Manifolds and Tangent Bundles

Seminar on deRham-Cohomology (S2D3)
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1 Manifolds

Definition 1.1 (Topological manifold). A topological manifold M is a Hausdorff space that has a countable base and is locally homeomorphic to an open subset of $\mathbb{R}^n$, i.e. for each point in M there exists an open neighbourhood U, such that U is homeomorphic to an open subset $\Omega \subset \mathbb{R}^n$.
The dimension of M is n.

Remark . The number n is well-defined:
Let $U_1, U_2 \subset M$ be open with $U_1 \cap U_2 \neq \emptyset$ and $h_1 \colon U_1 \to \Omega_1, h_2 \colon U_2 \to \Omega_2$ be the corresponding homeomorphisms, where $\Omega_1 \subset \mathbb{R}^{n_1}$ open, $\Omega_2 \subset \mathbb{R}^{n_2}$ open. We consider the restrictions $h_1|_{U_1 \cap U_2} \colon U_1 \cap U_2 \to \Omega_1'$ and $h_2|_{U_1 \cap U_2} \colon U_1 \cap U_2 \to \Omega_2'$. We have Ω_1' homeomorphic to Ω_2', $\Omega_1' \subset \mathbb{R}^{n_1}$ open and $\Omega_2' \subset \mathbb{R}^{n_2}$ open (since $U_1 \cap U_2 \subset M$ open). It follows from the theorem of invariance of domain, that $n_1 = n_2$.

Definition 1.2 (Chart, coordinates and atlas). Let M be an n-dimensional topological manifold.

1. A chart (U, h, Ω) consists of
   - an open subset $U \subset M$ (chart domain),
   - a homeomorphism $h \colon U \to \Omega$ (chart map), and
   - an open subset $\Omega \subset \mathbb{R}^n$ (chart image).
2. Given a chart (U, h, Ω), the coordinates of $x \in U$ with respect to h are $h(x) = (h^1(x), \dots, h^n(x)) \in \mathbb{R}^n$.
3. If $\{U_i \mid i \in I\}$ covers M, then an atlas is a system $A = \{(U_i, h_i, \Omega_i) \mid i \in I\}$ of charts.
4. A chart transition is a map $h_{ji} = h_j \circ h_i^{-1} \colon h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$ (see Figure 1). An atlas is smooth, when all chart transitions are smooth (as maps between subsets of $\mathbb{R}^n$).
5. Two smooth atlases A_1, A_2 are smoothly equivalent, if $A_1 \cup A_2$ is a smooth atlas. A smooth structure on M is an equivalence class $\mathcal{A}$ of smooth atlases.

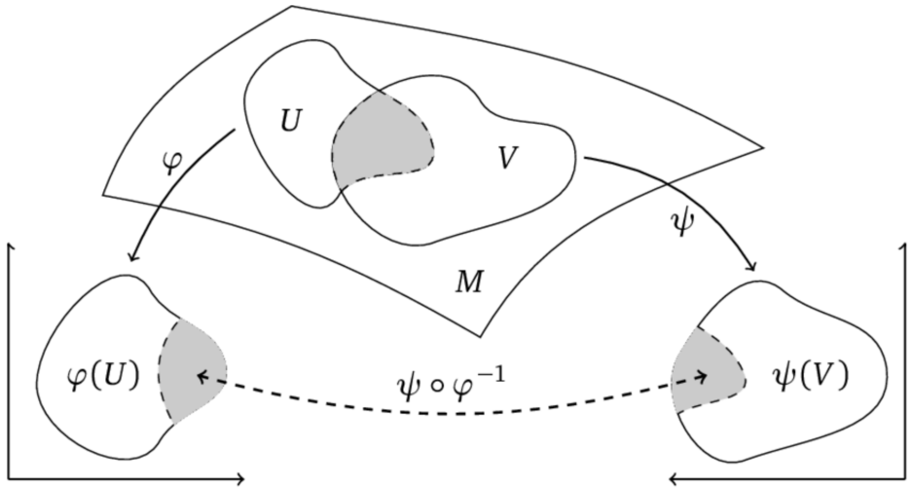

Figure 1: Two charts $(U, \varphi, \phi(U))$ and $(V, \psi, \psi(V))$ on M, and a chart transition $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$.

6. A maximal atlas $\mathcal{A}_{max}$ is a representative of the equivalence class $\mathcal{A}$, such that if a smooth atlas $A \in \mathcal{A}$, then $A \subset \mathcal{A}_{max}$. In other words, the maximal atlas is the maximal element with respect to $\subset$.

Definition 1.3 (Smooth manifold). A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a maximal atlas on M.
It suffices to give a representative of the smooth structure – the maximal atlas would be then automatically defined.

Example 1.4. Every open subset $U \subset \mathbb{R}^n$ has a canonical structure of a smooth n-dimensional manifold, which is given by the atlas with one chart – (U, id, U). Note that the corresponding maximal atlas contains the restriction $id|_V$ on every open subset $V \subset U$, and also $t \circ id|_V \colon V \to \mathbb{R}^n$, where $t\colon V \to \mathbb{R}^n$ represents a translation of V.

Example 1.5 (n-dim sphere $\mathbb{S}^n$). $\mathbb{S}^n$ is an n-dimensional smooth manifold:
Let $\mathring{D}^n$ be the open ball of radius 1 in $\mathbb{R}^n$ centered at 0. We define the sets $U_i^+ = \{x \in \mathbb{S}^n \mid x_i > 0\}, U_i^- = \{x \in \mathbb{S}^n \mid x_i < 0\}$ and the maps $h_i^\pm\colon U_i^\pm \to \mathring{D}^n$, $x \mapsto (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$, where $\hat{x}_i$ denotes the omitting of x_i. We have
$(h_i^\pm)^{-1}(u) = (u_1, \ldots, u_{i-1}, \pm\sqrt{1 - \|u\|^2}, u_i, \ldots, u_n)$, which means $(U_i^\pm, h_i^\pm, \mathring{D}^n)$ are charts (see Figure 2). (Note that all charts have the same chart image $\mathring{D}^n$.) The chart transitions are smooth and thus define a smooth structure on $\mathbb{S}^n$.

Example 1.6 (Projective space $\mathbb{RP}^n$). $\mathbb{RP}^n$ is an n-dimensional smooth manifold:
Let $\pi\colon \mathbb{S}^n \to \mathbb{RP}^n$ be the canonical projection. Using the notations in Example 1.5, $\pi(U_i^+) = \pi(U_i^-)$. We define $U_i = \pi(U_i^\pm) \subset \mathbb{RP}^n$ and see that $\pi\colon U_i^+ \to U_i$ is a homeomorphism (see Figure 3). The maps $h_i = h_i^+ \circ \pi^{-1}\colon U_i \to \mathring{D}^n$ form a smooth atlas on $\mathbb{RP}^n$.

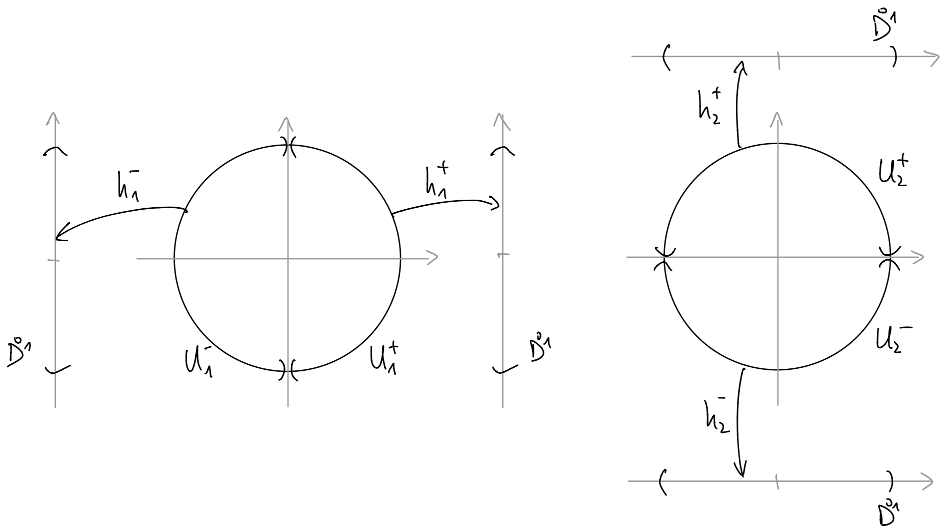

Figure 2: The charts $(U_i^\pm, h_i^\pm, \mathring{D}^n)$ on $\mathbb{S}^n$ in Example 1.5 with $n = 1$.

