Talk 13: Cohomology of Projective Bundles

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1 Stiefel Manifolds and Grassmannian Manifolds

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $\mathbb{F}^n$ be equipped with the standard inner product. The construction of Stiefel manifolds requires the following theorem from differential geometry.

Theorem 1.1 (Preimage Theorem)**.** Let $f : X \to Y$ be a smooth map between smooth manifolds with $\dim X \geq \dim Y$. Let $y \in Y$ be a regular value, then $f^{-1}(y)$ is a smooth submanifold of X. If $y \in f(X)$, then $\dim f^{-1}(y) = \dim X - \dim Y$.

Construction 1.2 (Stiefel manifold)**.** Let $V_k(\mathbb{F}^n)$ (or $V_{k,n}$) be the set of all orthonormal k-frames in $\mathbb{F}^n$. An orthonormal k-frame is an ordered set of k vectors which are orthonormal, so $V_k(\mathbb{F}^n)$ can be identified with $\{A \in M_{n\times k}(\mathbb{F}) : A^\dagger A = I_k\}$ (or equivalently the set $\mathbf{L}(\mathbb{F}^k, \mathbb{F}^n)$ of linear isometric embeddings). We may give $V_k(\mathbb{F}^n)$ the structure of a smooth manifold as follows. Consider the map $f : M_{n\times k}(\mathbb{F}) \to S_k(\mathbb{F})$, $A \mapsto A^\dagger A$, where $S_k(\mathbb{F})$ denotes the space of symmetric/Hermitian matrices. This is a smooth map and I_k is a regular value. To see this, note that if $f(A) = I_k$, then for each $C \in S_k(\mathbb{F})$, a solution to $Df|_A(H) = H^\dagger A + A^\dagger H = C$ is given by $H = \frac{1}{2}AC$. Thus by preimage theorem, $V_k(\mathbb{F}^n)$ is a smooth manifold, and $\dim V_k(\mathbb{R}^n) = nk - \frac{1}{2}k(k+1)$ and $\dim V_k(\mathbb{C}^n) = 2nk - k^2$. [Note that $f^{-1}(I_k)$ is closed and bounded, so $V_k(\mathbb{F}^n)$ is compact.]

Example 1.3.

(i) $V_1(\mathbb{R}^n) \cong S^{n-1}$, $V_1(\mathbb{C}^n) \cong S^{2n-1}$.

(ii) $V_n(\mathbb{R}^n) \cong O(n)$, $V_n(\mathbb{C}^n) \cong U(n)$.

(iii) $V_2(\mathbb{R}^n) \cong S(TS^{n-1})$, the total space of the unit sphere bundle of TS^{n-1}.

(iv) $V_{n-1}(\mathbb{R}^n) \cong SO(n)$. (For each orthonormal $(n-1)$-frame of $\mathbb{R}^n$, there is a unique choice of the nth vector to extend to an orthonormal basis with positive orientation.)

Lemma 1.4. There is a fiber bundle $V_{k-1}(\mathbb{F}^{n-1}) \to V_k(\mathbb{F}^n) \xrightarrow{p} V_1(\mathbb{F}^n)$ $(1 < k \leq n)$, where the projection map p is defined by extracting the first vector in an orthonormal k-frame.

Sketch of proof. We proceed as in [2]. Each $v \in V_1(\mathbb{R}^n)$ defines a hyperplane by the equation $\langle x, v\rangle = 0$. To construct a trivialization near v, we must identify the fibers over points near v with the standard $V_{k-1}(\mathbb{R}^n)$ smoothly. To this end, we extend v to an orthonormal basis $(v, w_2, ..., w_n)$ of V, so $(w_2, ..., w_n)$ form an orthonormal basis of $v^\perp$. For all u in the open hemisphere U containing v, the orthogonal projection $Q : v^\perp \to u^\perp$ onto $u^\perp$ is a linear isomorphism, so $\{Q(w_2), ..., Q(w_n)\}$ is linearly independent. One can apply Gram-Schmidt process to this linearly independent set to get an orthonormal sequence. This identifies $u^\perp$ with $\mathbb{R}^{n-1}$ isometrically for u in the open hemisphere containing v, and this depends smoothly on u. Under this identification a $(k-1)$-frame of $u^\perp$ is identified with an orthonormal $(k-1)$-frame of $\mathbb{R}^{n-1}$, and this depends smoothly on u. This is a local trivialization.

The complex case follows from similar argument.

□

Construction 1.5 (Grassmann manifold)**.** Note that $O(k)$ (resp. $U(k)$) acts on $V_k(\mathbb{R}^n)$ (resp. $V_k(\mathbb{C}^n)$) as follows. Given a linear isometry $f : \mathbb{R}^k \to \mathbb{R}^n$ (resp. $\mathbb{C}^k \to \mathbb{C}^n$) and an element $\alpha \in O(k)$ (resp. $U(k)$), α acts on f by precomposition. Let $G_k(\mathbb{F}^n)$ (or $G_{k,n}$) be the orbit space $V_{k,n}/O(k)$, which can be identified with the set of k-dimensional[1] linear subspaces of $\mathbb{F}^n$. $G_{k,n}$ can be given the structure

[1]Dimension here means $\dim_{\mathbb{F}}$.

of a smooth manifold as follows. For each subset $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, ..., n\}$ and $n \times k$ matrix A, we introduce the notation A_I to denote the submatrix of A obtained by deleting rows indexed by $\{1, ..., n\} \setminus I$ and $\hat{A}_I$ to denote the submatrix of A obtaiend by deleting rows indexed by I. Define coordinate neighborhoods

$$U_I = \{\text{linear subspace spanned by columns of } A \in M_{n\times k}(\mathbb{F}) : \det A_I \neq 0\}$$

Up to change of basis, each linear subspace in U_I has a unique representative A s.t. $A_I = I_k$, e.g., if $I = \{1, ..., k\}$, then

$$U_{\{1,...,k\}} = \left\{ \text{linear subspace spanned by the columns matrices of the form } \begin{pmatrix} I_k \\ * \end{pmatrix} \right\}$$

Define $\varphi_I : U_I \to \mathbb{F}^{k(n-k)}$ by picking out the entries of $\hat{A}_I$. Given $I, J \subseteq \{1, ..., n\}$. The transition function $\varphi_I \circ \varphi_J^{-1}$ picks $k(n-k)$ entries to give a matrix A s.t. $A_J = I_k$ and $\det A_I \neq 0$, then inverts A_I and map to $\mathbb{F}^{k(n-k)}$ by φ_I. Each coordinate is a rational function such that the denominator is $\det A_I$, so the transition maps are smooth.

We have $\dim \mathrm{Gr}_k(\mathbb{R}^n) = k(n-k)$ and $\dim \mathrm{Gr}_k(\mathbb{C}^n) = 2k(n-k)$.

Remark 1.6. The above construction of smooth atlas specializes to the standard affine charts on $\mathbb{F}P^{n-1}$ when $k = 1$.

Example 1.7.

(i) $G_1(\mathbb{R}^n) \cong \mathbb{R}P^{n-1}$, $G_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$.

(ii) $G_k(\mathbb{F}^n) \cong G_{n-k}(\mathbb{F}^n)$. Note that there is a map $\mathrm{Gr}_{k,n} \to \mathrm{Gr}_{n-k,n}$ sending a k-dimensional subspace to its unique orthogonal complement.

