

Introduction to differential forms and integration on manifolds

Talk in the seminar on the deRahm-Cohomology

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2.1 Cotangent bundles and vector fields

Definition: Tangent bundle (Repetition)

Let M be a smooth manifold. Its tangent bundle TM is a vector bundle and the disjoint union of the tangens spaces of M :

$$TM := \bigsqcup_{x \in M} T_x M.$$

The projection of the bundle is given by

$$\begin{aligned} \pi : TM &\longrightarrow M \\ (x, v) &\longmapsto x. \end{aligned}$$

Note that given an atlas of M , a bundle atlas can easily be derived. Consider a chart (U, h, Ω) . Define

$$\begin{aligned} H : p^{-1}(U) = TU &\longrightarrow U \times \mathbb{R}^n \\ (x, v) &\longmapsto H((x, v)) = (x, D_x h(v)) \end{aligned}$$

as the corresponding map in the bundle chart.

Definition: Cotangent bundle

Let M be a smooth manifold. We saw that $T_x M$ has a vector space structure. Let $T_x^\# M$ be the corresponding dual vectorspace. We define

$$T^\# M := \bigsqcup_{x \in M} T_x^\# M$$

as the cotangent bundle of M . The projection is given by

$$\begin{aligned} \pi^\# : T^\# M &\longrightarrow M \\ (x, v^\#) &\longmapsto x. \end{aligned}$$

Note that $v^\#$ is just some vector in the cotangent space and has nothing to do with the v from the tangent space.

Again, we can construct a bundle atlas given an atlas on M . Consider a chart (U, h, Ω) . Define

$$\begin{aligned} H : p^{-1}(U) = T^\# U &\longrightarrow U \times (\mathbb{R}^n)^\# \\ (x, v^\#) &\longmapsto H((x, v^\#)) = (x, (D_x h)^\#(v^\#)) \end{aligned}$$

as the corresponding map in the bundle chart.

Definition: Function (Repetition)

Let M be a smooth manifold. A map $f : M \rightarrow \mathbb{R}^n$ is called a function. The function is smooth if for every chart (U, h, Ω) , $f \circ h^{-1}$ is smooth.

Definition: Section (vector bundel)

Remark: One could define this for fiber bundles. For our purpose, the definition applied to vector bundles is enough.

Consider a vector bundle with total space E , base space B and projection $p : E \rightarrow B$. A section $s : B \rightarrow E$ in this vector bundle, is a right inverse of the projection:

$$p(s(x)) = x \quad \forall x \in B.$$

Definition: vector field

Let M be a smooth manifold. A vector field is a smooth section in the tangent bundle TM . It is a function

$$v : M \rightarrow TM \text{ with } p \circ v = \text{Id}_M.$$

Therefore, $v(x) \in T_x M \quad \forall x \in M$. We want v to be smooth (C^∞). We denote the set of all smooth vector fields with $\Gamma^\infty(TM)$.

Example (trivial vector field)

We always have the trivial vector field on a smooth manifold that maps every x to $(x, 0)$.

Example (vector fields on $\mathbb{S}^2$)

There is no vector field on $\mathbb{S}^2$ that does not vanish. Therefore, $T\mathbb{S}^2$ cannot be trivial (i.e. $T\mathbb{S}^2 \cong \mathbb{S}^2 \times \mathbb{R}^2$). If it was trivial, a non-vanishing vector field could easily be constructed.