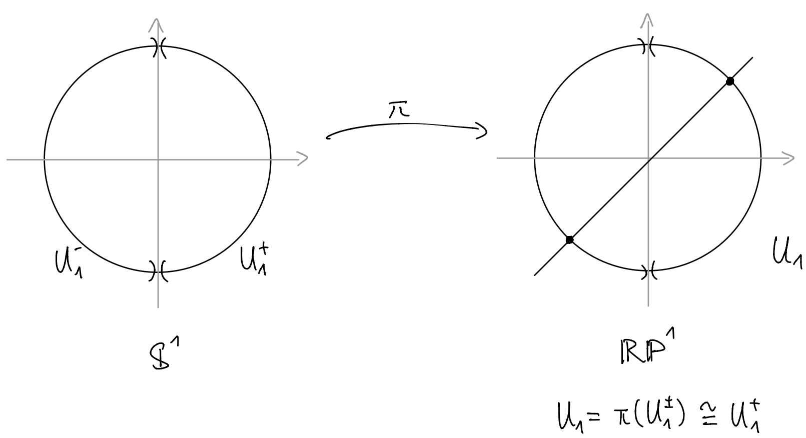

Figure 3: Example 1.6 with $n = 1$. The chart domains $U_i \subset \mathbb{RP}^n$ are defined by the canonical projection π by grouping every pair of antipodal points in U_i^+ and U_i^- together into one element.

Example 1.7 (2-dim sphere $\mathbb{S}^2$). Consider another atlas on $\mathbb{S}^2$:
Denote the north pole with $N_2 = (0,0,1)$ and the south pole with $S_2 = (0,0,-1)$. The "stereographic projection from a pole to a plane $z = z_0$" maps a point on the sphere to the $z = z_0$ intersection of the line that passes through the pole and the point itself. Let $\phi_2\colon \mathbb{S}^2 - N_2 \to \mathbb{R}^2, (x,y,z) \mapsto (2x/(1-z), 2y/(1-z))$ be the composition of first stereographically projecting from N_2 to the plane $z = -1$, and then translating to the plane $z = 0$ (see Figure 4). Similarly, let $\psi_2\colon \mathbb{S}^2 - S_2 \to \mathbb{R}^2, (x,y,z) \mapsto (2x/(z+1), 2y/(z+1))$ be the composition of first stereographically projecting from S_2 to the plane $z = 1$, and then translating to the plane $z = 0$. It is clear that $(\mathbb{S}^2 - N_2, \phi_2, \mathbb{R}^2), (\mathbb{S}^2 - S_2, \psi_2, \mathbb{R}^2)$ are charts on $\mathbb{S}^2$. Moreover, we have

$$\phi_2^{-1}(u,v) = (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, 1 - \frac{8}{u^2+v^2+4}),$$
$$\psi_2^{-1}(u,v) = (\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{8}{u^2+v^2+4} - 1).$$

Hence the chart transition $\psi_2 \circ \phi_2^{-1}\colon \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$ is given by $v \mapsto 4v/\|v\|^2$, which is smooth on $\mathbb{R}^2 - \{0\}$. Therefore, this defines a smooth structure on $\mathbb{S}^2$.

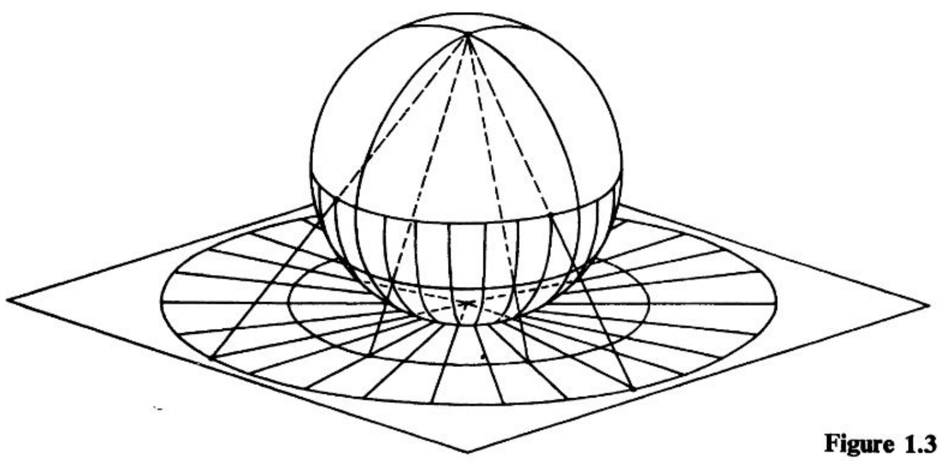

Figure 4: The stereographic projection of $\mathbb{S}^2$ from the north pole N_2 to the plane $z = -1$.

Remark . The smooth structures constructed in Example 1.5 and 1.7 are the same:
Using the above notations, we have $U_1^+ \cap (\mathbb{S}^2 - N_2) = U_1^+$. Consider the chart transition $\phi_2 \circ (h_1^+)^{-1}\colon \mathring{D}^2 \to \{v \in \mathbb{R}^2 \mid v_1 > 0\}, (u_1,u_2) \mapsto (2\sqrt{1-\|u\|^2}/(1-u_2), 2u_1/(1-u_2))$, which is smooth. Analogously, all such chart transitions are smooth and thus the two atlases belong to the same equivalence class.

Example 1.8 (Torus T). T is a 2-dimensional smooth manifold:
The torus T is defined as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. Let $\pi\colon \mathbb{R}^2 \to T$ be the canonical projection, $0 < \epsilon < 1/4$. Consider the open subsets $\Omega_1 = (\epsilon, 1-\epsilon) \times (\epsilon, 1-\epsilon), \Omega_2 = (\epsilon.1-\epsilon) \times (-2\epsilon, 2\epsilon), \Omega_3 = (-2\epsilon, 2\epsilon) \times (\epsilon, 1-\epsilon)$ and $\Omega_4 = (-2\epsilon, 2\epsilon) \times (-2\epsilon, 2\epsilon)$ in $\mathbb{R}^2$. We define $U_i = \pi(\Omega_i) \subset T$ for $i = 1,2,3,4$ and see that $\pi\colon \Omega_i \to U_i$ is a homeomorphism (see Figure 5). The maps $h_i = \pi^{-1}\colon U_i \to \Omega_i$ form a smooth atlas on T.

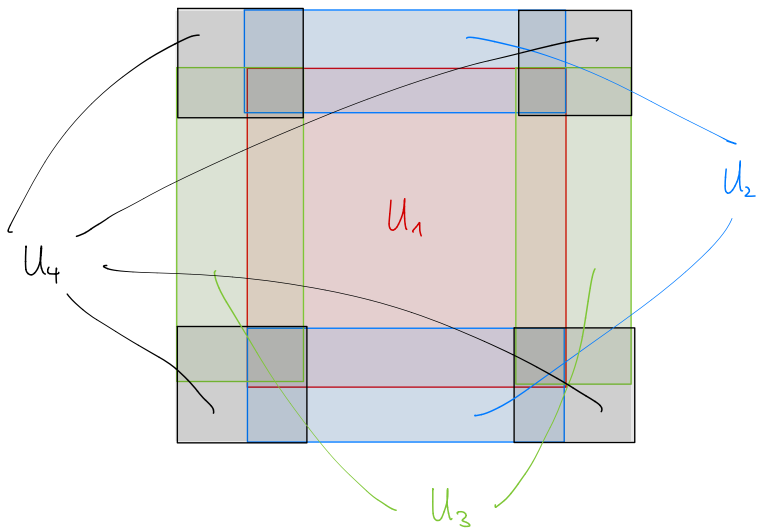

Figure 5: The quotient space $T = \mathbb{R}^2/\mathbb{Z}^2$. The 4 chart domains $U_i \subset T$ are shown in different colours.

Remark . An atlas on a compact manifold has at least 2 charts:
Assume there is an atlas with just one chart (U, h, Ω), where $\Omega \subset \mathbb{R}^n$ open. Then U has to be the entire compact space. Since h is a homeomorphism, $\Omega = h(U)$ must be compact, and using the theorem of Heine-Borel, Ω must be closed and bounded. $\mathbb{R}^n$ is connected, so the only non-empty open and closed subset is $\mathbb{R}^n$ itself, which is unbounded. This is a contradition to Ω being bounded.

Example 1.9. The restriction of a smooth manifold on an open subset is a smooth manifold: Let $(M, \mathcal{A})$ be a smooth manifold and $V \subset M$ open. Then a smooth atlas on V is $\mathcal{A}_V = \{(U \cap V, h|_{U\cap V}, h(U \cap V)) \mid (U, h, \Omega) \in \mathcal{A}\}$. (The smoothness of chart transitions is preserved under the restriction.) The dimension remains unchanged.