Construction 1.8 (Oriented Grassmann manifold)**.** Let $\tilde{G}_k(\mathbb{R}^n)$ (or $\tilde{G}_{k,n}$) be the orbit space $V_k(\mathbb{R}^n)/SO(n)$. To see that this is a smooth manifold, note that the cyclic group C_2 acts freely on $\tilde{G}_k(\mathbb{R}^n)$ by flipping the orientation. The quotient map $\tilde{G}_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ is a double cover, and one can lift the smooth structure on $G_k(\mathbb{R}^n)$ to a smooth structure on $\tilde{G}_k(\mathbb{R}^n)$ such that the covering map is smooth.

Example 1.9.

(i) $\tilde{G}_{1,n} \cong S^{n-1} \cong \tilde{G}_{n-1,n}$.

(ii) $\tilde{G}_{k,n} \cong \tilde{G}_{n-k,n}$.

2 Projectivization of Vector Bundles

Construction 2.1 (Projectivization)**.** Let $\xi : E \to M$ be a (smooth) $\mathbb{F}$-vector bundle of rank n over a smooth manifold M. We will construct a new fiber bundle called the projectivization of ξ, denoted by $P(\xi)$.

Define

$$E(P(\xi)) := \{(p, L) : p \in M,\ L \in P(\xi_p)\}$$

where $P(\xi_p)$ is the projective space of 1-dimensional $\mathbb{F}$-linear subspaces of the fiber over p, i.e., $\mathbb{F}P^n$. The projection onto the first coordinate is a fiber bundle (the projectivization of ξ), denoted by $P(\xi)$.

More generally, consider

$$E(G_k(\xi)) := \{(p, V) : p \in M,\ V \in G_k(\xi_p)\}$$

The projection onto the first coordinate is a fiber bundle over M with fiber $G_k(\mathbb{F}^n)$, and we denote this bundle by $G_k(\xi)$.

If ξ is an oriented real vector bundle of rank n, define

$$E(\tilde{G}_k(\xi)) := \{(p, V) : p \in M,\ V \in \tilde{G}_k(\xi_p)\}$$

The projection onto the first coordinate is a fiber bundle over M with fiber $\tilde{G}_k(\mathbb{R}^n)$, and we denote this bundle by $\tilde{G}_k(\xi)$.

We construct local trivializations for these constructions. Let (U, ϕ) be a local trivialization of ξ so that for each $p \in U \subseteq M$, we have an $\mathbb{F}$-linear isomorphism $\xi_p \xrightarrow{\phi} \mathbb{F}^n$. This induces a diffeomorphism $G_k(\xi_p) \cong G_k(\mathbb{F}^n)$ (resp. $\tilde{G}_k(\mathbb{R}^n)$). Define $\psi : E(G_k(\xi))|_U \to U \times G_k(\xi), (p, V) \mapsto (p, \phi(V))$, i.e., the map induced by ϕ by taking Grassmannian fiberwise. The topology on $E(\gamma_k(\xi))$ is induced by glueing together these local trivialization.

The total space is locally a product space, and one can give it a smooth structure such that the projection π is a smooth map. When M is a point, we recover the definition of Grassmannian manifolds.
[3] claims that π is proper and left it as an exercise for the readers. We spell out the proof here, but this will not be presented during the talk in the interest of time.

Lemma 2.2. The bundle projection $\pi : E(G_k(\xi) \to M$ (resp. $\pi : E(\tilde{G}_k(\xi)) \to M$ if ξ is orientable) is a proper map.

Proof. The proof is an exercise of point-set topology and consists of two steps.
Step 1: Suppose X, Y are topological spaces and $f : X \to Y$ a closed continuous map. If $f^{-1}(y)$ is compact for all $y \in Y$, then f is proper. Let $K \subseteq Y$ be compact. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of $f^{-1}(K)$. For each $y \in K$, $f^{-1}(y)$ is compact, so we have $f^{-1}(y) \subseteq W_y := \bigcup_{j=1}^{m_y} U_{i_{y,j}}$. The set $X \setminus W_y$ is closed in in X, so $f(X \setminus W_y)$ is closed in Y. Since $y \notin f(X \setminus W_y)$, we can find an open nbd V_y s.t. $V_y \cap f(X \setminus W_y) = \varnothing$, so $f^{-1}(V_y) \subseteq W_y$. We now have an open cover of K by $\{V_y : y \in K\}$. By compactness of K, we can reduce to a finite subcover, say $K \subseteq \bigcup_{k=1}^{m} V_{y_k}$. Then $f^{-1}(K) \subseteq \bigcup_{k=1}^{m} \bigcup_{j=1}^{m_y} U_{i_{y,j}}$.
Step 2: Thus, it suffices to prove that π is a closed map. Take a closed set $C \subseteq E(G_k(\xi))$. We want to prove that $M \setminus \pi(C)$ is open. Let $p \in M \setminus \pi(C)$. We take a local trivialization $U_p \ni p$ and $\phi : \pi^{-1}(U_p) \xrightarrow{\cong} U_p \times G_{k,n}$. Consider the set $A = \phi(C \cap \pi^{-1}(U_p)) \subseteq U_p \times G_{k,n}$ which is closed in the subspace topology. Since $p \notin \pi(C)$, we have $(p \times G_{k,n}) \cap A = \varnothing$. For each $x \in p \times G_{k,n}$, there exists an open nbd $V_x \times W_x$ disjoint from A (regularity). Since $G_{k,n}$ is compact, $p \times G_{k,n} \subseteq \bigcup_{i=1}^{m} V_{x_i} \times W_{x_i}$. Take $V_p = \bigcap_{i=1}^{n} V_{x_i}$, then $V \cap \pi(A) = \varnothing$. Hence p is an interior point of $M \setminus \pi(A)$. □

Remark. The proof of (ii) generalizes to prove that the bundle projection for any locally trivial fiber bundle with compact fiber is a closed map.

Construction 2.3 (Tautological vector bundle)**.** For each of the constructions above, one can define

$$E(\gamma_k(\xi)) := \{(p, V, v) : p \in M,\ V \in G_k(\xi_p),\ v \in V\}$$

then the projection $\gamma_k(\xi) : E(\gamma_k(\xi)) \to E(G_k(\xi)),\ (p, V, v) \mapsto (p, V)$ is a k-plane bundle over $E(G_k(\xi))$. If M is a point, the this recovers the construction of tautological vector bundle over $G_k(\xi)$. Similarly, if ξ is an oriented vector bundle, then let

$$E(\tilde{\gamma}_k(\xi)) := \{(p, V, v) : p \in M,\ V \in \tilde{G}_k(\xi_p),\ v \in V\}$$

and the projection $\tilde{\gamma}_k(\xi) : E(\tilde{\gamma}_k(\xi)) \to E(\tilde{G}(\xi))$ is a k-plane bundle.

3 Cohomology of Fiber Bundles

The following theorem gives us an important tool to compute the cohomology of the total space of a fiber bundle in terms of the cohomology of the fiber and the base. To simplify the notation, we will assume $H^*(-) = H^*_{dR}(-)$ unless stated otherwise.
If $\pi : \tilde{E} \to M$ is a smooth fiber bundle over a manifold M with fiber F, $H^*(\tilde{E})$ has the structure of a $H^*(M)$-module via the product $\pi^*(-) \wedge - : H^*(M) \otimes_{\mathbb{R}} H^*(E) \to H^*(E)$.

