Remark 1.5. Recall that an isometric embedding $f : X \to Y$ is a map with $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Lemma 1.6. *Clara Prop 5.1.10*

Lemma 1.7. *Let G be a group and let S, T be finite generated systems. Then (G, d_S) and (G, d_T) are quasi-isometric.*

Proof. By symmetry it suffices to show that there is $A \geq 1$ with $d_S(g, h) \leq A d_T(g, h)$. Let $A := \max\{l_T(s) \mid s \in S\}$. For $g \in G$ with $l_S(g) = n$ let $s_1, .., s_n \in S \cup S^{-1}$ be such that $g = s_1 \cdot .. \cdot s_n$. Then each s_i can be expressed as a word of length at most A in $T \cup T^{-1}$. Hence $l_T(g) \leq An$. It follows that

$$d_S(g, h) = l_S(g^{-1}h) \leq A l_T(g^{-1}h) = d_T(g, h)$$

as needed. □

Hence the word metric allows us to view G as a metric space even without specifying a generating system as long as we only consider properties that are invariant under quasi-isometries.

Definition 1.8. Let G be a group with finite generating set S. The Cayley graph $C(G, S)$ is the graph with vertex set G and an edge between g, h if and only if $g^{-1}h \in S \cup S^{-1}$.

The inclusion of the vertex set $G \to C(G, S)$ gives a quasi-isometry between (G, d_S) and $C(G, S)$ with the path metric in which every edge has length one. In particular, up to quasi-isometry the Cayley graph with the path metric is independent of the choice of S.

Example 1.9. Let G be a free group, freely generated by S. Then $C(G,S)$ is a tree: Any path without backtracking is a reduced word, hence can't be trivial/a loop.

Theorem 1.10. *Let G be a group and let S be a generating set of G. Then the left translation action on the Cayley graph $C(G,S)$ is free if and only if S does not contain any elements of order 2.*

Proof. The action on the vertices is nothing but the left translation action by G on itself, which is free. It therefore suffices to study under which conditions the action of G on the edges is free: If the action of G on the edges of the Cayley graph $C(G,S)$ is not free, then S contains an element of order 2: Let $g, v, v' \in G$ with $g \neq e$, such that g fixes the edge between v and v'. Then $gv = v'$ and $gv' = v$ hence g^2 is trivial and g has order 2. Conversely, if $s \in S$ has order 2, then s
fixes the edge between e and s. $\square$

Definition 1.11. geodesic space
quasi-geodesic + examples graphs and groups with word metric
Svarc-Milnor Lemma

2. Ends of groups

Definition 2.1. Let X be a connected, locally finite CW complex. If $K \subseteq K'$ are finite subcomplexes, we have the inclusion $X \setminus K' \to X \setminus K$. The *space of ends of* X is

$$\mathrm{Ends}(X) = \lim_{K \subseteq X \ fin.} \pi_0(X \setminus K)$$

with the inverse limit topology. (The system of finite subcomplexes with inclusion is filtered. Hence the dual system of complements is cofiltered, thus this is an inverse system.)

Lemma 2.2. *If $f\colon X \to Y$ is continuous and proper, i.e. inverse images of compact subspaces are compact, then it induces a (continuous) map $f_*\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$.*

Proof. For each $K \subseteq Y$ finite, we have a map $f\colon X \setminus f^{-1}(K) \to Y \setminus K$. This induces a continuous map on π_0 (since both spaces are discrete). Hence we obtain a continuous map

$$f_*\colon \lim_{K \subseteq Y \ fin.} \pi_0(X \setminus f^{-1}(K)) \to \lim_{K \subseteq Y \ fin.} \pi_0(Y \setminus K) = \mathrm{Ends}(Y).$$

The set $\{f^{-1}(K) \mid K \subseteq Y \ fin.\}$ is cofinal in $\{K' \mid K' \subseteq X \ fin.\}$, i.e. for each K' there exists K with $K' \subseteq f^{-1}(K)$ (e.g. $K = f(K')$). Hence the dual system of complements is final. Therefore, $\lim_{K \subseteq Y \ fin.} \pi_0(X \setminus f^{-1}(K)) \cong \mathrm{Ends}(X)$. $\square$

This turns Ends into a functor from connected, countable, locally finite CW complexes with continuous and proper maps to the category of spaces.

Lemma 2.3. *If $H\colon X \times [0,1] \to Y$ is a proper homotopy between f and g, then $f_* = g_*$.*

Proof. Let $i_j\colon X \to X \times [0,1]$ be the inclusion into the $j = 0,1$ coordinate. Since $H \circ i_j = f, g$ is suffices to show that $(i_j)_*$ is independent of j. This follows since $K \times I$ is cofinal in the collection of finite subcomplexes of $X \times I$ and $(i_j)_*\colon \pi_0(X \setminus K) \to \pi_0((X \times I) \setminus (K \times I))$ is independent of j. $\square$

Lemma 2.4. *Let $i\colon X^{(1)} \to X$ be the inclusion of the 1-skeleton, then i_* is a homeomorphism.*

Proof. This follows since $i_*\colon \pi_0(X^{(1)} \setminus (X^{(1)} \cap K)) \to \pi_0(X \setminus K)$ is a bijection for every finite subcomplex $K \subseteq X$. □

Lemma 2.5. *Let $f\colon X \to Y$ be a quasi-isometric embedding between proper, geodesic metric spaces, then we obtain a map $f_*\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$. This is again natural.*

Proof. Let $f\colon X \to Y$ be a quasi-isometry with

$$\tfrac{1}{A} d_R(x,y) - B \leq d_S(f(x), f(y)) \leq A d_R(x,y) + B$$

for all $x, y \in X$. Let $C := A(2B + A + 1)$. Fix $n \in \mathbb{R}$ and a base point $x_0 \in X$.

