Talk 12. Euler class

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The goal of this talk is to define Euler class of a real vector bundle $\xi\colon E \to M$, and show some of its properties. For this we need two additional tools - Pffafian polynomials and metric connections.

1 Pfaffian

Definition 1.1: Denote the set of skew-symmetric real $n\times n$ matrices by $\mathfrak{so}_n := \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid A^t = -A\}$ and the set of skew-hermitian by $\mathfrak{su}_n := \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \overline{A^t} = -A\}$. Clearly they are closed under scaling and become a group under addition. It is remarkable, that they are not closed under multiplication, but admit the structure of Lie-algebras under $[A, B] = AB - BA$.

Let $A = (A_{ij}) \in \mathfrak{so}_n$. Consider the determinant of A

- if n is odd, then $\det(A) = \det(A^t) = \det(-A) = -\det(A)$, and so $\det(A) = 0$.
- if n is even, say $n = 2k$, then we know from the linear algebra course, that A is similar to the diagonal block matrix
$$B = \mathrm{diag}\left(\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ -b_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_k \\ -b_k & 0 \end{pmatrix}\right)$$
Moreover, we can choose B such that the similarity transformation matrix G is orthogonal. In particular, $\det(A) = \det(B) = b_1^2 b_2^2 \dots b_k^2 = (b_1 \dots b_k)^2$, i.e. the square of some polynomial of degree k in variables $b_1, \dots, b_k$. Dramatic is the fact that the determinant of A is always the square of a polynomial depending on the entries A_{ij} and not only in the special case! Let us construct such a polynomial formally:

For $A \in \mathfrak{so}_{2k+1}$ we set $\mathrm{Pf} = 0$ and for $A \in \mathfrak{so}_{2k}$, we let
$$\omega(A) = \sum_{i,j} A_{ij} e_i \wedge e_j \in \Lambda^2(\mathbb{R}^{2k}),$$
and define $\mathrm{Pf}(A)$ by the equation
$$\underbrace{\omega(A) \wedge \dots \wedge \omega(A)}_{k\text{-times}} = 2^k k! \mathrm{Pf}(A) e_1 \wedge e_2 \wedge \dots \wedge e_{2k} \in \Lambda^{2k}(\mathbb{R}^{2k}) \cong \mathbb{R}.$$
For the block matrix
$$A = \mathrm{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ -a_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix}\right)$$
a simple calculation gives
$$\omega(A) = 2a_1 e_1 \wedge e_2 + 2a_2 e_3 \wedge e_4 + \dots + 2a_k e_{2k-1} e_{2k},$$
and one can see that
$$\omega(A) \wedge \dots \wedge \omega(A) = 2^k k! (a_1 a_2 \dots a_k) e_1 \wedge \dots \wedge e_{2k}$$
it follows that $\mathrm{Pf}(A) = a_1 a_2 \dots a_k$ and $\mathrm{Pf}(A)^2 = \det(A)$ in this case.

Theorem 1.2: If $A \in \mathfrak{so}_{2k}$ and B is an arbitrary matrix then

(i) $\mathrm{Pf}(A)^2 = \det(A)$

(ii) $\mathrm{Pf}(BAB^t) = \mathrm{Pf}(A)\det(B)$

Proof. Because of the remarks above, (i) follows immediately from (ii), namely choose a similar diagonal block matrix with orthogonal transition matrix B, then $\mathrm{Pf}(A)^2 = \mathrm{Pf}(BAB^{-1})^2 = \det(BAB^{-1}) = \det(A)$. In order to prove (ii), we consider the elements $f_i = Be_i = \sum_\nu B_{\nu i} e_i \in \mathbb{R}^{2k}$, we have that

$$\tau = \sum A_{ij} f_i \wedge f_j = \sum B_{\nu i} A_{ij} B_{\mu j} e_\nu \wedge e_\mu = \sum (BAB^t)_{\nu\mu} e_\nu \wedge e_\mu$$

so that $\tau = \omega(BAB^t)$. Hence

$$2^k k! \mathrm{Pf}(BAB^t) e_1 \wedge \cdots \wedge e_{2k} = \omega(BAB^t) \wedge \cdots \wedge \omega(BAB^t) = \tau \wedge \cdots \wedge \tau = 2^k k! \mathrm{Pf}(A) f_1 \wedge \cdots \wedge f_{2k}.$$

and from the linear algebra it is known that $f_1 \wedge \cdots \wedge f_{2k} = \det(B) e_1 \wedge \cdots \wedge e_{2k}$ (just from the definition of the determinant), hence $\mathrm{Pf}(BAB^t) = \mathrm{Pf}(A)\det(B)$.