Theorem 3.1 (Leray-Hirsch)**.** Let $\pi : \tilde{E} \to M$ be a smooth fiber bundle over M with fiber F. Suppose there exists cohomology classes $e_j \in H^{n_j}(\tilde{E})$ such that for all $p \in M$, $\{i_p^*(e_j)\}$ form a basis of $H^*(F_p)$. Then, the map

$$\Phi : H^*(M) \otimes_{\mathbb{R}} H^*(F) \to H^*(\tilde{E}),\ \sum_{k,j} \beta_k \otimes i^*(e_j) \mapsto \sum_{k,j} \pi^*(\beta_k) \wedge e_j$$

is an isomorphism of graded $\mathbb{R}$-vector spaces. Moreover, Φ is an isomorphism of $H^*(M)$-module which exhibits $H^*(\tilde{E})$ is a free $H^*(M)$-module with basis e_j.

Remark 3.2.

(i) The statement still holds if one replaces de Rham cohomology by singular cohomology with coefficients in a ring R and require $H^*(F;R)$ to be a free R-module.

(ii) The theorem does not assert that the map Φ is an isomorphism of graded rings.

(iii) The condition of the theorem is non-trivial. Consider the Hopf fibration $S^1 \to S^3 \to S^2$. Then the degree 1 class of S^1 does not arise as the restriction of any class on S^3.

The proof relies on the following technical lemma which we have used in the proof of Poincare duality.

Technical Lemma 1. Let M be a smooth n-manifold equipped with an open cover $\mathcal{V} = \{V_\beta : \beta \in B\}$. Suppose $\mathcal{U}$ is a collection of open subsets of M that satisfies the following conditions:

(i) $\varnothing \in \mathcal{U}$.

(ii) If $U \subseteq V_\beta$ is diffeomorphic to $\mathbb{R}^n$, then $U \in \mathcal{U}$.

(iii) If $U_1, U_2, U_1 \cap U_2 \in \mathcal{U}$, then $U_1 \cup U_2 \in \mathcal{U}$.

(iv) If $(U_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint open subsets with $U_n \in \mathcal{U}$ for all $n \in \mathbb{N}$, then $\bigcup_n U_n \in \mathcal{U}$

Then, $M^n \in \mathcal{U}$.

Proof of Leray-Hirsch. Let $\mathcal{V}$ be an open cover of M by trivializing neighborhoods. Let $\mathcal{U}$ be the collection of open subsets of M for which the theorem holds. Condition (i) is trivial.

To verify condition (ii), fix some $V_\beta \in \mathcal{V}$ and consider $U \subseteq V_\beta$ diffeomorphic to $\mathbb{R}^n$. In this case, we have $H^*(U) \cong H^*(\mathbb{R}^n)$, so $\Phi : \mathbb{R} \otimes H^*(F) \to H^*(\pi^{-1}U) \cong H^*(U \times F) \cong H^*(F)$ is given by $\sum_j \lambda_j \otimes i^*(e_j|_{\pi^{-1}U}) \mapsto \sum_j \lambda_j \wedge e_j|_{\pi^{-1}U}$, where $\lambda \in H^0(U)$. This is clearly an isomorphism.

To verify condition (iii), we use MV sequence. Suppose $U_1, U_2, U_1 \cap U_2 \in \mathcal{U}$. To simplify notation, write $U = U_1 \cup U_2$ and $U_{12} = U_1 \cap U_2$. Let E_1, E_2, E_{12}, E be the total space of the bundle when restricted U_1, U_2, U_{12}, U, respectively. Consider the MV sequence of the triad $(U; U_1, U_2)$.

$$\cdots \longrightarrow H^p(U) \xrightarrow{(j_1^*, j_2^*)} H^p(U_1) \oplus H^p(U_2) \xrightarrow{i_1^* - i_2^*} H^p(U_{12}) \xrightarrow{\delta^*} H^{p+1}(U) \longrightarrow \cdots$$

For each q, $H^q(F)$ is an $\mathbb{R}$-vector space, so tensoring with $H^q(F)$ preserves exactness, so we get the following LES

$$\begin{array}{l} \cdots \longrightarrow H^p(U) \otimes H^q(F) \longrightarrow (H^p(U_1) \otimes H^q(F)) \oplus (H^p(U_2) \otimes H^q(F)) \\ \qquad \swarrow \\ H^p(U_{12}) \otimes H^q(F) \longrightarrow H^{p+1}(U) \otimes H^q(F) \longrightarrow \cdots \end{array}$$

Taking direct sums over p and q, one obtains the following.[2]

$$\begin{array}{l} \cdots \longrightarrow \bigoplus_{p+q=n} H^p(U) \otimes H^q(F) \longrightarrow \bigoplus_{p+q=n} (H^p(U_1) \otimes H^q(F)) \oplus (H^p(U_2) \otimes H^q(F)) \\ \qquad \swarrow \\ \bigoplus_{p+q=n} H^p(U_{12}) \otimes H^q(F) \longrightarrow \bigoplus_{p+q=n} H^{p+1}(U) \otimes H^q(F) \longrightarrow \cdots \end{array}$$

We claim that the map Φ gives a map between the LES above and the MV sequence of the triad $(E; E_1, E_2)$. This amounts to checking the commutativity of some diagrams.

$$\begin{array}{ccccc} \bigoplus_{p+q=n} H^p(U) \otimes H^q(F) & \longrightarrow & \bigoplus_{p+q=n} (H^p(U_1) \otimes H^q(F)) \oplus (H^p(U_2) \otimes H^q(F)) & \longrightarrow & \bigoplus_{p+q=n} H^p(U_{12}) \otimes H^q(F) \\ \Phi \downarrow & & \Phi \downarrow & & \downarrow \Phi \\ H^n(E) & \longrightarrow & H^n(E_1) \oplus H^n(E_2) & \longrightarrow & H^n(E_{12}) \end{array}$$

We must check that this diagram commutes. By assumption, there exist cohomology classes $e_j \in H^*(\tilde{E})$ which restrict to a basis of $H^*(F)$ over any point $p \in M$. If we let $e_j|_E$ denote the class e_j restricted

[2] We allow p, q negative.

to E, then $e_j|_U$ also restricts to a basis of $H^*(F)$ over any $p \in U$. Take an elementary tensor $\beta \otimes \alpha \in H^p(U) \otimes H^q(F)$. We compute

$$\begin{aligned}(i_1^*, i_2^*)(\Phi(\beta \otimes i^*(e_j|_E))) &= (i_1^*, i_2^*)(\pi^*(\beta) \wedge e_j|_E) \\ &= (i_1^*\pi^*(\beta) \wedge i_1^*(e_j|_E), i_2^*\pi^*(\beta) \wedge i_2^*(e_j|_E)) \\ &= (\pi^*(i_1^*\beta) \wedge e_j|_{E_1}, \pi^*(i_2^*\beta) \wedge e_j|_{E_2})\end{aligned}$$

In the last equality, we used that the following diagram of spaces commute.

$$\begin{array}{ccc} E_i & \rightarrowtail & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U_i & \rightarrowtail & U \end{array}$$

This shows that the left square commutes. One can apply the same argument to show that the other square commutes as well.