Example (gradient)

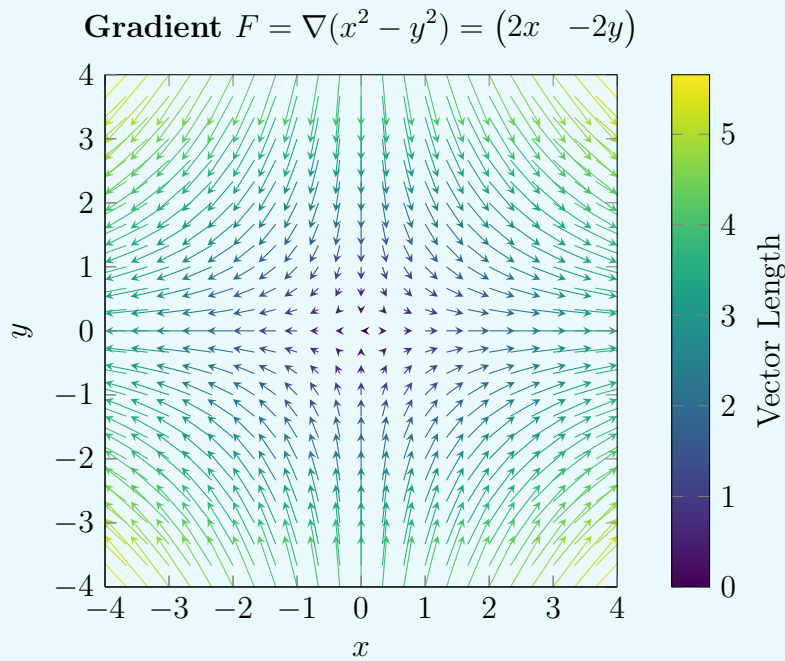
Consider $\mathbb{R}^n$ as a manifold. Each tangent space is $\mathbb{R}^n$ again. For a smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the map

$$\nabla f : x \mapsto \left(\frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right)$$

is a vector field. An example of this is the gradient of

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto x^2 - y^2.$$

The gradient of this function looks like this:



2.2 Differential forms

Definition: exterior algebra (repetition from Linear Algebra II)

Let V be a vector space. We define

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

Also, we define I as the ideal generated by all $x \otimes x$ for all $x \in V$. Then, we define

$$\bigwedge(V) := T(V)/I.$$

Note that $T(V)$ is a ring with the tensor product as a multiplication. Define

$$\alpha \wedge \beta = \alpha \otimes \beta \mod I.$$

We define $\bigwedge^k(V)$ as the vector subspace of $\bigwedge(V)$ spanned by all

$$x_1 \wedge x_2 \wedge \dots \wedge x_k \text{ for } x_i \in V, i = 1, 2, \dots, k.$$

Definition: differential form

Let M be a smooth manifold. A differential form of degree k or k -form ω is a smooth section in the k -th exterior power of the cotangent bundle of M :

$$\omega : M \rightarrow \bigwedge^k(T^{\#}M) := \bigsqcup_{x \in M} \bigwedge^k(T_x^{\#}M).$$

Because ω is a section, we get

$$\omega(x) \in \bigwedge^k(T_x^{\#}M).$$

We see that ω maps every point $x \in M$ to an alternating multilinear form in $T_x M$. We often write ω_x for $\omega(x)$. We see that

$$\omega_x : (T_x M)^k \rightarrow \mathbb{R}$$

is smoothly varying with x (we will see what that means).

Example (k -forms)

- smooth maps are 0-forms because $\Lambda^0(T_x^\# M) = \mathbb{R}$. Therefore $\Omega^0(M) = C^\infty(M, \mathbb{R})$.
- Pfaffian forms are 1-forms because $\Lambda^1(T_x^\# M) = T_x^\# M$. These are covector fields and each ω_x is just a linear map $T_x M \rightarrow \mathbb{R}$. Pfaffian forms are the dual concept for vector fields because they are sections $M \rightarrow T_x^\# M$ instead of sections $M \rightarrow T_x M$.