□

Corollary 1.3: For any $A, B \in \mathfrak{so}_k$, we have $\mathrm{Pf}(A \oplus B) = \mathrm{Pf}(A) \oplus \mathrm{Pf}(B)$

Remark: One can show, that for $A \in \mathfrak{so}_{2k}$ the Pffafian has explicit form

$$\mathrm{Pf}(A) = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} \mathrm{sgn}(\sigma) \prod_{i=1}^{k} a_{\sigma(2i-1),\sigma(2i)}$$

Consider now the subset of skew-hermitian matrices $\mathfrak{su}_n \subseteq M_n(\mathbb{C})$ (i.e. $\overline{A^t} = -A$). The realification map $M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ induces a map $\mathfrak{su}_n \to \mathfrak{so}_{2n}$, denoted by $A \mapsto A_\mathbb{R}$, and we have

Theorem 1.4: For $A \in \mathfrak{su}_n$, $\mathrm{Pf}(A_\mathbb{R}) = (-i)^n \det_\mathbb{C}(A)$.

Proof. It is well known that a skew-hermitian matrix has an orthonormal basis of eigenvectors, so we may assume that A is diagonal, namely $A = \mathrm{diag}(ia_1, \ldots, ia_n)$ with $a_i \in \mathbb{R}$. Now direct computation yields $\mathrm{Pf}(A_\mathbb{R}) = (-1)^n a_1 \ldots a_n$, and $\det_\mathbb{C}(A) = i^n a_1 \ldots a_n$, thus $\mathrm{Pf}(A_\mathbb{R}) = (-i)^n \det_\mathbb{C}(A)$. □

2 Metric connection and the Euler class

Let ξ be a smooth $2n$ real vector bundle over M with inner product $\langle\, , \rangle$. The inner product induces a pairing

$$\langle\, , \rangle \colon \Omega^i(\xi) \otimes \Omega^j(\xi) \to \Omega^{i+j}(M);$$
$$\langle \omega_1 \otimes s_1, \omega_2 \otimes s_2 \rangle = \langle s_1, s_2 \rangle\, \omega_1 \wedge \omega_2$$

where $\langle s_1, s_2 \rangle \in \Omega^0(M)$ defined by $p \mapsto \langle s_1(p), s_2(p) \rangle$, and $\omega_1 \in \Omega^i(M), \omega_2 \in \Omega^j(M)$.

Definition 2.1: A connection ∇ on $(\xi, \langle\, , \rangle)$ is said to be **metric** or **orthogonal** if

$$d\langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle.$$

We express this condition locally in terms of the connection form A associated to an orthonormal frame. Let $e_1, \ldots, e_{2n} \in \Omega^0(\xi)$ be sections over U, so that $e_1(p), \ldots, e_{2n}(p)$ forms an orthonormal basis of ξ for $p \in U$. Let A be the associated connection form,

$$\nabla(e_i) = \sum_j A_{ij} \otimes e_j.$$

For every pair (i,k) we have $\langle e_i, e_k \rangle = \delta_{ik}$ (on U), so $d\langle e_i, e_k \rangle = 0$. If ∇ is metric connection one gets

$$\begin{aligned} 0 = d\langle e_i, e_k \rangle &= \langle \Sigma_j A_{ij} \otimes e_j, e_k \rangle + \langle e_i, \Sigma_j A_{kj} \otimes e_j \\ &= \Sigma_j A_{ij} \langle e_j, e_k \rangle + \Sigma_j A_{kj} \langle e_i, e_j \rangle = A_{ik} + A_{ki}. \end{aligned}$$