Let $x, y \in X \setminus B_{Cn+C}(x_0)$ be in the same connected component. Then there is a path $x = y_0, y_1, .., y_k = y$ from x to y with $d(y_i, y_{i+1}) \leq 1$ $d(y_i, x_0) > Cn + C$. Hence

$$d(f(y_i), f(x_0)) \geq \tfrac{1}{A} d(y_i, x_0) - B > \tfrac{1}{A}(Cn + C) - B > n + A + B.$$

Since $d(y_i, y_{i+1}) \leq 1$, $d(f(y_i), f(y_{i+1})) \leq A + B$. Thus there is a geodesic path connecting $f(y_i)$ and $f(y_{i+1})$ of length at most $A + B$. This path then lies in $Y \setminus B_n(f(x_0))$. Hence $f(x)$ and $f(y)$ are in the same connected component of $Y \setminus B_n(f(x_0))$. This gives a map $f_*\colon \pi_0(X \setminus B_{Cn+C}(x_0)) \to \pi_0(Y \setminus B_n(f(x_0)))$. The systems $\{B_{Cn+C}(x_0)\}$ and $\{B_n(f(x_0))\}$ are cofinal in the compact/bounded subspaces of X and Y respectively. Hence $f_*\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ is well-defined. □

Lemma 2.6. *If $g, h\colon X \to Y$ are close quasi-isometric embeddings between proper, geodesic metric spaces. Then $g_* = h_*$.*

Proof. Let $d(f(x), g(x)) \leq C$ for all $x \in X$. For $K \subseteq Y$ bounded let $B_C(K)$ denote the C-neighborhood. Then $f_*, g_*\colon \pi_0(X \setminus f^{-1}(B_C(K))) \to \pi_0(Y \setminus K)$ are equal. This can be seen as follows. For any $x \in X \setminus f^{-1}(B_C(K))$ pick a geodesic between $f(x)$ and $g(x)$. Since it has length at most C and $f(x), g(X)$ are in $Y \setminus B_C(K)$, the geodesic is in $Y \setminus K$. Hence $f(x)$ and $g(x)$ are in the same connected component. □

Corollary 2.7. *If X, Y are quasi-isometric proper, geodesic spaces, then $\mathrm{Ends}(X) \cong \mathrm{Ends}(Y)$.*

Corollary 2.8. *Let G be a group with finite generating system S. Then $\mathrm{Ends}(G) := \mathrm{Ends}(C(G,S))$ is independent of S.*

Corollary 2.9. *If G, H are quasi-isometric groups, then $\mathrm{Ends}(G) \cong \mathrm{Ends}(H)$. For example, $H \leq G$ of finite index.*

Lemma 2.10. *If X is locally finite and connected, then for every finite subcomplex K the set $\pi_0(X \setminus K)$ is finite.*

Proof. We can again assume X is one-dimensional. Removing an edge can only split one connected component into at most two. Removing a vertex can only split a connected component into as many connected components as there were edges attached to the vertex. □

Lemma 2.11. *If X is locally finite and connected, then X is compact if and only if $\mathrm{Ends}(X)$ is empty.*

Proof. $\Rightarrow$ is obvious. If X is non-compact, fix an ascending, cofinal sequence of compact subspaces K_n. Then $\pi_0(X \setminus K_n)$ is non-empty and finite for each n. Then the inverse limit is non-empty since in each step we can pick a point with infinitely many pre-images. □

Definition 2.12. For a finitely generated group G we define $e(G) := |\operatorname{Ends}(G)|$.

Corollary 2.13. *G is finite if and only if $e(G) = 0$.*

Example 2.14. If G is infinite and virtually cyclic, i.e. contains a cyclic subgroup of finite index, then $e(G) = 2$.

Theorem 2.15 (Freudenthal-Hopf)**.** *For a finitely generated group G, $e(G) \in \{0, 1, 2, \infty\}$.*

Proof. Let G be a group with $e(G) > 2$. Then there exists a finite, connected subcomplex K of $X := C(G, S)$ such that $X \setminus K$ has $m \geq 3$ connected components V_i and all components are unbounded. Pick $g \in G$ with $gK \subseteq V_1$. This exists since G is infinite and acts freely on X. Since each gV_i is connected to gK, all but the one containing K are contained in V_1. Hence V_1 has at least 2 ends. The same argument shows that each V_i has at least 2 ends. Hence $e(G) \geq 2m$. Iterating the argument shows that $e(G) \geq 2^k m$ for every k. □

Theorem 2.16. *A group has two ends if and only if it is infinite and virtually cyclic, i.e. it contains $\mathbb{Z}$ as a subgroup of finite index.*

Proof. It is easy to see that $\mathbb{Z}$ has two ends. By Corollary 2.9, every group that contains $\mathbb{Z}$ as a subgroup of finite index has two ends.

The converse is more involved.

Let S be a finite generating set of G. Consider the action of G on $\operatorname{Ends}(C(G, S))$. Since $\operatorname{Ends}(C(G, S))$ consists of two points, the kernel of this action has index at most two. Hence we can pass to the kernel and thus assume that the action is trivial.