It remains to verify that Φ commutes with the coboundary operator.

$$\begin{array}{ccc} \bigoplus_{p+q=n} H^p(U_{12}) \otimes H^q(F) & \xrightarrow{\delta^*} & \bigoplus_{p+q=n} H^{p+1}(U) \otimes H^q(F) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ H^n(E_{12}) & \xrightarrow[\tilde{\delta}^*]{} & H^{n+1}(E) \end{array}$$

Take a class $\omega \otimes i^*\alpha \in \bigoplus_{p+q=n} H^p(U_{12}) \otimes H^q(F)$ and pick a smooth partition of unity $\{\rho_1, \rho_2\}$ subordinate to $\{U_1, U_2\}$ (so that $\{\pi^*\rho_1, \pi^*\rho_2\}$ is a smooth partition of unity subordinate to $\{E_1, E_2\}$). Recall[3] that

$$\delta^*\omega = \begin{cases} d(\rho_1\omega) & \text{on } U_1 \\ -d(\rho_2\omega) & \text{on } U_2 \end{cases}$$

By direct computation, we have

$$\Phi(\delta^*(\omega \otimes i^*\alpha)) = \pi^*\delta^*\omega \wedge \alpha = \begin{cases} \pi^*d(\rho_1\omega) \wedge \alpha = d\pi^*(\rho_1\omega) \wedge \alpha & \text{on } U_1 \\ -\pi^*d(\rho_2\omega) \wedge \alpha = d\pi^*(\rho_2\omega) \wedge \alpha & \text{on } U_2 \end{cases}$$

and

$$\tilde{\delta}^*(\Phi(\omega \otimes i^*\alpha)) = \tilde{\delta}^*(\pi^*\omega \wedge \alpha) = \begin{cases} d((\pi^*\rho_1)\pi^*\omega \wedge \alpha) = d(\pi^*(\rho_1\omega) \wedge \alpha) = d\pi^*(\rho_1\omega) \wedge \alpha & \text{on } E_1 \\ d((\pi^*\rho_2)\pi^*\omega \wedge \alpha) = d(\pi^*(\rho_2\omega) \wedge \alpha) = d\pi^*(\rho_2\omega) \wedge \alpha & \text{on } E_2 \end{cases}$$

In the last equality we used the fact that α is closed. This completes the proof of commutativity. By hypothesis, Φ is an isomorphism for U_1, U_2, U_{12}, so we conclude that Φ is an isomorphism for U by 5-lemma, so $U = U_1 \cup U_2 \in \mathcal{U}$.

To verify condition (iv), note that if $(U_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{U}$, then we have natural isomorphism of cohomology $H^k(\bigcup_n U_n) \cong \prod_n H^k(U_n)$, and the map Φ is an isomorphism on each factor, so $\bigcup_n U_n \in \mathcal{U}$.

Thus, $M \in \mathcal{U}$, and we are done. □

4 Applications

Calculation of the cohomology of projective bundles:

Proposition 4.1. Let $\xi : E(\xi) \to M$ be a complex vector bundle of rank n. Then $H^*(E(P(\xi)))$ is a free $H^*(M)$-module with basis $\{1, c, c^2, ..., c^{n-1}\}$, where $c = c_1(\gamma_1(\xi))$.

Proof. Consider the following pullback diagram.

$$\begin{array}{ccc} E(\gamma_1) & \longrightarrow & E(\gamma_1(\xi)) \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{C}P^{n-1} & \longrightarrow & E(P(\xi)) \end{array}$$

[3] It seems that this was not mentioned in talk 4.

We observe that the tautological line bundle over $E(P(\xi))$ restricts to tautological line bundle over $\mathbb{C}P^{n-1}$. By naturality of Chern class, we know that $c_1(\gamma_1(\xi)) \in H^2(E(P(\xi)))$ restricts to $c_1(\gamma_1) \in H^2(\mathbb{C}P^{n-1})$, so $c_1(\gamma_1(\xi))^j$ restricts to $c_1(\gamma_1)^j \in H^{2j}(\mathbb{C}P^{n-1})$. Since $H^*(\mathbb{C}P^{n-1}) \cong \mathbb{R}[c_1(\gamma_1)]/(c_1(\gamma)^n)$, we are exactly in the situation of Leray-Hirsch, and the result follows. ☐

Remark 4.2 (cf. Proposition 23.2 Bott-Tu [1])**.** In general, the tautological k-plane bundle γ_k over $G_k(\mathbb{C}^n)$ embeds as a subbundle of the trivial bundle $G_k(\mathbb{C}^n) \times \mathbb{C}^n$, so it makes sense to consider the orthogonal complement $\gamma_k^\perp$. In [1], some calculation using tools that we do not have time to introduce shows that $H^*(G_k(\mathbb{C}^n)) \cong \mathbb{R}[c_1, ..., c_k, \tilde{c}_1, ..., \tilde{c}_{n-k}]/((c_1 + \cdots c_k)(\tilde{c}_1 + \cdots + \tilde{c}_{n-k}) - 1)$, where $c_i = c_i(\gamma_k)$ and $\tilde{c}_i = c_i(\gamma_k^\perp)$. Since the cohomology is generated by Chern classes which are natural w.r.t. pullback, one can perform the same argument to show that $H^*(E(G_k(\xi)))$ is a free $H^*(M)$-module with basis given by Chern classes.

Theorem 4.3 (Splitting principle)**.** For any complex vector bundle $\xi : E \to M$, there exists a manifold $T = T(\xi)$ and a proper smooth map $f : T \to M$ s.t.

(i) $f^* : H^*(M) \to H^*(T)$ is injective;

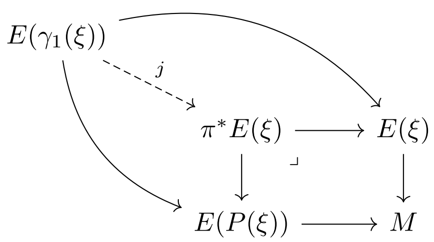
(ii) $f^*\xi$ splits as a direct sum of complex line bundles.

Proof. Observe that if we can show that there exist T and proper smooth map $f^* : T \to M$ s.t. f^* is injective and $f^*\xi \cong \xi' \oplus L$ for some line bundle L and vector bundle ξ', then we are done by repetitively applying this procedure.

Let's take $T = E(P(\xi))$ and $f = \pi : E(P(\xi)) \to M$, which is a smooth proper map (lemma 2.2). Consider the pullback square:

$$\begin{array}{ccc} \pi^*E(\xi) & \xrightarrow{\tilde{\pi}} & E(\xi) \\ \downarrow & \lrcorner & \downarrow \xi \\ E(P(\xi)) & \xrightarrow{\pi} & M \end{array}$$

(i): Proposition 4.2 implies that π^* is injective in every degree.
(ii): There is an injective bundle map $j : E(\gamma_1(\xi)) \to \pi^*E(\xi)$ given by universal property:



More explicitly the map j sends (p, L, v) to $((p, L), v)$. Its image $L \cong \gamma_1$ is a subbundle of $\pi^*\xi$, and it's a line bundle. If we let ξ' be the orthogonal complement of L in $\pi^*\xi$, then $\pi^*\xi \cong \xi' \oplus L$. ☐

Remark 4.4. By proposition 4.1, we have some relation[4]

$$(-1)^n c^n + \lambda_{n-1}(\xi)c^{n-1} + \lambda_{n-2}(\xi)c^{n-2} + \cdots + \lambda_0(\xi) = 0$$

In the proof above, we saw that $\pi^*\xi \cong \xi' \oplus \gamma_1(\xi)$, so the total chern class satisfies $\pi^*(c(\xi)) = c(\pi^*\xi) = c(\gamma(\xi))c(\xi') = (1 + c)c(\xi')$. We can solve for $c(\xi')$:

$$c(\xi') = \pi^*(c(\xi))(1 + c)^{-1} = \pi^*(c(\xi)) \sum_{j \geq 0} (-1)^j c^j$$