Thus the connection matrix with respect to an orthonormal frame is skew-symmetric. If conversely A is skew-symmetric with respect to an orthonormal frame, then ∇ is metric

$$\begin{aligned}\langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle &= \left\langle \nabla\left(\sum a_i e_i\right), \sum b_i e_i\right\rangle + \left\langle \sum a_i e_i, \nabla\left(\sum b_j e_j\right)\right\rangle \\ &= \left\langle \sum da_i e_i + \sum a_i \nabla e_i, \sum b_i e_i\right\rangle + \left\langle \sum a_i e_i, \sum db_j e_j + \sum b_j \nabla e_j\right\rangle \\ &= \sum_i (b_i da_i + a_i db_i) + \sum_{i.j} a_i b_j (\langle \nabla e_i, e_j\rangle + \langle e_i, \nabla e_j\rangle \\ &= \sum_i d(a_i b_i) = d\langle s_1, s_2\rangle\end{aligned}$$

where $\langle \nabla e_i, e_j\rangle + \langle e_i, \nabla e_j\rangle = 0$ because A is skew-symmetric.

Let $F^\nabla \in \Omega^2(\mathrm{HOM}(\xi,\xi))$ be the curvature form associated to a metric connection. After choice of an orthonormal frame e for ξ_U,

$$\Omega^2(\mathrm{HOM}(\xi,\xi)_{|U}) \cong M_{2n}(\Omega^2(U)).$$

In the Talk 10 we have seen that the corresponding matrix of 2-forms $F^\nabla(e)$ was calculated to be

$$F^\nabla(e) = dA - A \wedge A$$

where A is the connection form associated to e. In particular, $F^\nabla(e)$ is skew-symmetric, and we can apply the Pfaffian polynomial to $F^\nabla(e)$ to get

$$\mathrm{Pf}(F^\nabla(e)) \in \Omega^{2n}(U).$$

In another orthonormal frame e' over U, we have

$$F^\nabla(e')_p = B_p F^\nabla(e) B_p^{-1}$$

where B_p is the orthogonal transition matrix between $e(p)$ and $e'(p)$, in particular $B_p^{-1} = B_p^t$.

From now on, we suppose additionally that the vector bundle ξ is oriented, and that e and e' are in the orientation class, that is $B_p \in \mathrm{SO}_{2n}$, and by Theorem 1.2

$$\mathrm{Pf}(F^\nabla(e)) = \mathrm{Pf}(F^\nabla(e')).$$

It follows that $\mathrm{Pf}(F^\nabla)$ becomes a well-defined global $2n$-form on M.

Proposition 2.2: For the Pfaffian polynomial, and any metric connection ∇, $\mathrm{Pf}(F^\nabla) \in \Omega^{2n}(M)$ is a closed form.

Proof. We follow the same proof as for invariant polynomials in [MT97]. Choose a frame for ξ over U, and let ∇ have the connection matrix $A = (A_{ij})$, so that $F^\nabla = dA - A \wedge A = (F_{ij}^\nabla)$. Define a matrix

$$\mathrm{Pf}'(A) = \left(\frac{\partial \mathrm{Pf}}{\partial A_{ij}}(A)\right)^t,$$

then one can show the commutativity $\mathrm{Pf}'(B)B = B\mathrm{Pf}'(B)$ for all skew-symmetric matrices B. Together with Bianchi's identity $dF^\nabla = A \wedge F^\nabla - F^\nabla \wedge A$ we get

$$\begin{aligned} d\mathrm{Pf}(F^\nabla) &= \sum \frac{\partial \mathrm{Pf}}{\partial A_{ij}}(F^\nabla) \wedge dF_{ij}^\nabla = \mathrm{Tr}(\mathrm{Pf}'(F^\nabla) \wedge dF^\nabla) \\ &= \mathrm{Tr}\left(\mathrm{Pf}'(F^\nabla) \wedge A \wedge F^\nabla - \mathrm{Pf}'(F^\nabla) \wedge F^\nabla \wedge A\right) \\ &= \mathrm{Tr}\left(\mathrm{Pf}'(F^\nabla) \wedge A \wedge F^\nabla - F^\nabla \wedge \mathrm{Pf}'(F^\nabla) \wedge A\right) = 0\end{aligned}$$

□

We must verify that its cohomology class is independent of the choice of metric on ξ and of the metric connection. First note that connections can be glued together by a partition of unity:

Lemma 2.3: Let $(\nabla_i)_{i\in I}$ be a family of metric connections, and $(\rho_i)_{i\in I}$ be a smooth partition of unity on M, then $\nabla = \sum \rho_i \nabla_i$ defines a metric connection.