There exists a compact subset C of $C(G, S)$ such that the complement has two infinite components W_-, W_+ and no finite components. Since G is infinite and the action of G on $C(G, S)$ is proper, there exists $g \in G$ such that $C \cap gC = \emptyset$. We assume that $gC \subseteq W_+$. Then gW_+ is a proper subspace of W_+ since otherwise g would swap the ends of $C(G, S)$. Hence there is a proper decreasing chain

$$W_+ \supsetneq gW_+ \supsetneq g^2 W_+ \supsetneq \dots$$

and g has infinite order. Similarly,

$$W_- \supsetneq g^{-1} W_- \supsetneq g^{-2} W_- \supsetneq \dots$$

Let D be a compact subset containing C, gC and such that the complement has exactly two components both of which are infinite, i.e.

$$C(G, S) := W_+ \cup D \cup gW_+.$$

Suppose $x \in \bigcap_{i \in \mathbb{N}} g^i W_+$. Then there exists $w_i \in W_+$ with $x = g^i w_i$. This implies $w_i = g^{-i} x$. Since $g^{-i} x$ leaves any compact set it has to converge to an end and by the above it has to converge to the end represented by W_-. But $w_i \in W_+$, a contradiction. Hence $\bigcap_{i \in \mathbb{N}} g^i W_+$ is empty. Similarly, $\bigcap_{i \in \mathbb{N}} g^{-i} W_-$ is empty. We have

$$C(G, S) = g^{-1} W_- \cup \bigcup_{-i \leq j < i} g^j D \cup g^i W_+.$$

Thus,

$$C(G,S) = \bigcup_{j\in\mathbb{Z}} g^j D.$$

Therefore D contains a representative for each coset $\langle g\rangle h$ and $\langle g\rangle$ has finite index in G. □

Theorem 2.17 (Stallings)**.** *Let G be a finitely generated group with $e(G) = \infty$. Then one of the following holds:*

(1) *The group G admits a splitting $G = H *_C K$ as a free product with amalgamation where C is a finite group such that $C \neq H$ and $C \neq K$.*
(2) *The group G is an HNN extension $G = \langle H, t \mid t^{-1}ct = \phi(c)\rangle$ where $C \leq H$ is finite and $\phi\colon C \to H$ is injective.*

We will use the condition $e(G) = \infty$ to show that G acts on a tree with at most two orbits of vertices and one edge orbit. Furthermore, the edge orbit will have finite isotropy. So obtain the theorem from this, we will need some Bass-Serre theory. This will be developed in the next section before we return to the proof.

Corollary 2.18. *Let G be a torsion-free, finitely generated group with $e(G) = \infty$, then G is a free product of two non-trivial groups H, K.*

In particular, any torsion-free, finitely generated group is a finite free product of groups H_i with $e(H_i) = 1$ or $H_i \cong \mathbb{Z}$.

Proof. The finite subgroup C includes in both cases into G. If G is torsion-free, C has to be trivial. Hence $G = H * K$ with H, K non-trivial in the first case and in the second case $G \cong H * \mathbb{Z}$. Here H is non-trivial since otherwise $G \cong \mathbb{Z}$ would have two ends. □

3. Bass-Serre theory

Definition 3.1. A *graph of groups* (G, T) consists of an oriented graph T, a group G_P for every vertex $P \in V(T)$, a group G_y for every edge $y \in E(T)$, together with monomorphisms $\phi_{t,y}\colon G_y \to G_{t(y)}$ and $\phi_{o,y}\colon G_y \to G_{o(y)}$ where $o(y)$ and $t(y)$ denote the starting and ending vertex of y respectively.

Example 3.2. A segment

(an edge y from P to Q) consists of a pair of monomorphisms $\phi_{o,y}\colon G_y \to G_P$ and $\phi_{t,y}\colon G_y \to G_Q$.

Example 3.3. A loop

consists of two monomorphisms from G_y to G_P.

Definition 3.4. For a graph of groups (G, Y) we define the group

$$F(G,Y) := \langle \{G_P, y\} \mid y\phi_{t,y}(a)y^{-1} = \phi_{o,y}(a) \forall y \in E(Y), a \in G_y\rangle.$$

We will see that $G_P \to F(G,Y)$ is injective for every $P \in V(Y)$.

For an edge y we write y^{-1} for the edge with reversed orientation. Let c be a path in Y whose origin is a vertex P_0. We let $y_1^{\epsilon_1}, \ldots, y_n^{\epsilon_n}$ denote the edges of c, where $n = l(c)$, $\epsilon_i \in \{\pm 1\}$ and $\epsilon = -1$ if the path travels trough y_i against the chosen orientation. We put

$$P_i = t(y_i^{\epsilon_i}) = o(y_{i+1}^{\epsilon_{i+1}}).$$

Definition 3.5. A word of type c in $F(G, Y)$ is a pair (c, μ) where $\mu = (r_0, \ldots, r_n)$ is a sequence of elements $r_i \in G_{P_i}$. The element

$$|c, \mu| = r_0 y_0^{\epsilon_0} r_1 \ldots r_{n-1} y_n^{\epsilon_n} r_n \in F(G, Y)$$

is said to be associated with the word (c, μ).

We now give two definitions of the fundamental group of (G, Y).

Definition 3.6. Let P_0 be a vertex of Y. We let $\pi_1(G, Y, P_0)$ be the set of elements of $F(G, Y)$ of the form $|c, \mu|$, where c is a loop at P_0. Concatenation/Reversion of path shows that $\pi_1(G, Y, P_0)$ is a subgroup of $F(G, Y)$. It is called the fundamental group of (G, Y) at P_0. When G is trivial, i.e. $G_P = \{1\}$ for every $P \in V(P)$, the group $\pi_1(G, Y, P_0)$ coincides with the fundamental group (in the usual sense) $\pi_1(Y, P_0)$ of the graph Y at the point P_0.

In the general case, there is a surjective homomorphism $\pi_1(G, Y, P_0) \to \pi_1(Y, P_0)$; its kernel is the normal subgroup of $\pi_1(G, Y, P_0)$ generated by the G_P.

Definition 3.7. Let T be a maximal tree of Y. The fundamental group $\pi_1(G, Y, T)$ of (G, Y) at T is, by definition, the quotient of $F(G, Y)$ by the normal subgroup generated by the edges y of T. In particular we have $\phi_{t,y}(a) = \phi_{o,y}(a)$ for every edge y of T.

Example 3.8. For a segment

we have $\pi_1(G, Y, Y) \cong G_P \cong_{G_y} G_Q$ and for a loop

$\pi_1(G, Y, P)$ is the HNN extension of G_P along $\phi_{t,y}(G_y) \cong \phi_{o,y}(G_y)$.