Claim 2.1 Basis of $\Lambda^k(T_p^\# M)$

As we saw in the last talk, for a given chart (U, h, Ω) on an n -dimensional smooth manifold M , the directional derivatives form a base of the tangent space. Based on this basis, we construct the corresponding dual basis of $T_p^\# M$ with $(x_1 \cdots x_n)$ being a vector representation in the basis given by the partial derivatives:

$$\begin{aligned} dx_i : T_p M &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\longmapsto x_i. \end{aligned}$$

We see that $dx_i \in T_p^\# M$ and because every linear form $T_p M \rightarrow \mathbb{R}$ can be written as

$$\sum_{i=1}^n a_i x_i,$$

the dx_i are a basis of $T_p^\# M$. We therefore see, that a basis of the k -forms is

$$\{dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} \text{ for } 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$$

and that $\dim \Lambda^k(T_p^\# M) = \binom{n}{k}$. We see that every k -form can be written as

$$\omega_x = \sum_{i_1 < \dots < i_k} f_{i_1, \dots, i_k}(x) dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

The smoothness of the k -form now just means that all the $f_{i_1, \dots, i_k}$ are smooth for every chart.

We especially see that a basis of the n -forms is just

$$dx_1 \wedge \dots \wedge dx_n$$

and therefore every n -form can be written as

$$\omega_x = f(x) dx_1 \wedge \dots \wedge dx_n$$

for some smooth $f \in C^\infty(M, \mathbb{R})$.

Example (differential forms on $\mathbb{S}^2$)

- 0-Form: $\omega_p = f(p)$
- 1-Form: $\omega_p = f_1(p)dx_1 + f_2(p)dx_2$
- 2-Form: $\omega_p = f(p)dx_1 \wedge dx_2$
- 3-forms: constant 0 because $T_x M \cong \mathbb{R}^2$ so every alternating 3-form has already 2 linearly dependent parts that cancel

“Wedging”:

$$\begin{aligned}
 & (f_1(p)dx_1 + f_2(p)dx_2) \wedge (g_1(p)dx_1 + g_2(p)dx_2) \\
 &= ((f_1(p) \cdot g_1(p))(dx_1 \wedge dx_1) + (f_1(p) \cdot g_2(p))(dx_1 \wedge dx_2) \\
 &\quad + (f_2(p) \cdot g_1(p))(dx_2 \wedge dx_1) + (f_2(p) \cdot g_2(p))(dx_2 \wedge dx_2)) \\
 &= (f_1(p) \cdot g_2(p) - f_2(p) \cdot g_1(p))(dx_1 \wedge dx_2).
 \end{aligned}$$

Example ($dx_1 \wedge dx_2$)

What does this mean? This must be an alternating bilinearform that takes 2 vectors and only considers their first 2 components and should be normed.

$$dx_1 \wedge dx_2 : \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \mapsto ?$$

From Linear Algebra I we then know that this can only be the determinant. Therefore

$$\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \mapsto \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

We get analogue identities with determinantes of $k \times k$ -matrices that contain the components $i_1, \dots, i_k$ for $dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

Definition: $\Omega^k(M)$

The k -forms on M form a vector space as we can add them pointwise and multiply them by scalars pointwise as well. We call the usually infinite-dimensional vector space of the k -forms $\Omega^k(M)$. We set

$$\Omega(M) = \bigoplus_{k=1}^{\infty} \Omega^k(M).$$

Note that for finite dimensional manifolds, this sum is finite since $\Omega^k(M)$ is trivial for $k > \dim M$.

Definition: exterior power of the differential forms

The product in the exterior power defines maps

$$\Omega^k(M) \times \Omega^\ell(M) \rightarrow \Omega^{k+\ell}(M), (\omega, \mu) \mapsto \omega \wedge \mu$$

with

$$(\omega \wedge \mu)_x := \omega_x \wedge \mu_x.$$

Definition: orientable

A smooth manifold M of dimension n is called orientable if there exists $\omega \in \Omega^n(M)$ with $\omega_x \neq 0$ for all $x \in M$. Such an ω is called an orientation form on M . Two orientation forms ω, τ on M are equivalent if $\tau = f \cdot \omega$ for some $f \in \Omega^0(M)$ with $f(x) > 0$ for all $x \in M$. An orientation of M is an equivalence class of orientation forms on M .