Proof. straightforward, if

$$d\langle s_1, s_2\rangle = \langle \nabla_i s_1, s_2\rangle + \langle s_1, \nabla_i s_2\rangle$$

then

$$\begin{aligned}\langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle &= \sum \langle \rho_i \nabla_i s_1, s_2\rangle + \sum \langle s_1, \rho_i \nabla_i s_2\rangle \\ &= \sum \rho_i (\langle \nabla_i s_1, s_2\rangle + \langle s_1, \nabla_i s_2\rangle) \\ &= \sum \rho_i d\langle s_1, s_2\rangle = d\langle s_1, s_2\rangle.\end{aligned}$$

□

In the calculations above we have used only that ∇_i is metric on the $\mathrm{supp}_M(\rho_i)$, and not necessarily on all of M. This is crucial for us.

Corollary 2.4: For all real vector bundles $(\xi, \langle\, ,\rangle)$ over M there exists a compatible metric connection.

Proof. Pick an open covering $\{U_i\}_{i\in I}$ of M, such that ξ is trivial on each U_i. Define ∇_i such that over each U_i the connection matrix A_i is skew-symmetric, e.g. $A_i = 0$. The paracompactness of M and previous lemma end the proof. □

Consider the maps

$$M \underset{i_1}{\overset{i_0}{\rightrightarrows}} M \times \mathbb{R} \xrightarrow{\quad \pi \quad} M$$

with $i_\alpha(x) = (x, \alpha)$ and $\pi(x,t) = x$, and let $\widetilde{\xi} = \pi^*(\xi)$ be induced vector bundle over $M \times \mathbb{R}$. Then clear $i_\alpha^*(\widetilde{\xi}) \cong \xi$ for $\alpha = 0, 1$ and we have:

Lemma 2.5: For any choice of inner products and metric connections g_α, ∇_α (α = 0,1) on the smooth real vector bundle ξ over M, there is an inner product $\widetilde{g}$ on $\widetilde{\xi}$ and a metric connection $\widetilde{\nabla}$ compatible with $\widetilde{g}$ such that $i_\alpha^*(\widetilde{g}) = g_\alpha$ and $i_\alpha^*(\widetilde{\nabla}) = \nabla_\alpha$.

Proof. We can pullback by π^* the metric g_α and the metric connections ∇_α to $\widetilde{\xi}$. Let $\{\rho_0, \rho_1\}$ be a partition of unity on $M \times \mathbb{R}$ subordinate to the cover $M \times (-\infty, 3/4)$ and $M \times (1/4, \infty)$. Then $\widetilde{g} = \rho_0 \pi^*(g_0) + \rho_1 \pi^*(g_1)$ is a metric on $\widetilde{\xi}$ which agrees with $\pi^*(g_0)$ over $M \times (-\infty, 1/4)$ and with $\pi^*(g_1)$ on $M \times (3/4, \infty)$. In particular $i_\alpha^*(\widetilde{g}) = g_\alpha$.
Let $\hat{\nabla}$ be any metric connection on $\widetilde{\xi}$ compatible with $\widetilde{g}$. We have connections $\pi^*(\nabla_0)$, $\hat{\nabla}$ and $\pi^*(\nabla_1)$ compatible with $\widetilde{g}$ over $M \times (-\infty, 1/4)$, $M \times (1/8, 7/8)$ and $M \times (3/4, \infty)$ respectively. We use a partition of unity, subordinate to this cover, to glue together the three connections to construct a connection $\widetilde{\nabla}$ over $M \times \mathbb{R}$. This is metric ($\pi^*(\nabla_0)$ and $\pi^*(\nabla_1)$ are metric with respect to induced metric, for example because the induced connection matrix is also skew-symmetric) with respect to $\widetilde{g}$ by construction, and $i_\alpha^* \widetilde{\nabla} = \nabla_\alpha$. □

Corollary 2.6: The cohomology class $\left[\mathrm{Pf}(F^\nabla)\right] \in H^{2k}(M)$ is independent of the metric and the compatible metric connection.