Proposition 3.9. *Let (G, Y) be a graph of groups, P_0 a vertex and T a maximal tree. The projection $p\colon F(G, Y) \to \pi_1(G, Y, T)$ induces an isomorphism $\pi_1(G, Y, P_0) \to \pi_1(G, Y, T)$.*

Proof. For $P \in V(Y)$ let c be the geodesic path in T from P_0 to P with edges $y_1^{\epsilon_1}, \ldots, y_n^{\epsilon_n}$. Define

$$y_P := y_1^{\epsilon_1} \cdots y_n^{\epsilon_n} \in F(G, Y).$$

For $x \in G_P$ define

$$x' := y_P x y_P^{-1}$$

and for an edge $y \in Y$ define

$$y' := y_{o(y)} y y_{t(y)}^{-1}.$$

The latter is the word belonging to a loop at P_0 in Y that travels through y and only through T otherwise. If $y \in T$, then $y' = 1$ (easy). For $a \in G_y$ we have

$$\begin{aligned} y'\phi_{t,y}(a)'(y')^{-1} &= y_{o(y)}yy_{t(y)}^{-1}y_{t(y)}\phi_{t,y}(a)y_{t(y)}^{-1}y_{t(y)}y^{-1}y_{o(y)}^{-1} \\ &= y_{o(y)}y\phi_{t,y}(a)y^{-1}y_{o(y)}^{-1} \\ &= y_{o(y)}\phi_{o,y}(a)y_{o(y)}^{-1} \\ &= \phi_{o,y}(a)' \end{aligned}$$

Hence $(-)'$ defines a homomorphism

$$f\colon \pi_1(G,Y,T) \to \pi_1(G,Y,P_0).$$

Since y_P comes from a path in T for every $P \in V(Y)$, we have $p \circ f = \mathrm{id}$.

On the other hand, let c be a loop at P_0, with edges $y_0,\dots,y_n$ and vertices $P_j = o(y_{j+1}) = t(y_j)$; let (c,μ), with $\mu = (r_0,\dots,r_n)$, be a word of type c. We have $r_i' = y_{P_i}r_iy_{P_i}^{-1}$ and $y_i' = y_{P_{i-1}}y_iy_{P_i}^{-1}$. Furthermore $y_{P_n} = y_{P_0} = 1$. Hence

$$r_0'y_1'\cdots y_n'r_n' = y_{P_0}r_0y_{P_1}^{-1}y_{P_1}y_1\cdots y_ny_{P_n}^{-1}y_{P_n}r_ny_{P_n}^{-1} = r_0y_1\cdots y_nr_n.$$

We then have $f \circ p = \mathrm{id}$ and the proof is complete. $\square$

Definition 3.10. One says that (c,μ) is reduced if it satisfies the following condition:

(1) If $n = 0$, then $r_0 \neq 1$ and
(2) if $y_i = y_{i+1}, \epsilon_i = -1\epsilon_{i+1}$, then $r_i \notin \phi_{t,y_i^{\epsilon_i}}(G_{y_i^{\epsilon_i}})$.

In particular, every word whose type is a path of length ≥ 1 without backtracking is reduced.

Theorem 3.11 (Serre, Thm 11)**.** *If (c,μ) is a reduced word, the associated element $|c,\mu|$ of $F(G,Y)$ is nontrivial.*

The special case where the length is zero gives:

Corollary 3.12. *The homomorphisms $G_P \to F(G,Y)$ are injective.*

Corollary 3.13. *If (c,μ) is reduced and if $l(c) \geq 1$, then $|c,\mu| \notin G_{P_0}$, where $P_0 = o(c)$.*

Proof. If $|c,\mu| = x \in G_{P_0}$, the reduced word (c,μ') with $\mu' = (x^{-1}r_0,..,r_n)$ would have $|c,\mu'| = 1$. $\square$

Corollary 3.14. *Let T be a maximal tree of Y, and let (c,μ) be a reduced word whose type c is a closed path. Then the image of $|c,\mu|$ in $\pi_1(G,Y,T)$ is non-trivial.*

Proof. Let $P_0 = o(c)$. We have $1 \neq |c,\mu| \in \pi_1(G,Y,P_0)$. The corollary now follows from the statement that the projection

$$\pi_1(G,Y,P_0) \to \pi_1(G,Y,T)$$

is an isomorphism. $\square$

We now want to construct a graph $\widetilde{X} = \widetilde{X}(G,Y,T)$ such that $\pi := \pi_1(G,Y,T)$ acts on $\widetilde{X}$ with $\pi\backslash\widetilde{X} \cong Y$ and "vertex and edge stabilizers corresponding to G_P, G_y". For this we define

$$V(\widetilde{X}) := \coprod_{P \in V(Y)} \pi/G_P, \quad E(\widetilde{X}) = \coprod_{y \in E(Y)} \pi/\phi_{o,y}(G_y).$$

We define a section $Y \to \widetilde{X}$ by sending $P \to [1] \in \pi/G_p$ and $y \to [1] \in \pi/G_y$ and denote the images with $\widetilde{P}$ and $\widetilde{y}$. Then for the stabilizers, we have $\pi_{\widetilde{P}} = G_P$ and $\pi_{\widetilde{y}} = \phi_{o,y}(G_y)$. Define

$$o(g\widetilde{y}) = g\widetilde{o(y)}, \quad t(g\widetilde{y}) = gy\widetilde{t(y)}.$$

We have to check that this is independent of the choice of g: For $h = \phi_{o,y}(a) \in \pi_{\widetilde{y}} = \phi_{o,y}(G_y) \subseteq G_{o(y)}$, we have

$$o(h\widetilde{y}) = \phi_{t,y}(a)\widetilde{o(y)} = \widetilde{o(y)} = o(\widetilde{y})$$

and

$$t(h\widetilde{y}) = hy\widetilde{t(y)} = \phi_{o,y}(a)y\widetilde{t(y)} = y\phi_{t,y}(a)\widetilde{t(y)} = y\widetilde{t(y)} = t(\widetilde{y}).$$

If $y \in T$, then $1 = y \in \pi$. Hence $t(\widetilde{y}) = \widetilde{t(y)}$. So we have a lift $\widetilde{T}$ of T in $\widetilde{X}$.