By considering elements of degree $2n$, we get

$$0 = c_n(\xi') = \sum_{i+j=n} \pi^*(c_i(\xi)) \wedge (-1)^j c^j$$

By matching coefficients,

$$\lambda_j(\xi) = (-1)^j c_{n-j}(\xi)$$

[4]This differs from the book [3] by a sign

Corollary 4.5. There is a ring isomorphism

$$H^*(E(P(\xi))) \cong H^*(M)[c]/(c_n(\xi) - c_{n-1}(\xi)c + \cdots + (-1)^{n-1}c_1(\xi)c^{n-1} + (-1)^n c^n)$$

Proof. The relation derived in Remark 4.5 is the only non-trivial one. Suppose we have another relation given by some polynomial equation $p(c) = 0$ for $p \in H^*(M)[X]$, then it descends to a relation $\tilde{p}(c) = 0$, where $\tilde{p} = p \mod c_n(\xi) - c_{n-1}(\xi)X + \cdots + (-1)^{n-1}c_1(\xi)X^{n-1} + (-1)^n X^n$, then $\tilde{p} = 0$ since $\{1, c, ..., c^{n-1}\}$ is a free $H^*(M)$-basis. □

There is a splitting principle for oriented real vector bundles. This requires the calculation of $H^*(\tilde{G}_2(\mathbb{R}^n))$, which can be done by a geometric argument.

There is an embedding $j : \mathbb{R}P^{n-1} \hookrightarrow \mathbb{C}P^{n-1}$, $[x_1 : \cdots : x_n] \mapsto [x_1 : \cdots : x_n]$, so we can think of $\mathbb{R}P^{n-1}$ as a submanifold of $\mathbb{C}P^{n-1}$.

Proposition 4.6. There is a homotopy equivalence $\tilde{G}_2(\mathbb{R}^n) \simeq W_n := \mathbb{C}P^{n-1} \setminus \mathbb{R}P^{n-1}$

Lemma 4.7. Let $Q = \{A \in GL_2(\mathbb{R}) : A^T = A,\ \det A = 1,\ \operatorname{tr} A > 0\}$, i.e., the space of positive definite symmetric matrices with determinant 1.

(i) There is a homeomorphism $\psi : Q \times \mathbb{C}^\times \to GL_2^+(\mathbb{R})$, $(A, re^{i\theta}) \mapsto rAR_\theta$.

(ii) Q is contractible via $F(A, t) = A^t$ (exponentiation). Moreover, this homotopy is equivariant with respect to the action of $S^1 \cong SO(2)$ on Q by conjugation.

Sketch of proof. (i): This is the statement that polar decomposition exists and is unique.

(ii) If $A \in Q$, then by spectral theorem, $\mathbb{R}^2$ has orthogonal decomposition in terms of eigenspaces, i.e., $\mathbb{R}^2 = V_\lambda \overset{\perp}{\oplus} V_{\lambda^{-1}}$, where $\lambda > 0$. If P denotes the orthogonal projection onto V_λ, then $A = \lambda P + \lambda^{-1}(I - P)$. Then $A^t = \lambda^t P + \lambda^{-t}(I - P) = \lambda^{-t}I + (\lambda^t - \lambda^{-t})P$ at least away form I, and this has a continuous extension to I. To see that this map is equivariant w.r.t. the action of S^1, we compute

$$F(R_\theta^{-1}AR_\theta, t) = \lambda^{-t}I + (\lambda^t - \lambda^{-t})R_\theta^{-1}PR_\theta = R_\theta^{-1}F(A, t)R_\theta$$

The first equality follows from change of basis. □

Proof of 4.7 (Sketch). We work with the embedding $\varphi : V_{2,n} \to S^{2n-1} \subseteq \mathbb{C}^n \setminus 0$, $(x, y) \mapsto \frac{1}{\sqrt{2}}(x - iy)$. This map equivariant with respect to the action of S^1 on $V_{2,n}$ by rotation matrices and the action on S^{2n-1} by multiplication and it descends to an embedding $\tilde{\varphi} : \tilde{G}_{2,n} \to \mathbb{C}P^{n-1}$. This information can be summarized in the following diagram of maps of fiber bundles.

$$\begin{array}{ccccc}
S^1 & \xrightarrow{\text{id}} & S^1 & \xrightarrow{\text{incl}} & \mathbb{C}^\times \\
\downarrow & & \downarrow & & \downarrow \\
V_{2,n} & \xrightarrow{\varphi} & S^{2n-1} & \xrightarrow{\text{incl}} & \mathbb{C}^n \setminus 0 \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\
\tilde{G}_{2,n} & \xrightarrow[\tilde{\varphi}]{} & \mathbb{C}P^{n-1} & \xrightarrow[\text{id}]{} & \mathbb{C}P^{n-1}
\end{array}$$

To see that the image of the embedding $\tilde{\varphi}$ lies in W_n, we note that $\mathbb{C}P^{n-1}$ has an involution given by complex conjugation, whose fixed point set is the embedded copy of $\mathbb{R}P^{n-1}$ and the imaginary part of anything in $\tilde{\varphi}(\tilde{G}_{2,n})$ cannot be zero. Define

$$\Phi : V_{2,n} \times GL_2^+(\mathbb{R}) \to \mathbb{C}^n \setminus 0,$$
$$\left((x, y), \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) \mapsto (ax + by) - i(cx + dy)$$

Note that when the matrix is $2^{-1/2}I$, we recover φ, so we can think of φ as a slice of Φ. One can check that $\operatorname{im} \Phi = \pi^{-1}W_n$.

Note that S^1 acts on $V_{2,n} \times GL_2^+(\mathbb{R})$ by change of coordinates, which means $(x, y, A) \cdot R_\theta := ((x, y)R_\theta, R_\theta^{-1}A)$. Note that each orbit of this action parametrizes exactly the fiber over $\Phi(x, y, A)$, so we get a bijection $\Psi : V_{2,n} \times_{S^1} GL_2^+(\mathbb{R}) \to \pi^{-1}W_n \subseteq \mathbb{C}^n \setminus 0$. We note that $re^{i\theta} \in \mathbb{C}^\times$ acts on $V_{2,n} \times GL_2^+(\mathbb{R})$ by $(x, y, A) \mapsto (x, y, A(rR_\theta))$, which induces an action on the orbit space $V_{2,n} \times_{S^1} GL_2^+(\mathbb{R})$, and the map

Ψ is equivariant w.r.t. this action and the standard action on $\mathbb{C}^n \setminus 0$ so that this descends to a map to W_n. This gives the following commutative diagram, where the horizontal maps are diffeomorphisms.

$$\begin{array}{ccc} V_{2,n} \times_{S^1} GL_2^+(\mathbb{R}) & \xrightarrow{\Psi} & \pi^{-1}W_n \\ \downarrow & & \downarrow \\ (V_{2,n} \times GL_2^+(\mathbb{R}))/\mathbb{C}^\times & \xrightarrow[\tilde{\Psi}]{} & W_n \end{array}$$

Recall from lemma 4.8 that $\psi : Q \times \mathbb{C}^\times \cong GL_2^+(\mathbb{R})$, which induces a diffeomorphism $V_{2,n} \times GL_2^+(\mathbb{R}) \cong V_{2,n}\times(Q\times\mathbb{C}^\times)$. Define an action of S^1 on $V_{2,n}\times Q\times\mathbb{C}^\times$ by $((x,y),B,re^{i\gamma})\cdot R_\theta = ((x,y)R_\theta, R_\theta^{-1}BR_\theta, re^{i\gamma})$.