Proof. Let (g_0, ∇_0) and (g_1, ∇_1) be two different choices and let $(\widetilde{g}, \widetilde{\nabla})$ be the metric and connection of the previous lemma. Then $i_\alpha^*(F^{\widetilde{\nabla}}) = F^{\nabla_\alpha}$, and hence $i_\alpha^* \mathrm{Pf}(F^{\widetilde{\nabla}}) = \mathrm{Pf}(F^{\nabla_\alpha})$. The maps i_0 and i_1 are homotopic, so $i_0^* = i_1^* \colon H^n(M \times \mathbb{R}) \to H^n(M)$. Thus the cohomology classes of $\mathrm{Pf}(F^{\nabla_0})$ and $\mathrm{Pf}(F^{\nabla_1})$ agree:

$$\left[\mathrm{Pf}(F^{\nabla_0})\right] = i_0^* \left[\mathrm{Pf}(F^{\widetilde{\nabla}})\right] = i_1^* \left[\mathrm{Pf}(F^{\widetilde{\nabla}})\right] = \left[\mathrm{Pf}(F^{\nabla_1})\right]$$

□

Definition 2.7: Let ξ be a real oriented n-dimensional vector bundle. The cohomology class

$$e(\xi) = \left[\mathrm{Pf}\left(\frac{-F^{\nabla}}{2\pi} \right) \right] \in H^n(M)$$

is called the **Euler class**.

Example 2.8: Suppose M is an oriented surface with Riemannian metric and that $\xi = \tau^* \cong \tau_M$ is the cotangent bundle. Let e_1, e_2 be an oriented orthonormal frame for $\Omega^0(\tau^*_{|U}) = \Omega^1(U)$, such that $e_1 \wedge e_2 = \mathrm{vol}$ on U. Let a_1, a_2 be the smooth functions on U determined by

$$de_1 = a_1(e_1 \wedge e_2), de_2 = a_2(e_1 \wedge e_2)$$

and let $A_{12} = a_1 e_1 + a_2 e_2$. We give $\tau^*_{|U}$ the connection form

$$A = \begin{pmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{pmatrix}$$

so that $\nabla(e_1) = A_{12} \otimes e_2$ and $\nabla(e_2) = -A_{12} \otimes e_1$. This is so-called Levi-Civita connection. The associated curvature form is

$$F^{\nabla} = dA - A \wedge A = \begin{pmatrix} 0 & dA_{12} \\ -dA_{12} & 0 \end{pmatrix}$$

since $A \wedge A = (a_1 e_1 + a_2 e_2) \wedge (a_1 e_1 + a_2 e_2) = 0$. In this case $\mathrm{Pf}(F^{\nabla}) = dA_{12}$ is called the Gauss-Bonnet form, and the Gaussian curvature $\kappa \in \Omega^0(M)$ is defined by the formula

$$-\kappa\, e_1 \wedge e_2 = \mathrm{Pf}(F^{\nabla}).$$

Definition 2.9: Let ξ be a complex vector bundle over M equipped with a hermitian metric $\langle\, , \rangle_{\mathbb{C}}$. A connection on $(\xi, \langle\, , \rangle_{\mathbb{C}})$ is called **metric** or **hermitian**, if

$$d\langle s_1, s_2 \rangle_{\mathbb{C}} = \langle \nabla s_1, s_2 \rangle_{\mathbb{C}} + \langle s_1, \nabla s_2 \rangle_{\mathbb{C}}.$$

Analogously as for the inner product, the connection ∇ is hermitian with respect to $(\xi, \langle\, , \rangle_{\mathbb{C}})$ if and only if the connection form A is skew-hermitian, i.e., $A_{ij} + \overline{A_{ji}} = 0$ or in matrix terms $A^* + A = 0$.