Theorem 3.15. *Let (G, Y) be a connected graph of groups with at least one edge and let T be a maximal tree of Y. Then the graph $\widetilde{X} = \widetilde{X}(G, Y, T)$ constructed above is a tree.*

Proof. For every $y \in E(Y)$ we have $o(\widetilde{y}) = \widetilde{(o(y))} \in \widetilde{T}$. Hence the subgraph W containing all the $\widetilde{y}$ with $y \in E(Y)$ is connected. By construction $\pi W = \widetilde{X}$. It now suffices to show that there is a generating set S such that $W \cup sW$ is connected for all $s \in S$; since then

$$W \cup s_1W \cup s_1s_2W \cup \ldots \cup s_1 \cdots s_nW$$

is connected for all $s_1, .., s_n \in S \cup S^{-1}$. If $s \in G_P$ for $P \in V(Y)$, then W and sW both contain $\widetilde{P} \in \widetilde{T}$. Similarly,

$$yt(\widetilde{y}) = \widetilde{t(y)} \in yW \cap W.$$

This shows that $\widetilde{X}$ is connected.

To show that X is a tree, it now suffices to prove that X does not contain any closed path of length $n > 0$ without backtracking. Let c be such a path, let $(s_1\widetilde{y}_1^{\epsilon_1}, .., s_n\widetilde{y}_n^{\epsilon_n})$ be the sequence of its edges, and let $(P_0, .., P_n)$ be the sequence of vertices of the projection c of $\widetilde{c}$ in Y; we have $P_0 = P_n$. We have

$$t(s_k\widetilde{y}_k^{\epsilon_k}) = s_ky_k^{(1+\epsilon)/2}\widetilde{t(y_k^{\epsilon_k})}$$

and

$$o(s_k\widetilde{y}_k^{\epsilon_k}) = s_ky_k^{(1-\epsilon)/2}\widetilde{o(y_k^{\epsilon_k})}.$$

Define $q_k := s_ky_k^{(1-\epsilon_k)/2}$, then $q_k^{-1}q_{k+1} = y_k^{\epsilon_k}r_k$ for some $r_k \in G_{P_k}$ since c is a path.

Hence

$$y_1^{\epsilon_1}r_1 \cdots y_n^{\epsilon_n}r_n = 1.$$

To obtain a contradiction, it remains to show that (c, μ) with $\mu = (0, r_1, .., r_n)$ is reduced.

Suppose $y_i = y_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$. Then from

$$s_iy_i^{(1+\epsilon_i)/2}r_i = s_{i+1}y_{i+1}^{(1-\epsilon_{i+1})/2} = s_{i+1}y_i^{(1+\epsilon_i)/2}$$

it follows that

$$r_i = y_i^{-(1+\epsilon_i)/2}s_i^{-1}s_{i+1}y_i^{-(1+\epsilon_i)/2}.$$

We have to show that $r_i \notin \phi_{t,y_i^{\epsilon_i}}(G_{y_i^\epsilon})$. Hence that $s_i^{-1}s_{i+1} \notin \phi_{o,y_i^\epsilon}(G_{y_i^\epsilon})$. But if $s_i^{-1}s_{i+1} \in \phi_{o,y_i^\epsilon}(G_{y_i^\epsilon})$, then $s_i\widetilde{y_i^{\epsilon_i}}$ and $s_{i+1}\widetilde{y_{i+1}^{\epsilon_{i+1}}} = s_{i+1}\widetilde{y_i^{-\epsilon_i}}$ were inverse and the path $\widetilde{c}$ had backtracking. □

Let G be a group which acts without inversion on a connected graph X. We can pick an orientation on X compatible with the G action. We want to show that, if X is tree, then G can be identified with the fundamental group of a certain graph of groups (G, Y), where $Y = G\backslash X$ with the orientation induced from X. We first want to construct (G, Y). Let T be a maximal tree of Y and let $j : T \to X$ be a lift of T. For $y \in Y \setminus T$, pick $jy \in X$ with $o(jy) \in V(jT)$, then $o(jy) = jo(y)$. Choose $\gamma_y \in G$ with $t(jy) = \gamma_y jt(y)$. This is possible since $t(jy)$ and $jt(y)$ both project to $t(y)$ in Y. Put $\gamma_y = 1$ for $y \in T$. Then we have

$$o(jy) = jo(y), \quad t(jy) = \gamma_y jt(y)$$

for all $y \in Y$. Define $G_P := G_{jP}$, where G_{jP} denotes the stabilizer of jP and $G_y := G_{jy}$ with homomorphism $\phi_{o,y} : G_y \to G_{o(y)}$ given by the inclusion, while $\phi_{t,y} : G_y \to G_{t(y)}$ is given by $a \mapsto \gamma_y^{-1}a\gamma_y$. The latter is well defined since $G_{jy} \subseteq G_{\gamma_y jt(y)} = \gamma_y G_{jt(y)}\gamma_y^{-1}$.

Let $\psi : \pi_1(G, Y, T) \to G$ be defined by the inclusions $G_P \to G$ and $\psi(y) = \gamma_y$. Let $\Psi : \widetilde{X}(G, Y, T) \to X$ be defined by

$$\Psi(g\widetilde{P}) = \psi(g)jP, \quad \Psi(g\widetilde{y}) = \psi(g)jy.$$

By construction, Ψ is equivariant with respect to ψ.