$$\begin{array}{ccc} V_{2,n} \times Q \times \mathbb{C}^\times & \xrightarrow{\mathrm{id}\times\psi} & V_{2,n} \times GL_2^+(\mathbb{R}) \\ \downarrow & & \downarrow \\ V_{2,n} \times_{S^1} Q \times \mathbb{C}^\times & \longrightarrow & V_{2,n} \times_{S^1} GL_2^+(\mathbb{R}) \\ \downarrow & & \downarrow \\ V_{2,n} \times_{S^1} Q & \xrightarrow[\cong]{} & (V_{2,n} \times_{S^1} GL_2^+(\mathbb{R}))/\mathbb{C}^\times \end{array}$$

$$\begin{array}{ccc} V_{2,n} \times_{S^1} GL_2^+(\mathbb{R}) & \xrightarrow{\Psi} & \pi^{-1}W_n \\ \downarrow & & \downarrow \pi \\ (V_{2,n} \times_{S^1} GL_2^+(\mathbb{R}))/\mathbb{C}^\times & \xrightarrow{\tilde{\Psi}} & W_n \\ \cong \uparrow & \nearrow_{\tilde{\Phi}} & \\ V_{2,n} \times_{S^1} Q & & \end{array}$$

(Diagram on the left) The map $\mathrm{id}\times\psi$ is equivariant w.r.t. the actions of S^1. To see this, note that for $A \in GL_2^+(\mathbb{R})$, the symmetric part of the unique polar decomposition is given by $B = \det(A)^{-1}(AA^\dagger)^{1/2}$. The action of $e^{i\theta} \in S^1$ on $V_2 \times GL_2^+(\mathbb{R})$ on A translates to

$$\det(R_\theta^{-1}A)^{-1}((R_\theta^{-1}A)(R_\theta A)^\dagger)^{1/2} = \det(A)^{-1}(R_\theta^{-1}AA^\dagger R_\theta)^{1/2} = \det(A)^{-1}R_\theta^{-1}(AA^\dagger)^{1/2}R_\theta = R_\theta^{-1}BR_\theta$$

where the second to last equality uses the fact that $(R_\theta^{-1}AA^\dagger R_\theta)^{1/2}$ and $R_\theta(AA^\dagger)^{1/2}R_\theta$ are both positive definite square root of $R_\theta AA^\dagger R_\theta$, so they must be equal. This map first descends to a diffeomorphism $V_{2,n}\times_{S^1} Q\times\mathbb{C}^\times \cong V_{2,n}\times_{S^1} GL_2^+(\mathbb{R})$. Then we check that this map is equivariant w.r.t. the action of $\mathbb{C}^\times$ on the $\mathbb{C}^\times$ factor of the domain and the $\mathbb{C}^\times$-action on the target defined earlier and get a diffeomorphism $V_{2,n}\times_{S^1} Q \cong (V_{2,n}\times_{S^1} GL_2^+(\mathbb{R}))/\mathbb{C}^\times$. (Diagram on the right) We define $\tilde{\Phi}$ to be the map induced from $\tilde{\Psi}$ using the diffeomorphism obtained by the left diagram, so $\tilde{\Phi}$ is a diffeomorphism. Using the above construction, one can obtain an explicit formula for $\tilde{\Phi}$:

$$\tilde{\Phi}\left[x, y, \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}\right] = [(\alpha x + \beta y) - i(\beta x + \gamma y)] \in \mathbb{C}P^{n-1}$$

We now analyze the space $V_{2,n} \times_{S^1} Q$. The conjugation action on Q fixes the identity matrix, so we have an inclusion $i_0 : \tilde{G}_{2,n} \cong (V_{2,n} \times \{I\})/S^1 \hookrightarrow W_n$ given by $i_0[x, y, I] = [x - iy]$, which coincides with the map $\tilde{\varphi} : \tilde{G}_{2,n} \hookrightarrow \mathbb{C}P^{n-1}$ defined earlier. We claim that i_0 is a homotopy equivalence. For this purpose, we consider

$$\mathrm{id}\times F : V_{2,n} \times Q \times [0,1] \to V_{2,n} \times Q$$

Using the equivariance part of lemma 4.8(ii), this descends to a well-defined map

$$V_{2,n} \times_{S^1} Q \times [0,1] \to V_{2,n} \times_{S^1} Q$$

which is a homotopy between the identity and the inclusion of the subspace $(V_{2,n} \times \{I\})/S^1$. This proves the proposition. □

Lemma 4.8. Let $i : W_n \to \mathbb{C}P^{n-1}$ be the inclusion. We get an induced map $i_* : H_c^p(W_n) \to H^p(\mathbb{C}P^{n-1})$. i_* is an isomorphism except the following cases:

(i) if $p = 0$, then $H_c^p(W_n) = 0$;

(ii) if $p = n$ and n is even, then we have SES

$$0 \to \mathbb{R} \to H_c^n(W_n) \to H^n(\mathbb{C}P^{n-1}) \to 0$$

Proof (Sketch). We have exact sequence[5]

$$\cdots \longrightarrow H^p_c(W_n) \xrightarrow{i_*} H^p(\mathbb{C}P^{n-1}) \xrightarrow{j^*} H^p(\mathbb{R}P^{n-1}) \xrightarrow{\delta} H^{p+1}_c(W_n) \longrightarrow \cdots$$

(1) For $p \neq 0, n$, we have $H^p(\mathbb{R}P^{n-1}) = H^{p-1}(\mathbb{R}P^{n-1}) = 0$, so i_* is an iso.

(2) When $p = 0$, $H^p_c(W_n) = 0$ since W_n is not compact.

(3) For $p = n$ and n odd, $H^{n-1}(\mathbb{R}P^{n-1}) \cong 0$ and i_* is still an isomorphism.

(4) For $p = n$ and n even, $H^{n-1}(\mathbb{R}P^{n-1}) \cong \mathbb{R}$, so we get the desired SES.

□

Remark. If M is a smooth compact manifold and N is a smooth compact submanifold of M, then there is an exact sequence of (co)chain complexes:

$$0 \to \Omega^*(M, N) \to \Omega^*(M) \xrightarrow{j^*} \Omega^*(N) \to 0$$

where j is the inclusion and $\Omega^*(M, N)$ is defined to be $\ker j^*$, i.e., those forms on M which restrict to 0 on N. This gives rise to a LES of cohomology. The inclusion $i : M \setminus N \to M$ induces a map $i_* : \Omega^*_c(M \setminus N) \to \Omega^*(M, N)$ called extension by 0. It can be shown that i_* induces isomorphism on cohomology, so one can replace the term $H^*(\Omega^*(M, N))$ by $H^*_c(M \setminus N)$, which gives exactly the LES used in the proof of this lemma.