Remark

An orientation form is either everywhere positive or negative for a given basis. Therefore, on a connected orientable manifold, there are just 2 orientations. An orientation form orients a given basis by plugging the basis vectors into the form and checking the resulting value for its sign. If the determinant of a basis transformation is positive, the basis have the same orientations for a given orientation form.

2.3 Integration on manifolds

Definition: Integration on $\mathbb{R}^n$

Consider $\mathbb{R}^n$ as an n -dimensional manifold with the standard orientation (the canonical basis of the partial derivatives should be positively oriented). We saw that $dx_1 \wedge \dots \wedge dx_n$ is a basis of the n -forms. We write

$$\omega \in \Omega_c^n(\mathbb{R}^n) \text{ as } f(x)dx_1 \wedge \dots \wedge dx_n.$$

$f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ should have compact support here. We define

$$\int_{\mathbb{R}^n} f(x)dx_1 \wedge \dots \wedge dx_n = \int_{\mathbb{R}^n} f(x)d\mu_n$$

using the usual Lebesgue measure $d\mu_n$ on $\mathbb{R}^n$.

Definition: Integration on open subsets

We can extend the definition from before for $\omega \in \Omega_c^n(V)$ for $V \subseteq \mathbb{R}^n$ open because we can smoothly extend ω and f to $\mathbb{R}^n$ by setting them equal to 0 on $\mathbb{R}^n - \text{supp}_V(\omega)$.

Example (3.12(ii) in [M-T])

Let $\varphi : U_1 \rightarrow U_2$ be a smooth map between open sets in $\mathbb{R}^n$. Then

$$\varphi^*(dx_1 \wedge \dots \wedge dx_n) = \det(D_x \varphi) dx_1 \wedge \dots \wedge dx_n.$$

Lemma 2.2 Integrals under diffeomorphisms

Let $\varphi : V \rightarrow W$ be a diffeomorphism between open subsets V and W of $\mathbb{R}^n$ and assume that the Jacobi determinant $\det(D_x \varphi)$ is of constant sign $\delta = \pm 1$ for $x \in V$. For $\omega \in \Omega_c^n(W)$ we have

$$\int_V \varphi^*(\omega) = \delta \cdot \int_W \omega.$$

Note that $\varphi^*(\omega)$ is the pullback. It means that we express ω in the coordinates of the map φ .

Proof. If ω is written as

$$\omega = f(x) dx_1 \wedge \dots \wedge dx_n$$

with $f \in C_c^\infty(W, \mathbb{R})$ it follows from example 3.13(ii) in the book that

$$\varphi^*(\omega) = f(\varphi(x)) \det(D_x \varphi) dx_1 \wedge \dots \wedge dx_n = \delta f(\varphi(x)) |\det(D_x \varphi)| dx_1 \wedge \dots \wedge dx_n.$$

The lemma then follows from the transformation of integrals that states

$$\int_W f(x) d\mu_n = \int_V f(\varphi(x)) |\det(D_x \varphi)| d\mu_n.$$

□

Proposition 2.3 Integrals on manifolds

For an arbitrary oriented n -dimensional smooth manifold M of dimension n there exists a unique linear map

$$\int_M : \Omega_c^n(M) \rightarrow \mathbb{R}$$

with the following property: If $\omega \in \Omega_c^n(M)$ has support contained in U , where (U, h, Ω) is a positively oriented C^∞ -chart, then

$$\int_M \omega = \int_{h(U)} (h^{-1})^* \omega.$$

Again, this means expressing ω in coordinates from the chart h .