We first want to see that ψ and Ψ are surjective. Let W be the smallest subgraph of X containing jy for all $y \in E(Y)$. Then $o(jy) = jo(y) \in V(jT)$ and $GW = X$. By definition, W is contained in $\Psi(\widetilde{X})$ and ϕ induces an isomorphism between the stabilizers of corresponding vertices and edges in $\widetilde{X}$ and X. Because X is connected, it suffices to show that an edge w with origin/target in $\psi(\pi)W$ belongs to $\psi(\pi)W$. By translating w by an element of $\psi(\pi)$, if necessary, we can assume that $o(w)$ or $t(w)$ belongs to $V(jT)$. Since $GW = X$, there is $g \in G$ such that $gw \in W$, i.e. $gw = jy$ for some $y \in E(Y)$. It suffices to show that $g \in \psi(\pi)$.

We have $o(gw) \in V(jT)$. If $o(w) \in V(jT)$, then $o(w) = o(gw) = go(w)$ and $g \in G_{o(w)} \subseteq \psi(\pi)$. If $t(w) \in V(jT)$, then $\gamma_y^{-1}t(gw) = \gamma_y^{-1}\gamma_y j(t(y)) \in V(jT)$ and $t(w) = \gamma_y^{-1}t(gw) = \gamma_y^{-1}gt(w)$. Hence $\gamma_y^{-1}g \in G_{t(w)}$ and $g \in \psi(\pi)$.

Since ψ induces isomorphisms between the stabilizers, Ψ is locally injective: Suppose $\Psi(\widetilde{y}) = \Psi(g\widetilde{y}')$ with $\widetilde{P} := o(\widetilde{y}) = o(\widetilde{y}')$. Since the projection to Y agree, $\widetilde{y}' = \widetilde{y}$ and thus $g \in \pi_{\widetilde{P}}$ and $\psi(g) \in G_{jy}$. Hence there is $h \in \pi_{\widetilde{y}}$ with $\psi(h) = \psi(g)$. But since the inclusion $G_{jy} \to G_{jP}$ is injective and $\psi : \pi_{\widetilde{P}} \to G_{jP}$ is an isomorphism, $h = g$, hence $\widetilde{y} = g\widetilde{y}'$.

Theorem 3.16. *The following are equivalent:*

(1) *X is a tree.*
(2) *$\Psi : \widetilde{X} \to X$ is an isomorphism.*
(3) *$\psi : \pi_1(G, Y, T) \to G$ is an isomorphism.*

We are mostly interested in (1) ⇒ (3).

Proof. (1) ⇒ (2):
We already know that Ψ is surjective. It remains to show that it is injective. It

suffices to show that for every injective path c in $\widetilde{X}$ also $\Psi \circ c$ is injective. Since X is a tree, it suffices to check that $\Psi \circ c$ has no backtracking. But the latter property follows immediately from the fact that c is injective and Ψ is locally injective.

(2) $\Rightarrow$ (1):
Follows from Theorem 3.15.

(2) $\Rightarrow$ (3):
It remains to show injectivity. Let N be the kernel of ψ. Let $P \in V(Y)$, we have $N \cap G_P = \{1\}$ because ψ defines an isomorphism between G_P and G_{jP}. If $n \in N$ is non-trivial, $\widetilde{P}$ and $n\widetilde{P}$ are hence distinct but have the same image under Ψ.

(3) $\Rightarrow$ (2):
clear ∎

Corollary 3.17. *A group which acts freely on a tree is free.*

Proof. If the action is free, all stabilizers are trivial by definition. Hence $\pi_1(G, Y, T)$ is the free groups generated by the edges of Y that don't belong to T. ∎

Corollary 3.18 (Nielsen-Schreier Theorem)**.** *Subgroups of free groups are free.*

Proof. For a group G with free generating set S the Cayley graph $C(G, S)$ is a tree with a free G action. Restricting the action to any subgroup H yields a free action of H on a tree. Hence H is free by the last corollary. ∎

The following is a strengthened version of the Nielsen-Schreier Theorem.

Lemma 3.19. *Let F be a free group of rank n, and let $G \leq F$ be a subgroup of index k. Then G is a free group of rank $(n-1)k+1$. In particular, finite index subgroups of free groups of finite rank are finitely generated.*

Proof. Let S be a free generating set of F. Then $C(F, S)$ is a regular tree of degree $2n$. Since G has index k in F, there are k G-orbits of vertices in $C(F, S)$. Hence $Y := G\backslash C(F, S)$ has k vertices. Pick one representative for each orbit in $C(F, S)$. Each edge y of $C(F, S)$ can be translated by a unique element of G so that $o(y)$ is a representative and another unique element (it might be the same) so that $t(y)$ is a representative. Hence from the 2nk edges going out of the representative, 2 always belong to the same orbit (or are the same edge). Hence Y has nk edges. Since Y has k vertices, a maximal tree has $k-1$ edges. Hence $\pi_1(G, Y, T)$ is freely generated by $nk - k + 1$ elements. ∎

Corollary 3.20. *A free group of rank $n \geq 2$ contains a free group of rank k for any $k \in \mathbb{N}$.*

Proof. Consider any surjection onto $\mathbb{Z}$ and the preimage of $k'\mathbb{Z}$ to obtain a subgroup of index k' and thus a free subgroup of rank $(n-1)k'+1$. Since a free group obviously contains a free group of any lower rank, this proves the corollary. ∎

Corollary 3.21. *Finite index subgroups of finitely generated groups are finitely generated.*

Proof. Let G be a finitely generated group, and let H be a finite index subgroup of G. If S is a finite generating set of G, then the universal property of the free group F(S) freely generated by S provides us with a surjective homomorphism $\phi\colon F(S) \to G$. Let H_0 be the preimage of H under ϕ; so H_0 is a subgroup of $F(S)$ of index $[G : H]$. Hence H_0 is finite generated; but then also the image $H = \phi(H_0)$ is finitely generated. ∎

4. Proof of Stalling's theorem

We will now prove Theorem 2.17 using Theorem 3.16. Recall Theorem 2.17:

Theorem (Stallings)**.** *Let G be a finitely generated group with $e(G) = \infty$. Then one of the following holds:*

(1) *The group G admits a splitting $G = H *_C K$ as a free product with amalgamation where C is a finite group such that $C \neq H$ and $C \neq K$.*
(2) *The group G is an HNN extension $G = \langle H, t \mid t^{-1}ct = \phi(c)\rangle$ where $C \leq H$ is finite and $\phi\colon C \to H$ is injective.*

Fix a group G with finite, symmetric generating set S and let $X := C(G, S)$ be the Cayley graph.