Proposition 4.9. We have $H^{2p-1}(W_n) = 0$ for all p, and

$$H^{2p}(W_n) \cong \begin{cases} \mathbb{R}^2 & 2p = m-2 \\ \mathbb{R} & 0 \leq 2p \leq 2m-4 \text{ and } 2p \neq 2m-2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Note that W_n is an open subset of $\mathbb{C}P^{n-1}$, so an orientation of $\mathbb{C}P^{n-1}$ restricts to an orientation on W_n. This means that Poincare duality applies. Consider the following commutative diagram[6]

$$\begin{array}{ccc} H^p(\mathbb{C}P^{n-1}) & \xrightarrow{i^*} & H^p(W_n) \\ \downarrow \cong & & \downarrow \cong \\ H^{2n-2-p}(\mathbb{C}P^{n-1})^* & \xrightarrow{(i_*)^*} & H^{2n-2-p}_c(W_n)^* \end{array}$$

(1) If $p \neq 2n-2,\ n-2$, lemma 4.9 implies that $H^p(\mathbb{C}P^{n-1}) \cong H^p(W_n)$.

(2) W_n is path-connected by the homotopy equivalence proved earlier (Proposition 4.7), so $H^0(W_n) \cong \mathbb{R}$.

(3) If $p = n-2$ and n odd, then i^* is still an isomorphism.

(4) If $p = n-2$ and n even, then we note that the SES from the preceding lemma splits, so $H^n_c(W_n) \cong \mathbb{R}^2$. By Poincare duality, $H^n(W_n) \cong \mathbb{R}^2$.

By (1), (3), the odd cohomology groups are trivial. By (1), (2), (4), we get the desired pattern of even cohomology groups. □

To prove the next result about $H^*(\tilde{G}_2(\mathbb{R}^n))$, we need Pontrjagin classes.[7]

Definition 4.10. Suppose ξ is a real vector bundle of rank n over M. We define the k-th Pontrjagin class of ξ by

$$p_k(\xi) = (-1)^k c_{2k}(\xi_{\mathbb{C}}) \in H^{4k}(M)$$

The total Pontrjagin class is defined as the formal sum

$$p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \cdots$$

[5]Proposition 13.11 in [3] but was not introduced in the previous talks.

[6]cf. lemma 3.3.3 talk 7, or chapter 13 of [3].

[7]Various properties of Pontrjagin classes are introduced in chapter 18 and 19 of [3]. They were not mentioned in the preceding talks, so we will quickly define the concepts and collect some useful properties.

Recall from talk 11 that the odd Chern classes of the complexification of a real vector bundle are trivial, so we only use the even Chern classes in the definition above.

We collect some useful properties of Pontrjagin classes in the following proposition.

Proposition 4.11.

(i) The Pontrjagin class is natural with respect to pullback. The total Pontrjagin class is exponential, that is, for real vector bundles ξ and η, $p(\xi \oplus \eta) = p(\xi)p(\eta)$.

(ii) If ξ is an oriented real vector bundle of rank $2k$, then $p_k(\xi) = e(\xi)^2$

Sketch of proof. (i) is essentially a consequence of the same statement for Chern classes. (ii) is proposition 19.9 of [3]. □

Proposition 4.12. Let $c = e(\tilde{\gamma}_2)$ be the Euler class of the tautological 2-plane bundle over $\tilde{G}_2(\mathbb{R}^n)$. When n is even, let $e = e(\tilde{\gamma}_2^\perp)$.

(i) If n is odd and $n \geq 3$, then $H^*(\tilde{G}_{2,n}) \cong \mathbb{R}[c]/(c^{n-1})$.

(ii) If n is even and $n \geq 4$, then $H^*(\tilde{G}_{2,n}) \cong \mathbb{R}[c, e]/(c^{n-1}, ce, e^2 + (-1)^{n/2}c^{n-2})$

Proof. (i): Let n be odd and $n \geq 3$. We discover from the proof of the preceding proposition that $i^* : H^*(\mathbb{C}P^{n-1}) \to H^*(W_n)$ is bijective other than in degree $2n - 2$. This immediately gives us the cohomology ring structure on W_n and thus $\tilde{G}_{2,n}$ by homotopy invariance. It suffices to show that the first Chern class $c_1(\gamma_1) \in H^2(\mathbb{C}P^{n-1})$ pulls back to $c = e(\tilde{\gamma}_2)$. For this purpose we consider the following diagram.

$$\begin{array}{ccc} E(\tilde{\gamma}_2) & \longrightarrow & E(\gamma_1) \\ \downarrow & & \downarrow \\ \tilde{G}_{2,n} & \rightarrowtail & \mathbb{C}P^{n-1} \end{array}$$

The top map is given by $(V = \text{span}(v_1, v_2), \lambda v_1 + \mu v_2 \in V) \mapsto (\mathbb{C}\langle v_1 - iv_2\rangle, (\lambda + i\mu)(v_1 - iv_2))$ In fact, this is a pullback square, given by restricting the tautological line bundle (regarded as a real 2-plane bundle) to the embedded copy of $\tilde{G}_{2,n}$. By naturality, the Euler class $c_1(\gamma_1)$ pulls back to $c = e(\tilde{\gamma}_2)$ along the embedding.

(ii): By the same argument as (i), the relation $c^{m-1} = 0$ holds in $H^*(\tilde{G}_{2,n})$. The relation $ce = 0$ follows from the fact that $\tilde{\gamma}_2 \oplus \tilde{\gamma}_2^\perp = \varepsilon^n$. To establish $e^2 + (-1)^{n/2}c^{n-2} = 0$, we use Pontrjagin class. We have

$$(1 + p_1(\tilde{\gamma}_2))(1 + p_1(\tilde{\gamma}_2^\perp) + \cdots + p_{n/2-1}(\tilde{\gamma}_2^\perp)) = 1$$

since $\tilde{\gamma}_2 \oplus \tilde{\gamma}_2^\perp = \varepsilon^n$. This gives us the relation $p_j(\tilde{\gamma}_2^\perp) = (-1)^j p_1(\tilde{\gamma}_2)^j$. By proposition 4.11(ii), we also have

$$e^2 = p_{n/2-1}(\tilde{\gamma}_2^\perp) = (-1)^{n/2-1}p_1(\tilde{\gamma}_2)^{n/2-1}$$

By naturality, $p_1(\tilde{\gamma}_2)$ is the pullback of the first Pontrjagin class of $(\gamma_1)_\mathbb{R}$ over $\mathbb{C}P^{n-1}$, regarded as a real vector bundle. We have the direct sum decomposition $(\gamma_1)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \cong \gamma_1 \oplus \gamma_1^*$ (lemma 16.19(ii) of [3]), so we can compute $p_1((\gamma_1)_\mathbb{R}) = -c_1(\gamma_1)(-c_1(\gamma_1)) = c_1(\gamma_1)^2$. Hence, $p_1(\tilde{\gamma}_2) = c^2$. Substitute, and get $e^2 = (-1)^{n/2-1}c^{n-2}$, i.e., $e^2 + (-1)^{n/2}c^{n-2} = 0$.

We claim that $1, c, c^2, ..., c^{n-2}, e$ form an additive basis of $H^*(\tilde{G}_{2,n})$. To see this, note that the only way this could fail is when e and the class $c^{n/2-1}$ are linearly dependent, so let's assume $c^{n/2-1} = \lambda e$ for some $\lambda \in \mathbb{R}$. Then $c^{n/2} = \lambda ce = 0$, but this is a contradiction. Now, by comparing vector space dimension, we deduce that $H^*(\tilde{G}_{2,n}) \cong \mathbb{R}[c, e]/(c^{n-1}, ce, e^2 + (-1)^{n/2}c^{n-2})$. □

Proposition 4.13. For any oriented real vector bundle ξ over M, $H^*(\tilde{G}_2(\xi))$ is a free $H^*(M)$-module with basis

$$\begin{cases} 1, e(\xi), e(\xi)^2, ..., e(\xi)^{m-2}, e(\xi^\perp) & \text{if } m = 2n \geq 4 \\ 1, e(\xi), e(\xi)^2, ..., e(\xi)^{m-2} & \text{if } m = 2n+1 \geq 3 \end{cases}$$

Proof. Apply Leray-Hirsch. □

Theorem 4.14 (Real splitting principle)**.** For any oriented real vector bundle ξ over M, there exists a manifold $T = T(\xi)$ and a smooth proper map $f : T \to M$ s.t.