Definition 1.10 (Smooth maps between smooth manifolds). Let M, M' be smooth manifolds and $f\colon M \to M'$ be a continuous map. f is smooth at $x \in M$, if there exist charts (U, h, Ω) on M and (U', h', Ω') on M' with $x \in U$, $f(x) \in U'$ such that $h' \circ f \circ h^{-1}\colon h(f^{-1}(U')) \to \Omega'$ is smooth in a neighbourhood of $h(x)$.
f is smooth, when f is smooth at all points of M.

Remark . The definition is independent of choice of charts, since chart transitions are smooth for smooth manifolds. A composition of two smooth maps is smooth. Hence the condition in the definition would be fulfilled for all charts (with suitable domains), given the existence of h and h'.

Example 1.11. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^2$ be the map, which maps $(x, y) \in \mathbb{S}^1$ to the equator $(x, y, 0) \in \mathbb{S}^2$. This is a smooth map:

Similar to Example 1.7, we define a smooth atlas on $\mathbb{S}^1$ with the help of stereographic projection. Denote the north pole of $\mathbb{S}^1$ with $N_1 = (0, 1)$ and the south pole of $\mathbb{S}^1$ with $S_1 = (0, -1)$. Let $\phi_1 \colon \mathbb{S}^1 - N_1 \to \mathbb{R}, (x, y) \mapsto 2x/(1 - y)$ be the composition of first stereographically projecting from N_1 to the line $y = -1$, and then translating to the line $y = 0$. Similarly, let $\psi_1 \colon \mathbb{S}^1 - S_1 \to \mathbb{R}, (x, y) \mapsto 2x/(y+1)$ be the composition of first stereographically projecting from S_1 to the line $y = 1$, and then translating to the line $y = 0$. The inverses are given by

$$\phi_1^{-1}(u) = (\frac{4u^2}{u^2+4}, 1 - \frac{8u}{u^2+4}), \psi_1^{-1}(u) = (\frac{4u^2}{u^2+4}, \frac{8u}{u^2+4} - 1).$$

Hence $(\mathbb{S}^1 - N_1, \phi_1, \mathbb{R})$ and $(\mathbb{S}^1 - S_1, \psi_1, \mathbb{R})$ form a smooth atlas on $\mathbb{S}^1$. The composition

$$\phi_2 \circ f \circ \phi_1^{-1} \colon \mathbb{R} \to \mathbb{R}^2, u \mapsto (\frac{8u^2}{u^2+4}, 2 - \frac{16u}{u^2+4})$$

is smooth, which means f is smooth in $\mathbb{S}^1 - N_1$. The same can be done for the north pole N_1, if we replace ϕ_1 with ψ_1.
Analogously, the map $g \colon \mathbb{S}^1 \to \mathbb{S}^2$, which maps $\mathbb{S}^1$ to any great circle on $\mathbb{S}^2$, is also smooth: We can write $g = R \circ f$, where $R \colon \mathbb{S}^2 \to \mathbb{S}^2$ is a proper rotation around an axis. Using the corresponding matrix representation, one sees that R is a smooth map between manifolds, and thus g is smooth. We look at the example $g(x, y) = (0, x, y)$ (the great circle passes through both poles). The rotation R in this case is a rotation of 90 $^\circ$ about the y -axis, followed by a rotation of 90 $^\circ$ about the x-axis. It is given by the matrix

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The composition

$$\phi_2 \circ R \circ \phi_2^{-1} \colon \mathbb{R}^2 - \{(0, 2)\} \to \mathbb{R}^2, (u, v) \mapsto (\frac{2u^2 + 2v^2 - 8}{u^2 + v^2 - 4v + 4}, \frac{8u}{u^2 + v^2 - 4v + 4})$$

is a smooth map. Therefore, R is smooth.

Definition 1.12 (Diffeomorphism). A diffeomorphism is a smooth map $f \colon M \to M'$ between smooth manifolds that has a smooth inverse. M is then diffeomorphic to M'.
A diffeomorphism is in particular a homeomorphism.

Example 1.13. The smooth map $f \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is a homeomorphism but not a diffeomorphism, since the inverse is continuous but not smooth.

Lemma 1.14. Suppose we have chosen a smooth atlas A on a manifold M. The maximal atlas $\mathcal{A}_{max}$ is equal to $\{(U, f, \Omega) \mid U \subset M \text{ open}, \Omega \subset \mathbb{R}^n \text{ open}, f \colon U \to \Omega \text{ diffeomorphism}\}$.

Proof. "$\subset$": Let $(U, h, \Omega) \in \mathcal{A}_{max}$ be a chart. We use the canonical smooth structure on $\Omega \subset \mathbb{R}^n$ given by the atlas $\{(\Omega, id, \Omega)\}$. Then h is a smooth map between smooth manifolds, since $id \circ h \circ h^{-1} = id \colon \Omega \to \Omega$ is smooth. Its inverse $h^{-1} \colon \Omega \to U$ is also smooth, shown by

the smooth composition $h \circ h^{-1} \circ id^{-1} = id\colon \Omega \to \Omega$. Thus h is a diffeomorphism.
"$\supset$": Let $f\colon U \to \Omega$ be a diffeomorphism, where $U \subset M$ open, $\Omega \subset \mathbb{R}^n$ open. It is clear that (U, f, Ω) is a chart. For all charts $(U_0, h_o, \Omega_0) \in \mathcal{A}_{max}$, the compositions $id \circ f \circ h_0^{-1}\colon h_0(U \cap U_0) \to f(U \cap U_0)$ and $h_0 \circ f^{-1} \circ id^{-1}\colon f(U \cap U_0) \to h_0(U \cap U_0)$ are smooth by definition. This means all chart transitions related to f are smooth. Therefore, (U, f, Ω) is an element of $\mathcal{A}_{max}$. □

Remark . The lemma shows that a smooth manifold is locally diffeomorphic to an open subset of $\mathbb{R}^n$. The inverse diffeomorphisms $f^{-1}\colon \Omega \to U$ are called local parametrizations.

Definition 1.15 (Product structure). Let M and M' be smooth manifolds with maximal atlases $\mathcal{A}_M$ and $\mathcal{A}_{M'}$. We define an atlas on $M \times M'$:
$\mathcal{A}_{M\times M'} = \{(U \times U', h \times h', \Omega \times \Omega') \mid (U, h, \Omega) \in \mathcal{A}_M, (U', h', \Omega') \in \mathcal{A}_{M'}\}$. $\mathcal{A}_{M\times M'}$ is smooth and defines the so-called product structure on $M \times M'$.
The projections $M \times M' \to M$ and $M \times M' \to M'$ are smooth.

2 Submanifolds and Embedding

Definition 2.1 (Topological submanifold). Let M be an n-dimensional topological manifold.

1. A subspace $N \subset M$ is a topological submanifold, if for every $x \in N$ there exists a chart (U, h, Ω) on M with $x \in U$, such that $h(U \cap N) = \Omega \cap (\mathbb{R}^k \times \{0\})$ (see Figure 6). The dimension of N is k. The number $n - k$ is called the codimension of N in M.

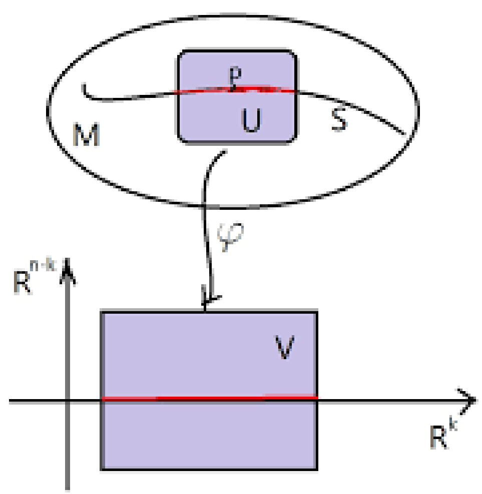

Figure 6: Let $S \subset M$ be the submanifold. For $p \in S$, the chart (U, ϕ, V) maps $U \cap S$ to $V \cap (\mathbb{R}^k \times \{0\})$, which is the flat red surface marked in the figure.

2. If $(M, \mathcal{A})$ is a smooth manifold and every chart (U, h, Ω) used is contained in the maximal atlas $\mathcal{A}$, then N is a smooth submanifold of M.

A smooth submanifold is in particular a smooth manifold, with a smooth atlas $\mathcal{A}_N = \{(U \cap N, h|_{U\cap N}, \Omega \cap (\mathbb{R}^k \times \{0\})) \mid (U, h, \Omega) \in \mathcal{A}, x \in U\}$.