Given a hermitian smooth vector bundle $(\xi, \langle\, , \rangle_{\mathbb{C}})$ of complex dimension n with a hermitian connection $\nabla_{\mathbb{C}}$, the underlying real vector bundle $\xi_{\mathbb{R}}$ is $2n$ naturally oriented real vector bundle, and inherits an inner product $\langle\, , \rangle_{\mathbb{R}} = \mathrm{Re}(\langle\, , \rangle_{\mathbb{C}})$ and an orthogonal connection $\nabla_{\mathbb{R}} = \mathrm{Re}(\nabla_{\mathbb{C}})$.
If $A_{\mathbb{C}}$ is the hermitian connection form of $(\xi, \langle\, , \rangle_{\mathbb{C}})$ with respect to an orthonormal frame e, then the connection form associated with the underlying real situation is $A_{\mathbb{R}}$, the matrix of 1-forms given by the usual embedding of $M_n(\mathbb{C}) \hookrightarrow M_{2n}(\mathbb{R})$. This embedding sends skew-hermitian matrices into skew-symmetric matrices, and

$$\mathrm{Pf}(F^{\nabla}(e)_{\mathbb{R}}) = (-i)^n \det(F^{\nabla}(e)) \tag{1}$$

by Theorem 1.4. For a complex vector bundle we write $e(\xi)$ instead of $e(\xi_{\mathbb{R}})$ for the Euler class. Then we have

Theorem 2.10: (i) For a complex n-dimensional vector bundle ζ, $e(\zeta) = c_n(\zeta)$.

(ii) For oriented real vector bundles ξ_1 and ξ_2, $e(\xi_1 \oplus \xi_2) = e(\xi_1)e(\xi_2)$.

(iii) For oriented real vector bundle ξ, $e(f^*(\xi)) = f^* e(\xi)$.

Proof. For the first assertion, recall that

$$c_k(\zeta) = \left[\sigma_k \left(\frac{-1}{2\pi i} F^{\nabla} \right) \right] \in H^{2k}(M; \mathbb{C})$$

and for $k = n$, we have $\sigma_n(F^\nabla) = \det(F^\nabla)$ so by (1)

$$\begin{aligned}\mathrm{Pf}(-F^\nabla_{\mathbb{R}}/2\pi) &= (-1)^n/(2\pi)^n \mathrm{Pf}(F^\nabla_{\mathbb{R}}) \\ &= i^n/(2\pi)^n \det(F^\nabla) \\ &= i^n/(2\pi)^n \sigma_n(F^\nabla)\end{aligned}$$

when F^∇ is the curvature of a hermitian connection on $(\zeta, \langle\, ,\rangle_{\mathbb{C}})$. Thus

$$\mathrm{Pf}(-F^\nabla_{\mathbb{R}}/2\pi) = \sigma_k(iF^\nabla/2\pi).$$

This proves (i).

The second assertion is similar to the case of the Chern class. Let ∇_1, ∇_2 be metric connections for ξ_1 and ξ_2. Then $\nabla_1 \oplus \nabla_2$ is a metric connection for $\xi_1 \oplus \xi_2$, and the same for the curvature

$$F^\nabla = F^{\nabla_1} \oplus F^{\nabla_2}.$$

We end the proof by noting $\mathrm{Pf}(F^{\nabla_1}F^{\nabla_2}) = \mathrm{Pf}(F^{\nabla_1})\mathrm{Pf}(F^{\nabla_2})$

Finally assertion (iii) follows from the 10-th Talk, namely from the fact that $f^*(F^\nabla) = F^{f^*(\nabla)}$, so

$$e(f^*(\xi)) = \left[\mathrm{Pf}\left(\frac{-F^{f^*\nabla}}{2\pi}\right)\right] = \left[\mathrm{Pf}\left(\frac{-f^*F^\nabla}{2\pi}\right)\right] = f^*\left[\mathrm{Pf}\left(\frac{-F^\nabla}{2\pi}\right)\right] = f^*e(\xi).$$

□

Proposition 2.11: Let ξ be an oriented $2n$-dimensional vector bundle over M. And let $\widetilde{\xi}$ be the same vector bundle, with the opposite orientation. Then $e(\widetilde{\xi}) = -e(\xi) \in H^{2n}(M)$

Proof. Immediately from the fact that $\mathrm{Pf}(F^\nabla(e')) = \mathrm{Pf}(F^\nabla(e))\det(B) = -\mathrm{Pf}(F^\nabla(e))$, where B is the orthogonal transition matrix between e and e'. □

Proposition 2.12: Let ξ be an oriented vector bundle over M that possesses a nowhere zero section $s\colon M \to E$, then the Euler class $e(\xi)$ must be zero.