Definition 4.1. For a subset $U \subseteq G$ we define

- $|U| \subseteq X$ to be the full subgraph with vertex set U.
- ∂U to be the set of edges connecting U and $U^c := G \setminus U$.

We call U *almost invariant* if ∂U is finite and U, U^c are infinite.

Lemma 4.2. *If $e(G) \geq 2$, then there exists an almost invariant subset.*

Proof. Let $K \subseteq X$ be a finite subgraph such that $X \setminus K$ has at least two connected unbounded component and let U be one of them. Then U is almost invariant since each edge between U and U^c has to have at least one vertex in K, i.e. ∂U is finite. □

Definition 4.3. We define

$$\mathrm{width}(G, S) = \inf\{k \mid \exists U \text{ almost invariant with } \partial U = k\}.$$

An almost invariant subset U is *narrow* if it realizes the width, i.e. if ∂U has $\mathrm{width}(G, S)$ many elements.

Lemma 4.4. *If U is narrow, $|U|$ and $|U^c|$ are connected.*

Proof. $|U|$ can only have finitely many components since each has to be connected to U^c and ∂U is finite. One of them has to be infinite since U is infinite. This component with vertex set U' is also almost invariant since $\partial U' \subseteq \partial U$. If $U' \neq U$, then $\partial U'$ is smaller than ∂U since every other component of U has to be connected to U^c with at least one edge. This contradicts the assumption that U was narrow. □

Lemma 4.5. *Let $U_1 \supsetneq U_2 \supsetneq \ldots$ be a strictly decreasing sequence of narrow sets. Then $U_\infty := \bigcap_{n=1}^{\infty} U_n$ is empty.*

Proof. Let $k := \mathrm{width}(G, S)$. If U_∞ is non-empty, then so is ∂U_∞. Since $U_\infty \subsetneq U_1$ and U_1 is connected, there exists at least one edge e_1 from U_∞ to $U_1 \setminus U_\infty$.

There exists $i > 1$ such that for all $j \geq i$, e_1 is an edge from U_j to $U_1 \setminus U_j$. Hence $e_1 \in \partial U_j$ for $j \geq i$. Now pick an edge e_2 from U_∞ to $U_i \setminus U_\infty$. Again there is some i' such that e_2 is an edge between U_j and $U_i \setminus U_j$ for every $j \geq i'$. Hence $e_2 \in \partial U_j$ for $j \geq i'$. Moreover $e_1 \neq e_2$ since e_1 has a vertex in $U_1 \setminus U_i$ while $e_2 \subseteq U_i$.

After $k + 1$ steps we have found $k + 1$ distinct edges contained in ∂U_j for j big enough. This contradicts the assumption that U_j is narrow. □

Lemma 4.6. *If $e(G) \geq 2$, then for each vertex $v \in V(X)$ there is a minimal narrow subset U containing v.*

Proof. Let N be the set of narrow subsets containing v; ordered by inclusion. Let U be narrow, then $v \in U$ or $v \in U^c$ (which is also narrow). Hence N is non-empty. By Lemma 4.5, every chain in N has a lower bound. By Zorn's Lemma, there exists a minimal element as claimed. $\square$

We will now see how the tree is constructed. It is easier to work with graph with the set of oriented edges, i.e. each edge appears twice; each with different orientation. Let U be a minimal narrow set containing e. It exists by Lemma 4.6. Let $\mathcal{E} := \{gU, gU^c \mid g \in G\}$ and define an equivalence relation $S \sim S'$ if $S \cap (S')^c$ and $S' \cap S^c$ are finite. The set $E := \mathcal{E}/\sim$ will be the set of oriented edges. Here $[S^c]$ will be $\overline{[S]}$, the edge with the other orientation. Define $[S] \leq [T]$ if $S \cap T^c$ is finite.

Remark 4.7. Note that $\leq$ is transitive and reflexive and $[S] \leq [T], [T] \leq [S]$ implies $[S] = [T]$ by definition of $\sim$.

Also note that $[S] \leq [T]$ means $T^c \cap S = S \cap T^c$ is finite. Hence $[T^c] \leq [S^c]$.

Lemma 4.8. *For each $[S], [T] \in E$ one of $[S] \leq [T], [S] \leq [T^c], [S^c] \leq [T]$ or $[S^c] \leq [T^c]$ holds.*

Proof. Let $v \in S$ be such that S is minimal containing v. Suppose $W_1 := S \cap T, W_2 := S \cap T^c, W_3 := S^c \cap T$ and $W_4 := S^c \cap T^c$ are all infinite. Note that they partition G. Let $k = \#\partial S = \#\partial T$. Each edge in ∂S and ∂T appears in exactly two of the sets ∂W_i. Furthermore, $\partial W_i \subseteq \partial S \cup \partial T$.