Example 2.2 (n-dim sphere $\mathbb{S}^n$). $\mathbb{S}^n$ is an n-dimensional smooth submanifold of $\mathbb{R}^{n+1}$: We define the sets $V_i^+ = \{x \in \mathbb{R}^{n+1} \mid x_i > 0\}, V_i^- = \{x \in \mathbb{R}^{n+1} \mid x_i < 0\}$ and the maps $h_i^\pm \colon V_i^\pm \to h_i^\pm(V_i^\pm) \subset \mathbb{R}^{n+1}$, $x \mapsto (x_1, \ldots, \hat{x_i}, \ldots, x_{n+1}, \|x\|^2 - 1)$, where $\hat{x_i}$ denotes the omitting of x_i. Let $\Omega_i^\pm = h_i^\pm(V_i^\pm)$. For $u \in \Omega_i^\pm$, we have $(h_i^\pm)^{-1}(u) = (u_1, \ldots, u_{i-1}, \pm(u_{n+1} + 1 - \sum_{j=1}^n u_j^2)^{1/2}, u_i, \ldots, u_n)$, which means $(V_i^\pm, h_i^\pm, \Omega_i^\pm)$ are charts on $\mathbb{R}^{n+1}$. Consider the atlas $\mathcal{A} = \{(\mathring{D}^{n+1}, id, \mathring{D}^{n+1}), (V_i^\pm, h_i^\pm, \Omega_i^\pm) \mid i = 1, 2, \ldots, n+1\}$. The chart transitions are smooth and thus define a smooth structure on $\mathbb{R}^{n+1}$.
Note that the charts $(V_i^\pm, h_i^\pm, \Omega_i^\pm)$ are essentially extensions of the charts $(U_i^\pm, h_i^\pm, \mathring{D}^n)$ onto $V_i^\pm$ (see Example 1.5). Since $V_i^\pm \cap \mathbb{S}^n = U_i^\pm$, we have $h_i^\pm(V_i^\pm \cap \mathbb{S}^n) = h_i^\pm(U_i^\pm) = \mathring{D}^n \times \{0\} = \Omega_i^\pm \cap (\mathbb{R}^n \times \{0\})$. The sets $V_i^\pm$ cover $\mathbb{S}^n$.

Definition 2.3 (Embedding). An embedding is a smooth map $f \colon N \to M$ between smooth manifolds, such that $f(N) \subset M$ is a smooth submanifold and $f \colon N \to f(N)$ is a diffeomorphism. $f(N)$ is then an embedded submanifold of M.

Example 2.4 (Graph of function). Let $U \subset \mathbb{R}^m$ be an open subset and $f \colon U \to \mathbb{R}^n$ be a smooth map. Its graph $Graph(f) = \{(x, f(x)) \mid x \in U\}$ is a smooth m-dimensional submanifold of the smooth $(m+n)$-dimensional manifold $U \times \mathbb{R}^n$. The desired chart as in Definition 2.1 is $h \colon U \times \mathbb{R}^n \to U \times \mathbb{R}^n \subset \mathbb{R}^{m+n}, (x, y) \mapsto (x, y - f(x))$, satisfying $h(Graph(f)) = (U \times \{0\})$. An embedding is the map $i \colon U \to U \times \mathbb{R}^n, x \mapsto (x, f(x))$.

Theorem 2.5 (Smooth partition of unity). Let $U \subset \mathbb{R}^n$ be open and $\mathcal{V} = (V_i)_{i\in I}$ a cover of U by open sets V_i. Then there exist smooth functions $\phi_i \colon U \to [0, 1]$, satisfying

1. $supp(\phi_i) \subset V_i$ for all $i \in I$
2. Every point in U has a neighbourhood on which only finitely many ϕ_i do not vanish.
3. For every $x \in U$: $\sum_{i\in I} \phi_i(x) = 1$

Proof. Proved in Analysis 3. □

Lemma 2.6. Let $A \subset \mathbb{R}^n$ be closed and $U \subset \mathbb{R}^n$ open with $A \subset U$. Then there exists a smooth function $\psi \colon \mathbb{R}^n \to [0, 1]$ with $supp(\psi) \subset U$ and $\psi(x) = 1$ for all $x \in A$.

Proof. Apply Theorem 2.5 to the cover of $\mathbb{R}^n$ consisting of the open sets $V_1 = U, V_2 = \mathbb{R}^n - A$. Then $\psi = \phi_1$ has the desired properties. □

Lemma 2.7. Let M be an n-dimensional smooth manifold. For $x \in M$ there exist smooth maps $\phi_x \colon M \to \mathbb{R}$ and $f_x \colon M \to \mathbb{R}^n$, such that $\phi_x(x) > 0$ and f_x maps the open set $M - \phi_x^{-1}(0)$ diffeomorphically onto an open subset of $\mathbb{R}^n$.

Proof. Choose a chart (V, h, Ω) with $x \in V$. Using Lemma 2.6, we have a function $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $supp(\psi) \subset \Omega$, such that ψ is equal to 1 on a neighbourhood $\Omega_0 \subset \Omega$ of $h(x)$. We define

$$f_x(y) = \begin{cases} \psi(h(y))h(y) & \text{if } y \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Set $V_0 = h^{-1}(\Omega_0)$. Since $f_x|_{V_0} = h|_{V_0}$, f_x maps V_0 diffeomorphically onto Ω_0. We choose $\psi_0 \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $supp(\psi_0) \subset \Omega_0$ and $\psi_0(h(x)) > 0$, and define

$$\phi_x(y) = \begin{cases} \psi_0(h(y)) & \text{if } y \in V, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\phi_x(x) > 0$. Since $M - \phi_x^{-1}(0) \subset V_0$, f_x also fulfils the last requirement. □

Theorem 2.8 (Whitney Embedding Theorem). Let M be a smooth n-dimensional manifold. Then there exists an embedding of M into $\mathbb{R}^{n+k}$. In other words, every smooth manifold is diffeomorphic to an embedded manifold.

Proof. (for M compact.)
For every $x \in M$ we have ϕ_x and f_x as in Lemma 2.7. By compactness M can be covered by a finite number of the sets $M - \phi_x^{-1}(0)$. After a change of notation, we receive smooth functions $\phi_i \colon M \to \mathbb{R}$, $f_i \colon M \to \mathbb{R}^n$ $(i = 1, 2, \ldots, d)$ which fulfil

i. The open sets $U_i = M - \phi_i^{-1}(0)$ cover M.

ii. $f_i|_{U_i}$ maps U_i diffeomorphically onto an open set $\Omega_i \subset \mathbb{R}^n$.

We define a smooth map $f \colon M \to \mathbb{R}^{nd+d}$, $f(y) = (f_1(y), \ldots, f_d(y), \phi_1(y), \ldots, \phi_d(y))$. We check that f is injective and since M is compact, f is a homeomorphism from M to $f(M)$. We prove that f is the desired embedding:

1. Prove that for every $x \in f(M)$ there exists a chart (V, h, Π) on $\mathbb{R}^{nd+d}$ with $x \in V$, such that $h(V \cap f(M)) = \Pi \cap (\mathbb{R}^k \times \{0\})$.
   Denote with $\pi_1 \colon \mathbb{R}^{nd+d} \to \mathbb{R}^n$, $\pi_2 \colon \mathbb{R}^{nd+d} \to \mathbb{R}^{n(d-1)+d}$ the projections on the first n coordinates and the last $n(d-1)+d$ coordinates respectively. By (ii), $\pi_1 \circ f = f_1$ is a diffeomorphism from U_1 to Ω_1, which means π_1 maps $f(U_1)$ bijectively onto Ω_1. Consider now the smooth map $g_1 = \pi_2 \circ f \circ f_1|_{U_1}^{-1} \colon \Omega_1 \to \mathbb{R}^{n(d-1)+d}$. We have $Graph(g_1) = f(U_1)$. Similar to Example 2.4, we define a diffeomorphism $h_1 \colon \Omega_1 \times \mathbb{R}^{n(d-1)+d} \to \Omega_1 \times \mathbb{R}^{n(d-1)+d}$, $(u, v) \mapsto (u, v - g_1(u))$. We see that h_1 maps $f(U_1)$ bijectively onto $\Omega_1 \times \{0\}$. Let $V_1 \subset \Omega_1 \times \mathbb{R}^{n(d-1)+d}$ be the open set in $\mathbb{R}^{nd+d}$ satisfying $f(U_1) = f(M) \cap V_1$. The restriction $h_1|_{V_1}$ is a diffeomorphism from V_1 onto an open set $\Pi_1 \subset \Omega_1 \times \mathbb{R}^{n(d-1)+d}$, and maps $f(M) \cap V_1$ bijectively onto $\Pi_1 \cap (\mathbb{R}^n \times \{0\})$. Hence $(V_1, h_1|_{V_1}, \Pi_1)$ is the desired chart on $\mathbb{R}^{nd+d}$.
   The remaining $f(U_i)$ are treated analogously. Note that $(f(U_i))_{1 \le i \le d}$ is a cover of $f(M)$ by open sets, since $f \colon M \to f(M)$ is a homeomorphism.