Proof. Let $\langle\, ,\rangle$ be an inner product on ξ. Denote by ε the line bundle spanned by the nowhere vanishing section s of ξ. Then $\xi = \varepsilon \oplus \varepsilon^\perp$, where $\varepsilon^\perp$ is orthogonal vector bundle to ε in $(\xi, \langle\, ,\rangle)$. Hence by Theorem 2.10

$$e(\xi) = e(\varepsilon \oplus \varepsilon^\perp) = e(\varepsilon)e(\varepsilon^\perp) = 0.$$

□

Theorem 2.13 (Chern–Gauss–Bonnet theorem)**:** Let M be a connected orientable compact smooth manifold, and TM its tangent bundle. Then

$$D(e(TM)) = \int_M \mathrm{Pf}\left(\frac{-F^\nabla}{2\pi}\right) = \chi(M)$$

where $\chi(M)$ is the Euler characteristic of M, and $D\colon H^n(M) \to H^0(M)^* \cong \mathbb{R}$ is the isomorphism from the Poincaré-duality. For the oriented surface from Example 2.8 and its Gaussian curvature we have

$$\int_M \kappa\, e_1 \wedge e_2 = 2\pi\chi(M)$$

Proof. Omitted. See for example [Mor01]. □

Corollary 2.14: Let M be a connected orientable compact smooth manifold of dimension n, and TM its tangent bundle. Denote by $[\omega_M] \in H^n(M)$ the volume form, then

$$e(TM) = \chi(M)[\omega_M].$$

Corollary 2.15: The tangent bundles of manifolds with $\chi(M) \neq 0$ do not admit a nowhere zero section.

2.1 Uniqueness of Euler classes

To prove the uniqueness of Euler classes we need a version of the splitting principle for real oriented vector bundles, namely

Theorem 2.16 (Real splitting principle)**:** For any oriented real vector bundle ζ over M there exists a manifold $T(\zeta)$ and a smooth proper map $f\colon T(\zeta) \to M$ such that

(i) $f^*\colon H^*(M) \to H^*(T)$ is injective.

(ii) $f^*(\zeta) = \gamma_1 \oplus \cdots \oplus \gamma_n$ when $\dim(\zeta) = 2n$, and $f^*(\zeta) = \gamma_1 \oplus \cdots \oplus \gamma_n \oplus \varepsilon^1$ when $\dim(\zeta) = 2n+1$, where $\gamma_1, \ldots, \gamma_n$ are oriented 2-plane bundles, and ε^1 is the trivial line bundle.

Theorem 2.17 (Uniqueness of Euler classes)**:** Suppose that to each oriented isomorphism class of $2n$-dimensional oriented real vector bundle ζ we have associated a class $\hat{e}(\zeta) \in H^{2n}(M)$ that satisfies

(i) $f^*(\hat{e}(\zeta)) = \hat{e}(f^*(\zeta))$ for a smooth map $f\colon N \to M$

(ii) $\hat{e}(\zeta_1 \oplus \zeta_2) = \hat{e}(\zeta_1)\hat{e}(\zeta_2)$ for oriented even-dimensional vector bundles ζ_1, ζ_2 over the same base space.

Then there exists a real constant $a \in \mathbb{R}$ such that $\hat{e}(\zeta) = a^n e(\zeta)$.

Proof. Given a complex line bundle L over M, we can define $c(L) = \hat{e}(L_{\mathbb{R}})$. Then $f^*c(L) = c(f^*L)$, and the argument for uniqueness of Chern classes shows that $c(L) = ac_1(L)$. Thus $\hat{e}(\gamma) = ae(\gamma)$ for each oriented 2-plane bundle γ. Indeed, an oriented 2-plane bundle is of the form $L_{\mathbb{R}}$ for a complex line bundle which is uniquely determined up to isomorphism. One simply defines multiplication by i to be a positive rotation by $\pi/2$.

If $\zeta = \gamma_1 \oplus \cdots \oplus \gamma_n$ is a sum of oriented 2-plane bundles then we can use (ii) and Theorem2.10 to see that $\hat{e}(\zeta) = a^n e(\zeta)$. Finally Theorem 2.16 implies the result in general. □

References

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