Hence $\sum \#\partial W_i \leq 4k$. Since all W_i are infinite, they are almost invariant and hence $\#\partial W_i \geq k$. Thus, $\#\partial W_i = k$ for all i and the sets are narrow. Either $U \cap V$ or $U \cap V^c$ contains v. Since both are infinite, they are proper subsets of U. A contradiction to the minimality of U. $\square$

If $[S] \leq [S']$, define
$$[[S], [S']] := \{[T] \in E \mid [S] \leq [T] \leq [S']\}.$$
And $[S] \dashv [T]$ if $[S] \leq [T]$, $[S] \neq [T]$ and $[[S], [T]] = \{[S], [T]\}$. Now define $[S] \sim_V [T]$ if $[S] = [T]$ or $[S] \dashv [T^c]$.

Lemma 4.9. *$\sim_V$ is an equivalence relation.*

Proof. Obviously, $[S] \sim_V [S]$ and $[S] \sim_V [T] \Rightarrow [T] \sim_V [S]$. For simplicity call the elements of E e, f, g. Suppose $e \sim_V f$ and $f \sim_V g$ with $e \neq f$ and $f \neq g$. Then $e \vdash f^c$ and $f \vdash g^c$.

Claim $e \leq g^c$ or $e = g$:
If $e \leq g^c$ doesn't hold, then one of the following holds:

(1) $e \leq g$: From $f \leq g^c$ it follows that $g \leq f^c$. Hence $e \leq g \leq f^c$. By $[e, f^c] = \{e, f^c\}$ we have $e = g$ or $g = f^c$. But $f \vdash g^c$ implies $f \neq g^c$. Hence $e = g$.
(2) $e^c \leq g^c$: Then $g \leq e \leq f^c$ and thus $f \leq e^c \leq g^c$. Since $[f, g^c] = \{f, g^c\}$ and $e \neq f^c$, this implies $e^c = g^c$ and thus $e = g$.
(3) $e^c \leq g$: Then $f \leq g^c \leq e \leq f^c$. This can't be true since f is infinite.

So we can assume $e \leq g^c$. It remains to show that $e \neq g^c$ and $[e, g^c] = \{e, g^c\}$. We already saw that $e^c \leq g$ can't hold, hence $e \neq g^c$.

Let $e \leq b \leq g^c$. By the previous lemma one of the following conditions hold: $b \leq f$, $b \leq f^c$, $b^c \leq f$ or $b^c \leq f^c$.

- $b \leq f$ implies $e \leq b \leq f$ hence $e \cap f^c$ is finite. But by assumption $e \leq f^c$ and $e \cap f$ is finite. This contradicts e infinite.
- $b^c \leq f$ implies $f^c \leq b \leq g^c$ which contradicts $f \leq g^c$ as above.
- $b \leq f^c$ implies $e \leq b \leq f^c$. Hence either $e = b$ or $b = f^c$. In the latter case $f^c \leq g^c$, same contradiction as above.
- $b^c \leq f^c$ implies $f \leq b \leq g^c$ and hence $f = b$ or $b = g^c$. In the first case $e \leq f$ contradicts $e \leq f^c$ as above. □

Define the vertex set $V := E/\sim_V$ and let $t([S]) = [[S]]$. This implies $o([S]) = [[S^c]]$. Since not both $[S] \leq [T]$ and $[S] \leq [T^c]$ hold, $E \xrightarrow{(o,t)} V \times V$ is injective. Hence (V, E) is a graph.

By definition (E, V) is a graph with a G-action which is transitive on the edge set. It remains to show that (E, V) is a tree and the edge stabilizer is finite.

Lemma 4.10. *Let U, V, W be narrow. Then there are only finitely many $g \in G$ with $U \cap gW^c$ and $gW \cap V^c$ finite.*

In particular ($U = V = W$), the edge stabilizer is finite.

Proof. Arguing as in the proof of Freudenthal-Hopf, both U and V^c contain at least two ends. Let $L := \partial U \cup \partial V$. Then there exists $L \subseteq C \subseteq X$ finite, connected such that $U \cap (X \setminus C)$ and $V^c \cap (X \setminus C)$ both contain at least two unbounded path components.

Let K be a finite, connected complex containing ∂W. For almost all $g \in G$, $gK \subseteq X \setminus C$ and gK is contained in an unbounded component Z_g of $X \setminus C$. Since C is connected and $gK \cap C = \emptyset$, only gW or gW^c can intersect C. Hence either $gW \subseteq Z_g$ or $gW^c \subseteq Z_g$. Hence for almost all g one of the following holds.

1. If $gW \subseteq Z_g \subseteq X \setminus C$, then gW^c must contain one unbounded component of $U \cap C^c$ since there are two of them and gW lies in at most one of them. Thus, $gW^c \cap U$ is infinite.
2. If $gW^c \subseteq Z_g$, then gW must contain one unbounded component of $V^c \cap C^c$. Hence $gW \cap V^c$ is infinite. □

The following lemma completes the proof of Stalling's theorem.

Lemma 4.11. (E, V) *is a tree.*

Proof. By an edge path, we mean a path of edges $e_1, .., e_n$ with $t(e_i) = o(e_{i+1})$ and without backtracking.

First we show that (E, V) is connected. By Lemma 4.8, it suffices to show that $[S] \leq [T]$ implies that there is an edge path from $[S]$ to $[T]$. By Lemma 4.10, $[[S], [T]]$ is finite and forms a chain $[S] = e_1 \vdash e_2 \vdash \ldots \vdash e_n = [T]$. By definition, $t(e_i) = t(e_{i+1}^c) = o(t_{i+1})$.

If (E, V) is not a tree, there exists an edge path $e_1, .., e_n = e_1$. Then $t(e_i) = o(e_{i+1}) = t(e_{i+1}^c)$. Hence $e_i \sim_V e_{i+1}^c$. Since there is no backtracking, $e_i \vdash e_{i+1}$ and hence $e_i \leq e_{i+1}$. We can't have $t(e_1) = o(e_1)$ by definition: Would imply $e_1 = e_1^c$ (not true) or $e_1 \vdash e_1$ which is excluded by definition. Hence there is $f \neq e_1$ with $e_1 \leq f \leq e_1$ but this can't happen. □