2. Prove that $f\colon M \to f(M)$ is a diffeomorphism.
   We know that $f|_{U_1} = \pi_1|^{-1}_{f(U_1)} \circ f_1|_{U_1}\colon U_1 \to f(U_1)$ is a diffeomorphism. The same statement is also true for the other U_i. Therefore, f is a diffeomorphism.

□

Theorem 2.9. Every compact topological n-dimensional manifold is homeomorphic to a topological submanifold of $\mathbb{R}^{n+k}$.

Proof. Use the above proof, replacing "smooth manifold" with "topological manifold", "smooth map" with "continuous map" and "diffeomorphism" with "homeomorphism". □

3 Tangent Space

Definition 3.1 (Tangent space). Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $x \in M$.

1. Let I_1, I_2 be two open intervals around 0 and $\gamma_1\colon I_1 \to M, \gamma_2\colon I_2 \to M$ be two smooth paths with $\gamma_1(0) = \gamma_2(0) = x$. Let $(U, h, \Omega) \in \mathcal{A}$ be a chart with $x \in U$. Then γ_1 and γ_2 are equivalent, if $(h \circ \gamma_1)'(0) = (h \circ \gamma_2)'(0)$.

2. A tangent vector of M at x is an equivalence class $[\gamma]_x$. The tangent space of M at x is the set $T_xM = \{[\gamma]_x \mid \gamma\colon I \to M \text{ smooth}, \gamma(0) = x\}$.

Remark . The definition is independent of choice of charts:
Let $(U_1, h_1, \Omega_1), (U_2, h_2, \Omega_2) \in \mathcal{A}$ be two charts with $x \in U_1 \cap U_2$. Suppose $(h_1 \circ \gamma_1)'(0) = (h_1 \circ \gamma_2)'(0)$. The chart transition $h_2 \circ h_1^{-1}\colon h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2)$ is a smooth map between open subsets of $\mathbb{R}^n$. Denote with $J_{h_1(x)}(h_2 \circ h_1^{-1})$ its Jacobian matrix evaluated at $h_1(x)$. Using the chain rule, we have $(h_2 \circ h_1^{-1} \circ h_1 \circ \gamma_1)'(0) = J_{h_1(x)}(h_2 \circ h_1^{-1}) \cdot (h_1 \circ \gamma_1)'(0) = J_{h_1(x)}(h_2 \circ h_1^{-1}) \cdot (h_1 \circ \gamma_2)'(0) = (h_2 \circ h_1^{-1} \circ h_1 \circ \gamma_2)'(0)$, and thus $(h_2 \circ \gamma_1)'(0) = (h_2 \circ \gamma_2)'(0)$. The other direction follows similarly with the chart transition $h_1 \circ h_2^{-1}$.

Example 3.2 ($U \subset \mathbb{R}^n$ open). One identifies T_xU with $\mathbb{R}^n$:
We use the canonical structure on U given by the chart (U, id, U). The equivalence relation is then $\gamma_1 \sim \gamma_2 \Leftrightarrow \gamma_1'(0) = \gamma_2'(0)$. The map $T_xU \to \mathbb{R}^n$, $[\gamma]_x \mapsto \gamma'(0)$ is defined to be injective. It is also surjective, since for $v \in \mathbb{R}^n$ we can define the smooth path $\gamma(t) = x + tv$. Hence it is a bijection between T_xU and $\mathbb{R}^n$. By abuse of notation, we often regard $T_xU = \mathbb{R}^n$.

Definition 3.3 (Addition and scalar multiplication). Let T_xM be the tangent space of $(M, \mathcal{A})$ at x. Let $(U, h, \Omega) \in \mathcal{A}$ be a chart with $x \in U$.

1. We define the addition $+\colon T_xM \times T_xM \to T_xM$ as $[\gamma_1]_x + [\gamma_2]_x = [h^{-1} \circ \frac{1}{2}(h \circ \alpha_1 + h \circ \alpha_2)]_x$, where $\alpha_1(t) = \gamma_1(2t), \alpha_2(t) = \gamma_2(2t)$.

2. We define the scalar multiplication $\cdot\colon \mathbb{R} \times T_xM \to T_xM$ as $\lambda \cdot [\gamma]_x = [\gamma_\lambda]_x$, where $\gamma_\lambda(t) = \gamma(\lambda t)$.

The two operations are well-defined and define a real vector space structure on T_xM.

Lemma 3.4. Using the same chart as in Definition 3.3, the map $\Phi_h\colon T_xM \to \mathbb{R}^n$, $[\gamma]_x \mapsto (h \circ \gamma)'(0)$ is a linear isomorphism.

Proof. Bijectivity: Φ_h is injective by the definition of the equivalence relation. For surjectivity, let $v \in \mathbb{R}^n$. We define the smooth path $\gamma(t) = h^{-1}(h(x) + tv)$. Then we have $\Phi_h([\gamma]_x) = v$.
Linearity: For addition, we have $\Phi_h([\gamma_1]_x + [\gamma_2]_x) = \Phi_h([h^{-1} \circ \frac{1}{2}(h \circ \alpha_1 + h \circ \alpha_2)]_x) = \frac{1}{2}(h\circ\alpha_1)'(0) + \frac{1}{2}(h\circ\alpha_2)'(0) = (h\circ\gamma_1)'(0) + (h\circ\gamma_2)'(0) = \Phi_h([\gamma_1]_x) + \Phi_h([\gamma_2]_x)$. For scalar multiplication, we use the chain rule to get $\Phi_h(\lambda \cdot [\gamma]_x) = \Phi_h([\gamma_\lambda]_x) = (h\circ\gamma_\lambda)'(0) = \lambda(h\circ\gamma)'(0) = \lambda\Phi_h([\gamma]_x)$. □

Remark . From Lemma 3.4, we see that the linear structure defined on T_xM does not depend on choice of charts:
Let $(U_1, h_1, \Omega_1), (U_2, h_2, \Omega_2)$ be two suitable charts. Consider the following commutative diagram

$$\begin{array}{ccc} T_xM & \xrightarrow{\Phi_{h_1}} & \mathbb{R}^n \\ & \underset{\Phi_{h_2}}{\searrow} & \downarrow F \\ & & \mathbb{R}^n \end{array}$$

where F is the isomorphism defined by the Jacobian matrix $J_{h_1(x)}(h_2 \circ h_1^{-1})$. Let $V_x, W_x \in T_xM$ be two tangent vectors of M at x. Using Definition 3.3 with the chart map h_1, we have $V_x + W_x = \Phi_{h_1}^{-1}(v_1 + w_1)$ for some $v_1, w_1 \in \mathbb{R}^n$. For $v_2 = F(v_1), w_2 = F(w_1)$ we then have $V_x + W_x = \Phi_{h_2}^{-1}(v_2 + w_2)$, which is independent from the chart map h_1. Thus the addition on T_xM is well-defined.

Corollary 3.5. Given a chart (U, h, Ω) in the maximal atlas with $x \in U$, we write $(\frac{\partial}{\partial y_i})_{(x,h)} = \Phi_h^{-1}(e_i)$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n$ is the i-th standard basis vector. The system $(\frac{\partial}{\partial y_1})_{(x,h)}, \ldots, (\frac{\partial}{\partial y_n})_{(x,h)}$ forms a basis for T_xM.

Proof. This follows directly from Lemma 3.4. □

Example 3.6. Let $[\gamma]_x \in T_xM$ be the tangent vector given by the smooth path $\gamma\colon I \to U$ with $\gamma(0) = x$. Let $(a_1, \ldots, a_n) = (h\circ\gamma)'(0) \in \mathbb{R}^n$. Since $\Phi_h([\gamma]_x) = (h\circ\gamma)'(0) = \sum_{i=1}^n a_i \cdot e_i$, we can apply the linear map Φ_h^{-1} on both sides and receive $[\gamma]_x = \sum_{i=1}^n a_i(\frac{\partial}{\partial y_i})_{(x,h)}$.