Proof: First, consider $\omega \in \Omega_c^n(M)$ with support such that $\text{supp}_M(\omega)$ is contained in a coordinate patch. Then (U, h, Ω) can be chosen as above and the integral is determined. We must show that the right-hand side is independent of the choice of chart. Assume $(\tilde{U}, \tilde{h}, \tilde{\Omega})$ is another positively oriented C^∞ -chart with $\text{supp}_M(\omega) \subseteq \tilde{U}$.

The diffeomorphism $\varphi : V \rightarrow W$ from $V = h(U \cap \tilde{U})$ to $W = \tilde{h}(U \cap \tilde{U})$ given by $\varphi = \tilde{h} \circ h^{-1}$ has everywhere positive Jacobi determinant. Since

$$\text{supp}_{h(U)}((h^{-1})^* \omega) \subseteq V, \text{supp}_{\tilde{h}(\tilde{U})}((\tilde{h}^{-1})^* \omega) \subseteq W$$

and $\varphi^*(\tilde{h}^{-1})^*\omega$. The Lemma about the diffeomorphisms shows that

$$\int_{h(U)} (h^{-1})^*\omega = \int_{\tilde{h}(\tilde{U})} (\tilde{h}^{-1})^*\omega.$$

So for $\omega \in \Omega_c^n(M)$ with support in a coordinate patch, the integral defined in the proposition is independent of the chart.

Now choose a smooth partition of unity $(\rho_\alpha)_{\alpha \in \mathcal{A}}$ on M subordinate to an oriented C^∞ -atlas on M . For $\omega \in \Omega_c^n(M)$ we have that

$$\omega = \sum_{\alpha \in \mathcal{A}} \rho_\alpha \omega$$

where every term $\rho_\alpha \omega \in \Omega_c^n(M)$ has support in a coordinate patch and where only finitely many terms are non-zero. We define

$$I(\omega) = \sum_{\alpha \in \mathcal{A}} \int_M \rho_\alpha \omega$$

where the term associated to $\alpha \in \mathcal{A}$ is given by the integral defined in the proposition, applied to a U_α with $\text{supp}_M(\rho_\alpha) \subseteq U_\alpha$. It is obvious that I is a linear operator on $\Omega_c^n(M)$. If, in particular, $\text{supp}_M(\omega) \subseteq U$ where (U, h, Ω) is a positively oriented C^∞ -chart, the terms of the sum can be calculated by the integral applied to (U, h, Ω) . This yields

$$I(\omega) = \int_M \omega$$

which shows that I is a linear operator with the desired properties. Uniqueness follows analogously. $\square$

Remark

Note that in the proof we showed that by a partition of unity, we can integrate over more than a single coordinate patch.

Example

Consider $\mathbb{S}^1$ and the map $h^{-1} : (0, 2\pi) \rightarrow \mathbb{S}^1, t \mapsto (\cos(t), \sin(t))$. Let $\omega = xdy$ restricted to the upper semicircle ($\text{supp}(\omega) = [0, \pi]$). We want to integrate ω over $\mathbb{S}^1$. We see that

$$x = \cos t \quad y = \sin t \quad dy = \cos t dt.$$

Therefore, in the coordinates of our chart

$$(h^{-1})^*\omega = \cos(t) \cos(t) dt.$$

Then

$$\int_{\mathbb{S}^1} \omega = \int_0^\pi \cos^2(t) dt = \frac{\pi}{2}.$$

Lemma 2.4 properties of the Integral

(i) $\int_M \omega$ changes sign when the orientation of M is reversed.

(ii) If $\omega \in \Omega_c^n(M)$ has support contained in an open set $W \subset M$ then

$$\int_M \omega = \int_W \omega$$

when W is given the orientation induced by M .

(iii) If $\varphi : N \rightarrow M$ is an orientation-preserving diffeomorphism, then we have that

$$\int_M \omega = \int_N \varphi^*(\omega)$$

for $\omega \in \Omega_c^n(M)$.

Literature:

- [M-T]: I. Madsen, J. Tornehave From Calculus to Cohomology. Cambridge University Press (1978). (1982).