(i) $f^* : H^*(M) \to H^*(T)$ is injective,

(ii) $f^*\xi \cong L_1 \oplus \cdots \oplus L_n$ if $\mathrm{rank}(\xi)$ is even, and $f^*\xi \cong L_1 \oplus \cdots \oplus L_n \oplus \varepsilon^1$ if $\mathrm{rank}(\xi)$ is odd, where each L_i is an oriented 2-plane bundle.

Proof. The proof is the same as complex splitting principle. We take $f = \pi : E(\tilde{G}_2(\xi)) \to M$ which is a smooth proper map. The pullback bundle along this map splits as the direct sum of a 2-plane bundle (the image of the tautological bundle) and another vector bundle. Depending on the parity of the rank, we can iterate this procedure until the bundle splits as direct sums of oriented 2-plane bundles or there is a line bundle left, say ξ.

Argument 1: At the last step, we take one more pullback along the projection π of the total space of the unit sphere bundle of ξ. The pullback $\pi^*\xi$ has a nowhere vanishing global section given by $((p, \epsilon), \lambda\epsilon) \mapsto \epsilon$, where $\epsilon = \pm 1$ and $(p, \epsilon) \in S(\xi)$. Hence, after one more pullback, the vector bundle splits as a direct sum of oriented 2-plane bundle and a trivial line bundle.

Argument 2: Alternatively, an orientation of a vector bundle E amounts to a continuous choice of oriented basis of fibers. This is equivalent to the existence of a nowhere-vanishing global section of the bundle $\Lambda^{\mathrm{rk}(E)}(E) \cong \det(E)$. If we have two bundles E and F such that E and $E \oplus F$ are both orientable, then $\varepsilon^1 \cong \Lambda^{\mathrm{rk}(E)+\mathrm{rk}(F)}(E \oplus F) \cong \Lambda^{\mathrm{rk}(E)}(E) \otimes \Lambda^{\mathrm{rk}(F)}(F) \cong \Lambda^{\mathrm{rk}(F)}(F)$, so F is also orientable. Back to this theorem, ξ is an orientable line bundle by the argument using determinant bundle, so it has to be trivial as orientation in the case of line bundle precisely means the existence of a nowhere vanishing global section. □

If time permits, we will also calculate the cohomology ring of some Stiefel manifolds.

Proposition 4.15. For $1 \leq k \leq n$, there exists ring isomorphism

$$H^*(V_k(\mathbb{C}^n)) \cong \Lambda[x_{2n-2k+1}, x_{2n-2k+3}, ..., x_{2n-1}]$$

where $|x_j| = j$.

Proof. We proceed by induction on k. The base case is clear.

Suppose the statement is true for $V_{k-1,n-1}$, i.e., $H^*_{dR}(V_{k-1,n-1}) \cong \Lambda[x_{2n-2k+1}, x_{2n-2k+3}, ..., x_{2n-3}]$.

We claim without proof that the fiber bundle $V_{k-1,n-1} \to V_{k,n} \to S^{2n-1}$ satisfies the hypothesis of Leray-Hirsch theorem. Moreover, the inclusion of fiber $i : V_{k-1,n-1} \to V_{k,n}$ induces isomorphisms $i^* : H^j(V_{k,n}) \to H^j(V_{k-1,n-1})$ for $j \leq 2n-3$. Therefore, the inclusion of fiber completely determines the subring of $H^*(V_{k,n})$ generated by $y_{2n-2k+1}, ..., y_{2n-3}$ as $\Lambda[y_{2n-2k+1}, ..., y_{2n-3}]$. In particular, there exists cohomology classes $y_{2n-2k+1}, ..., y_{2n-3}$ on $V_{k,n}$ which restrict to $x_{2n-2k+1}, ..., x_{2n-3}$, so we are exactly in the situation of Leray-Hirsch. Therefore, $H^*(V_{k,n})$ is a free $H^*(S^{2n-1})$-module with basis given by products of distinct elements from $\{y_{2n-2k+1}, .., y_{2n-3}\}$. Let y_{2n-1} be the pullback of a generator of $H^{2n-1}_{dR}(S^{2n-1})$, then an additive basis of $H^*_{dR}(V_{k,n})$ is given by products of distinct elements from $\{y_{2n-2k+1}, ..., y_{2n-1}\}$.

We claim that $H^*_{dR}(V_{k,n})$ is the exterior algebra on $y_{2n-2k+1}, ..., y_{2n-1}$. There is a surjective ring homomorphism $\Lambda[y_{2n-2k+1}, ..., y_{2n-1}] \to H^*(V_{k,n})$ by sending y_j to the cohomology class with the same label. By comparing $\dim_{\mathbb{R}}$, we see that this is an isomorphism. □

Remark 4.16. It takes a non-trivial amount of work to prove that $V_{k-1,n-1} \to V_{k,n} \to S^{2n-1}$ satisfies the hypothesis of Leray-Hirsch. This is mainly because we do not have the cohomology structure of $V_{k,n}$ to begin with. However, with some help from homotopy theory, one can deduce certain information on the homotopy groups of $V_{k,n}$. Using the Hurewicz map, which relates homotopy groups and integral homology groups (and hence cohomology via universal coefficient theorem), we can actually show that the inclusion of fiber $V_{k-1,n-1} \to V_{k,n}$ induces isomorphism on cohomology groups up to sufficiently high degree so that Leray-Hirsch applies.

Consider the fiber sequence $V_{k-1,n-1} \to V_{k,n} \to S^{2n-1}$. The long exact sequence reads

$$\cdots \to \pi_{j+1}(S^{2n-1}) \to \pi_j(V_{k-1,n-1}) \xrightarrow{i_*} \pi_j(V_{k,n}) \to \pi_j(S^{2n-1}) \to \cdots$$

Claim 1: The space $V_{k,n}$ is simple (the action of π_1 on π_n is trivial for all $n \geq 1$) for $1 \leq k \leq n$. If $k = n$, then both spaces are H-spaces and hence simple. Otherwise, by induction on k (using the LES above), we see that $V_{k,n}$ is simply connected.

The map $i_* : \pi_j(V_{k-1,n-1}) \to \pi_j(V_{k,n})$ is an isomorphism if $j < 2n-2$ and an epimorphism if $j = 2n-2$, so the pair $(V_{k,n}, V_{k-1,n-1})$ is $(2n-2)$-connected. By claim 1, the relative Hurewicz map $h : \pi_j(V_{k,n}, V_{k-1,n-1}) \to H_j(V_{k,n}, V_{k-1,n-1}; \mathbb{Z})$ is an isomorphism if $j \leq 2n-3$.

This argument does not compute $H^*(V_k(\mathbb{R}^n))$. The connectedness of the pair $(V_k(\mathbb{R}^n), V_{k-1}(\mathbb{R}^{n-1}))$ is not sufficient to deduce that the analogous fiber bundle satisfies the hypothesis of Leray-Hirsch.

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