Definition 3.7 (Differential of smooth map at a point). Let $f\colon M \to M'$ be a smooth map between smooth manifolds and $x \in M$. The differential of f at x is the map $D_xf\colon T_xM \to T_{f(x)}M'$, $[\gamma]_x \mapsto [f\circ\gamma]_{f(x)}$.

Lemma 3.8. Let $f\colon M \to M', g\colon M' \to M''$ be smooth maps between smooth manifolds and $x \in M$.

1. The differential D_xf of f at x is well-defined and linear.

2. For the composition $g \circ f$ the chain rule is fulfilled, i.e. $D_x(g \circ f) = D_{f(x)}g \circ D_x f$.

3. We have $D_x id_M = id_{T_xM}$.

4. For a constant map $const\colon M \to M'$, we have $D_x const([\gamma]_x) = [0]_{const(x)}$, where $[0]_{const(x)}$ denotes the neutral element of $T_{const(x)}M'$ with respect to addition.

5. Let (U, h, Ω) be a chart around x in M. Then $D_x h\colon T_xU \to T_{h(x)}\Omega$ is an isomorphism and its inverse is given by $(D_x h)^{-1} = D_{h(x)}h^{-1}$.

Proof. (1): Let (U, h, Ω) be a chart around x in M and (U', h', Ω') be a chart around $f(x)$ in M'. Consider the following commutative diagram

$$\begin{array}{ccc} T_xM & \xrightarrow{D_xf} & T_{f(x)}M' \\ {\scriptstyle\Phi_h}\downarrow & & \downarrow{\scriptstyle\Phi_{h'}} \\ \mathbb{R}^m & \xrightarrow{F} & \mathbb{R}^n \end{array}$$

where F is the linear map defined by the Jacobian matrix $J_{h(x)}(h' \circ f \circ h^{-1})$. Since Φ_h and $\Phi_{h'}$ are linear isomorphisms, there exists a linear map $T_xM \to T_{f(x)}M'$ so that the diagram is commutative. The exact formula in Definition 3.7 is given by the composition $\Phi_{h'}^{-1} \circ F \circ \Phi_h$.
(2), (3): Follows directly from Definition 3.7.
(4): The equivalence class $[const \circ \gamma]_{const(x)}$ is represented by the smooth path $\alpha\colon I \to M'$, $t \mapsto const(x)$. Using the linear map $\Phi_{h'}^{-1}$ for a suitable chart (U', h', Ω'), we have $[\alpha]_{const(x)} = \Phi_{h'}^{-1}((h' \circ \alpha)'(0)) = \Phi_{h'}^{-1}(0)$.
(5): Let the dimension of M be n. By Example 1.9, U and Ω are both n-dimensional smooth manifolds and thus T_xU and $T_{h(x)}\Omega$ have the same dimension as vector spaces. We show that D_xh is injective through the following commutative diagram:

$$\begin{array}{ccc} T_xU & \xrightarrow{D_xh} & T_{h(x)}\Omega \\ & {\scriptstyle\Phi_h}\searrow & \downarrow{\scriptstyle d} \\ & & \mathbb{R}^n \end{array}$$

Φ_h is bijective by Lemma 3.4 and d is the bijection given in Example 3.2. Thus D_xh must be injective, and hence bijective. The formula for the inverse can be deduced using (2) and (3) with $f = h, g = h^{-1}$. □

Example 3.9. Let $N \subset \mathbb{R}^n$ be a smooth k-dimensional submanifold and (U, h, Ω) be a chart in $\mathbb{R}^n$ as in Definition 2.1. Then $(U', h', \Omega') := (U \cap N, h|_{U \cap N}, \Omega \cap (\mathbb{R}^k \times \{0\}))$ is a chart in N. Let $i\colon N \to \mathbb{R}^n$ be the inclusion map and $x \in U'$. Since $i = h^{-1} \circ j \circ h'$, where $j\colon \mathbb{R}^k \to \mathbb{R}^n, (v_1, \dots, v_k) \mapsto (v_1, \dots, v_k, 0, \dots, 0)$, the rank of D_xi is determined by the rank of $J_{h'(x)}j$, which is k. Hence $D_xi\colon T_xN \to T_x\mathbb{R}^n \cong \mathbb{R}^n$ is injective.
One usually identifies T_xN with $D_xi(T_xN) \subset \mathbb{R}^n$, which is the set of vectors $\{\gamma'(0) \mid \gamma\colon I \to N \in \mathbb{R}^n \text{ smooth}, \gamma(0) = x\}$ (see Example 3.2).

Lemma 3.10. Let $f\colon M \to M'$ be a smooth map between smooth manifolds and $x \in M$. Let $(U, h, \Omega) \in \mathcal{A}_M$ be a chart with $x \in U$ and $(U', h', \Omega') \in \mathcal{A}_{M'}$ be a chart with $f(x) \in U'$. Let $\{(\frac{\partial}{\partial y_i})_{(x,h)} \mid i = 1, 2, \ldots, n\}$ be a basis of T_xM and $\{(\frac{\partial}{\partial z_i})_{(f(x),h')} \mid i = 1, 2, \ldots, n'\}$ be a basis of $T_{x'}M'$. The matrix representation of the differential D_xf of f at x is the Jacobian matrix $J_{h(x)}(h' \circ f \circ h^{-1})$.

Proof. This follows directly from the first commutative diagram in Lemma 3.8 and the definition of the bases. □

Remark . In particular, if $M = M'$ and $f = id_M$, we have the matrix $J_h(x)(h' \circ h^{-1})$, i.e. $(\frac{\partial}{\partial y_i})_{(x,h)} = \sum_{j=1}^{n} \partial_j \phi_i(h(x)) \cdot (\frac{\partial}{\partial z_j})_{(x,h')}$, where $\phi = h' \circ h^{-1}$ denotes the smooth chart transition which expresses z_j-coordinates in terms of y_i-coordinates.

Lemma 3.11. There is a natural isomorphism $T_{(x_0,x'_0)}(M \times M') \cong T_{x_0}M \times T_{x'_0}M'$.

Proof. Denote the projections with $\pi\colon M \times M' \to M$ and $\pi'\colon M \times M' \to M'$, and let $i\colon M \to M \times M'$, $x \mapsto (x, x'_0)$ and $i'\colon M' \to M \times M'$, $x' \mapsto (x_0, x')$ be the inclusions. Consider the map $f = D_{(x_0,x'_0)}\pi \times D_{(x_0,x'_0)}\pi'\colon T_{(x_0,x'_0)}(M \times M') \to T_{x_0}M \times T_{x'_0}M'$, which is linear since $D_{(x_0,x'_0)}\pi$ and $D_{(x_0,x'_0)}\pi'$ are linear. Further we consider $g\colon T_{x_0}M \times T_{x'_0}M' \to T_{(x_0,x'_0)}(M \times M')$, $(V_{x_0}, V_{x'_0}) \mapsto D_{x_0}i(V_{x_0}) + D_{x'_0}i'(V_{x'_0})$. Then by Lemma 3.8, we have

$$(f \circ g)(V_{x_0}, V_{x'_0}) = f(D_{x_0}i(V_{x_0}) + D_{x'_0}i'(V_{x'_0}))$$

$$= (D_{(x_0,x'_0)}\pi(D_{x_0}i(V_{x_0}) + D_{x'_0}i'(V_{x'_0})), D_{(x_0,x'_0)}\pi'(D_{x_0}i(V_{x_0}) + D_{x'_0}i'(V_{x'_0})))$$
$$= (D_{x_0}(\pi \circ i)(V_{x_0}) + D_{x'_0}(\pi \circ i')(V_{x'_0}), D_{x_0}(\pi' \circ i)(V_{x_0}) + D_{x'_0}(\pi' \circ i')(V_{x'_0}))$$
$$= (D_{x_0}id_M(V_{x_0}) + D_{x'_0}const(V_{x'_0}), D_{x_0}const(V_{x_0}) + D_{x'_0}id_{M'}(V_{x'_0})) = (V_{x_0}, V_{x'_0}).$$

Since f has a right inverse, f is surjective. Thus f is a linear isomorphism, since the dimensions of both vector spaces are equal. □

Definition 3.12 (Directional derivative). Let $f \in C^\infty(M, \mathbb{R})$ be a smooth function and $D_xf\colon T_xM \to T_{f(x)}\mathbb{R} \cong \mathbb{R}$ be the differential of f at $x \in M$. For a tangent vector $V_x = [\gamma]_x \in T_xM$, the directional derivative is $V_x(f) = D_xf(V_x) \in \mathbb{R}$. In other words, $V_x(f) = (f \circ \gamma)'(0)$.

4 Vector Bundles

Definition 4.1 (Vector bundle). An n-dimensional vector bundle ξ over a topological space B (base space) consists of:

1. a topological space E (total space)

2. a continuous map $p\colon E \to B$ (projection, the preimage $F_b = p^{-1}(b)$ is called fiber of $b \in B$)

3. For every point $b \in B$ there exist a neighbourhood U of b and a homeomorphism $H\colon p^{-1}(U) \to U \times \mathbb{R}^n$ (bundle chart map), such that the following diagram is commutative

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{H} & U \times \mathbb{R}^n \\ & \overset{p}{\searrow} & \big\downarrow pr_U \\ & & U \end{array}$$

   and the following condition is fulfilled:
   Let U_1, U_2 be two neighbourhoods with $U_1 \cap U_2 \neq \emptyset$ and $H_1\colon p^{-1}(U_1) \to U_1 \times \mathbb{R}^n, H_2\colon p^{-1}(U_2) \to U_2 \times \mathbb{R}^n$ be the corresponding homeomorphisms. We consider the restrictions $H_1|_{p^{-1}(U_1 \cap U_2)}, H_2|_{p^{-1}(U_1 \cap U_2)}\colon p^{-1}(U_1 \cap U_2) \to (U_1 \cap U_2) \times \mathbb{R}^n$. Let $H_{21} = H_2 \circ H_1^{-1}\colon (U_1 \cap U_2) \times \mathbb{R}^n \to (U_1 \cap U_2) \times \mathbb{R}^n$ be the "bundle chart transition". Since the diagram above is commutative, we have $H_{21}(x, v) = (x, \tilde{H}_{21}(x, v))$ for some map $\tilde{H}_{21}(x, \_)\colon \mathbb{R}^n \to \mathbb{R}^n$.
   For every $x \in U_1 \cap U_2$, we require $\tilde{H}_{21}(x, \_)$ to be linear, i.e. $\tilde{H}_{21}(x, \_) \in GL_n(\mathbb{R})$.

A bundle chart is a pair (U, H), where $U \subset B$ and the homeomorphism $H\colon p^{-1}(U) \to U \times \mathbb{R}^n$ are defined as above. If $B = \bigcup_{i \in I} U_i$, then a bundle atlas is a set $\mathcal{A} = \{(U_i, H_i) \mid i \in I\}$ of bundle charts. For two charts (U_1, H_1) and (U_2, H_2) with $U_1 \cap U_2 \neq \emptyset$, we define the so-called transition function $u_{21}\colon U_1 \cap U_2 \to GL_n(\mathbb{R})$, $x \mapsto \tilde{H}_{21}(x, \_)$, which is automatically continuous.

Remark . Since the diagram above is commutative, the fiber F_b is automatically homeomorphic to $\{b\} \times \mathbb{R}^n$ through the homeomorphism $H|_{F_b}$.

Example 4.2 (Trivial vector bundle). The trivial vector bundle over a topological space B is given by the projection $p\colon B \times \mathbb{R}^n \to B$. A bundle atlas is then $\mathcal{A} = \{(U, id) \mid U \subset B$ open$\}$.

Example 4.3. The restriction of a vector bundle is a vector bundle:
Let ξ be the vector bundle given by $p\colon E \to B$ and $A \subset B$ be a subspace. Then the restriction ξ_A of ξ on A is the vector bundle given by $p|_{p^{-1}(A)}\colon p^{-1}(A) \to A$. If $\mathcal{A} = \{(U_i, H_i) \mid i \in I\}$ is a bundle atlas for B, then a bundle atlas for A is $\mathcal{A}_A = \{(U_i \cap A, H_i|_{p^{-1}(U_i \cap A)}) \mid (U_i, H_i) \in \mathcal{A}\}$. (The linearity of the homeomorphisms $\tilde{H}_{21}$ is preserved under the restriction.)

Definition 4.4 (Smooth vector bundle). Let ξ be a vector bundle over a smooth manifold M given by $p\colon E \to M$.

1. A bundle atlas $\mathcal{A}$ is smooth, if all the transition functions u_{ji} are smooth maps (between smooth manifolds).

2. We define the term "maximal bundle atlas" similarly as in Definition 1.2. A smooth vector bundle is then a pair $(\xi, \mathcal{A})$, where $\mathcal{A}$ is a maximal bundle atlas.

Example 4.5 (Möbius bundle). The Möbius bundle is an example of a non-trivial vector bundle:

Consider the projective space $\mathbb{RP}^1$ as the set of 1-dimensional subspaces $L \subset \mathbb{R}^2$. Define the total space $E = \{(L, v) \mid L \in \mathbb{RP}^1, v \in L\} \subset \mathbb{RP}^1 \times \mathbb{R}^2$ with the subspace topology. Let $p\colon E \to \mathbb{RP}^1$, $(L, v) \mapsto L$ be the projection. This defines a 1-dimensional smooth vector bundle τ_1 on $\mathbb{RP}^1$. (A smooth bundle atlas is $\mathcal{A} = \{(U, id) \mid U \subset \mathbb{RP}^1 \text{ open}\}$.)
Non-trivial: We define the zero-section of a vector bundle as the map $\zeta\colon B \to E$, $b \mapsto (b, 0)$. For the trivial vector bundle given by $pr\colon \mathbb{RP}^1 \times \mathbb{R} \to \mathbb{RP}^1$, we have $(\mathbb{RP}^1 \times \mathbb{R}) - \zeta(\mathbb{RP}^1) = \mathbb{RP}^1 \times (\mathbb{R} - 0)$, which is not connected. On the other hand for $p\colon E \to \mathbb{RP}^1$ defined above, let $(L_1, v_1), (L_2, v_2) \in E - \zeta(\mathbb{RP}^1)$, i.e. $v_1 \neq 0 \neq v_2$. Denote with $\pi\colon \mathbb{S}^1 \to \mathbb{RP}^1$ the canonical projection. Without restriction of generality, let $\|v_1\| = \|v_2\| = 1$. Then there is a path $w\colon [0, 1] \to \pi(\mathbb{S}^1) \times \mathbb{S}^1$ from (L_1, v_1) to (L_2, v_2), which means $E - \zeta(\mathbb{RP}^1)$ is connected (see Figure 7).

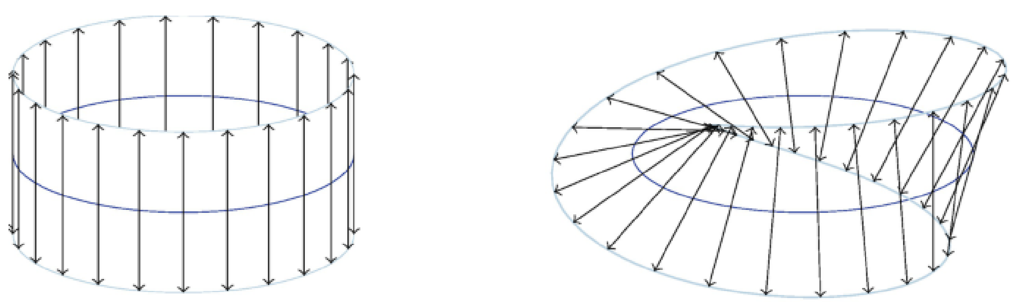

Figure 7: On the left is the trivial vector bundle, and on the right the Möbius bundle. The images of the zero-section $Im(\zeta)$ are marked with blue. We can see that for the trivial vector bundle, the complement of $Im(\zeta)$ is not connected. On the contrary, the complement of $Im(\zeta)$ is connected for the Möbius bundle.

Definition 4.6 (Map between vector bundles). Let ξ, ξ' be two vector bundles given by $p\colon E \to B$ and $p'\colon E' \to B'$ respectively.

1. A map between vector bundles $(\theta, \beta)\colon \xi \to \xi'$ is a pair of maps $\theta\colon E \to E'$ and $\beta\colon B \to B'$, such that $p' \circ \theta = \beta \circ p$ and for every $b \in B$ the induced map $\theta_b\colon F_b \to F'_{\beta(b)}$ is linear.

2. If additionally $B = B'$ and $\beta = id_B$, then θ is called a bundle map over B.

Lemma 4.7. Given a set E, a topological space B, a surjective map $p\colon E \to B$ and a set $\mathcal{A} = \{H_i\colon p^{-1}(U_i) \to U_i \times \mathbb{R}^n \mid U_i \subset B \text{ open}, B = \bigcup_{i \in I} U_i, p|_{p^{-1}(U_i)} = pr_{U_i} \circ H_i\}$ of maps, such that the transition functions u_{ji} are continuous.
Then there exists exactly one topology on E such that $p\colon E \to B$ defines an n-dimensional vector bundle, where $\mathcal{A}$ is a bundle atlas.

Proof. For a given map $H\colon p^{-1}(U) \to U \times \mathbb{R}^n$, let $e \in p^{-1}(U)$. Since B is a topological space, the topology on $U \times \mathbb{R}^n$ is defined. For the point $H(e) \in U \times \mathbb{R}^n$, we find a neighbourhood base $\{V_j\}$ at $H(e)$. To make H a homeomorphism, the preimages $\{H^{-1}(V_j)\}$ must form a neighbourhood base of e in $p^{-1}(U)$. We define a "neighbourhood" of e as a superset of an element in $\{H^{-1}(V_j)\}$. Note that this definition is independent of choice of H, since the "bundle chart transitions" $H_{ji} = H_j \circ H_i^{-1}$ need to be homeomorphisms. We do the same for all $e \in p^{-1}(U)$ and then for all $H \in \mathcal{A}$. Since a subset is open if and only if it is a neighbourhood of every element in itself, this defines (the only possible) topology on E. □

Remark . A subbasis of the topology on E is given by $S = \{H_i^{-1}(V_i) \mid i \in I, H_i \in \mathcal{A}, V_i \subset U_i \times \mathbb{R}^n \text{ open}\}$.

Corollary 4.8. We use the assumptions in Lemma 4.7 with the exceptions, that $B = M$ is a smooth manifold and the transition functions u_{ji} are smooth. Then there exists exactly one topology on E such that $p\colon E \to M$ defines a smooth vector bundle, where $\mathcal{A}$ is a smooth bundle atlas.

Proof. Use the same proof as above. □

Definition 4.9 (Tangent bundle). Let M be a smooth manifold. We define the set $TM = \bigsqcup_{x\in M} T_xM$ and the surjective map $p\colon TM \to M$, $(x, V_x) \mapsto x$.
Let $\mathcal{A}_{max} = \{(U_i, h_i, \Omega_i) \mid i \in I\}$ be the maximal atlas on M. For $i \in I$, we have $p^{-1}(U_i) = \{(x, V_x) \mid x \in U_i, V_x \in T_xU_i\} = TU_i$ and $\Omega_i \times \mathbb{R}^n = \{(h_i(x), V_{h_i(x)}) \mid x \in U_i, V_{h_i(x)} \in T_{h_i(x)}\Omega_i\} = T\Omega_i$ because of Example 3.2. Consider the maps $\mathcal{H}_i\colon p^{-1}(U_i) \to \Omega_i \times \mathbb{R}^n$, $(x, V_x) \mapsto (h_i(x), D_xh_i(V_x))$. We define $H_i = (h_i^{-1} \times id) \circ \mathcal{H}_i$, which means $H_i\colon p^{-1}(U_i) \to U_i \times \mathbb{R}^n$, $(x, V_x) \mapsto (x, D_xh_i(V_x))$.
We check that for $i \in I$, we have $p(x, V_x) = x = pr_{U_i}(x, D_xh_i(V_x)) = pr_{U_i}(H_i(x, V_x))$ for $(x, V_x) \in p^{-1}(U_i)$, which means the diagram below is commutative.

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{H_i} & U_i \times \mathbb{R}^n \\ & \underset{p|_{p^{-1}(U_i)}}{\searrow} & \downarrow pr_{U_i} \\ & & U_i \end{array}$$

Now let $(U_1, h_1, \Omega_1), (U_2, h_2, \Omega_2) \in \mathcal{A}_{max}$ be two charts on M with $U_1 \cap U_2 \neq \emptyset$. By Lemma 3.8(5), the inverse map $H_1^{-1}\colon U_1 \times \mathbb{R}^n \to p^{-1}(U_1)$ maps $(x, v) \mapsto (x, D_{h_1(x)}h_1^{-1}(v))$. Thus $H_{21} = H_2 \circ H_1^{-1}$ is defined as $H_{21}(x, v) = (x, D_xh_2(D_{h_1(x)}h_1^{-1}(v)))$, i.e. $\tilde{H}_{21}(x, v) = D_xh_2(D_{h_1(x)}h_1^{-1}(v)) = D_{h_1(x)}(h_2 \circ h_1^{-1})(v)$ with Lemma 3.8(2). We can then use the matrix representation in Lemma 3.10 on $\tilde{H}_{21}(x, \_)$ and receive $\tilde{H}_{21}(x, \_) = J_{h_1(x)}(h_2 \circ h_1^{-1}) \in GL_n(\mathbb{R})$. The transition function $u_{21}(x) = J_{h_1(x)}(h_2 \circ h_1^{-1})$ is a smooth map.
We use the topology on TM given in Corollary 4.8. Then $p\colon TM \to M$ defines a smooth vector bundle. We name it the tangent bundle of M.

Remark . In the above definition, a smooth bundle atlas is given by $\mathcal{A} = \{(U_i, H_i) \mid (U_i, h_i, \Omega_i) \in \mathcal{A}_{max}\}$. We see that TM gets a smooth structure from the smooth structure of M.

Example 4.10 ($T\mathbb{S}^1$ is trivial). We show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$:
For a point $x \in \mathbb{S}^1$, we write $v(x) = (-x_2, x_1) \in T_x\mathbb{S}^1$ for the tangent vector of $\mathbb{S}^1$ at x. Note that $T_x\mathbb{S}^1 \cong \mathbb{R}$, since $T_x\mathbb{S}^1$ is a 1-dimensional real vector space. Consider the map

$f\colon T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$, $(x, V_x) \mapsto (x, v(x) \cdot V_x)$. It is smooth and bijective. The inverse is given by $f^{-1}\colon \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1$, $(x, \lambda) \mapsto (x, \lambda v(x))$, which is also smooth.
For better visualization, consider the smooth vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(v_1, v_2) \mapsto (-v_2, v_1)$. For a point $x \in \mathbb{S}^1$, F rotates the tangent space $T_x\mathbb{S}^1$ by 90° in the anticlockwise direction, giving us $x \in F(T_x\mathbb{S}^1)$. The tangent spaces form the curved surface of the (infinitely long) cylinder around $\mathbb{S}^1$ (see Figure 8). Thus, we have $T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$.

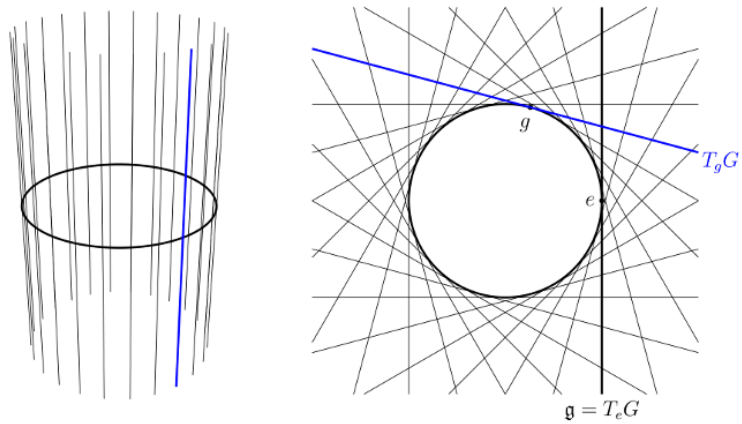

Figure 8: The blue line on the right represents the tangent space $T_g\mathbb{S}^1$ of $\mathbb{S}^1$ at the point g. Using the vector field F in the Example, $T_g\mathbb{S}^1$ will be rotated around g until it covers the origin $(0, 0)$. We can then use the interpretation on the left to understand the blue line as part of the cylindrical structure around $\mathbb{S}^1$. Doing this for every $g \in \mathbb{S}^1$ gives us the result $T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$.

Example 4.11 ($T\mathbb{S}^2$ is non-trivial). This is a result of the hairy ball theorem / hedgehog theorem.

Definition 4.12 (Differential of smooth map). Let $f\colon M \to M'$ be a smooth map between smooth manifolds. The differential of f is the smooth map between vector bundles $Df\colon TM \to TM'$ over f, which is defined by the different differentials $D_xf\colon T_xM \to T_{f(x)}M'$ at the points $x \in M$.

5 Cited Sources/Literature

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- `https://www.researchgate.net/figure/Two-overlapping-charts-and-their-transition-map_fig1_273327918`: Figure 01

- https://ncatlab.org/nlab/show/stereographic+projection: Figure 04
- http://staff.ustc.edu.cn/~wangzuoq/Courses/18F-Manifolds/Notes/Lec08.pdf: Figure 06
- https://link.springer.com/chapter/10.1007/978-3-0348-0983-2_2: Figure 07
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