Graduate Seminar on Topology (S4D2 Master)

Sommeremester 2022

Mapping Class Groups

Prof. Dr. Carl-Friedrich Bödigheimer

Tuesdays, 14:15 – 16:00 Uhr, seminar room N 0.008

Please, observe the registration procedure for seminars.

Content. A mapping class group $\Gamma = \Gamma(S)$ of an orientable surface S is the group of isotopy classes of orientation-preserving homeomorphisms (or diffeomorphisms) $f\colon S \to S$. This is a discrete group and it has many applications in low-dimensional topology and the theory of Riemann surfaces. For example, its classifying space $B\Gamma(S)$ classifies surface bundles with fibre S and so the cohomology of $B\Gamma(S)$ is the natural place to look for characteristic classes of such bundles. Or, as another example, if S is a Riemann surface, its group of automorphisms $\mathrm{Aut}(S)$ is a finite subgroup of $\Gamma(S)$, except in a few cases.

Obviously, the mapping class group $\Gamma(S)$ depends only on homeomorphism type of S; and so we write Γ_g if S is closed of genus g. We may also study the groups of mapping classes keeping m marked points fixed or keeping a boundary curve pointwise fixed. We write $\Gamma^m_{g,r}$ in case we have m marked points and r boundary curves (and we drop these parameters in case they are zero).

Famous examples of mapping class groups are the torus, when $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, or the m-times punctured disk, when $\Gamma^m_{0,1}$ is the m-th braid group.

Many facts are known, but many questions are still wide open. For example, we will see the Dehn twists as their generators; and we will see a finite presentation of $\Gamma(S)$. The Dehn twists along two intersecting curves satisfy the braid relation, another example for this important relation.

We will prove the theorem of Dehn-Nielsen-Baer which identifies $\Gamma(S)$ with an index two subgroup of the outer automorphism group of the fundamental group $\pi_1(S)$. We will calculate the first and second homology or cohomology groups of $\Gamma(S)$. Only a few higher homology groups are known for a few low genera $g = 0, 1, 2, 3$ and 4. However, we will learn the important stability theorem of Harer. (Another important theorem of Harer about the homological dimension of $\Gamma(S)$ will be proven in the lecture course.)

The seminar has some connection to my lecture course in the summer term 2022 on moduli spaces of Riemann surfaces. Although independent, it may be helpful for some talks to also attend the lecture course.

For all talks I have sketched the content and listed the literature. The material to be learned is certainly too much for a single talk; so we need – after you have learned the entire material – to make a selection what to present in detail, what to report on, and what to skip.

Prerequisites. The lecture courses *Topologie I* and *Topologie II* are indispensable. The courses *Algebraic Topology I + II* are certainly helpful and for some of the talks necessary. We assume a fair understanding of groups and group actions, as well as classifying spaces of groups and (co)homology of groups.

Literature. The book *A Primer on Mapping Class Groups* by Farb-Margalit will be used for many talks. I recommend to read the chapter *Overview* in this book to get some taste of the seminar. For some talks we must use research and survey articles.

Preparation of talks. The talks are supposed to be 90 minutes long. That means, you should prepare about 70 minutes and allow time for questions during the talk.

You should consult me about the talk at least two weeks before the day of the talk. If a talk is shared by two students, you both prepare together the entire talk and at the beginning of each seminar session we decide by throwing a coin who starts with the first half and who gives the second half of the talk.

Talks

(1) **Basics** Anton Ablov — 05.04.2022

Definitions. Examples, in particular symmetries of Riemann surfaces. Alexander-Lemma for the disk. More examples: annulus, punctured spheres, torus, punctured torus, torus with boundary. Punctures versus marked points, boundaries versus marked points, directions versus boundaries. Detecting mapping classes.

[F-M, 2].

(2) **Dehn Twists** Tianyi Feng — 12.04.2022

Definition. Properties. Geometric intersection number, homological intersection number. Braid relation. Lantern relation. Examples.

[F-M, 3, 5.1].

(3) **Inclusions and Surjections** Matthias Gessinger — 19.04.2022

Forgetful maps, capping maps. Birman exact sequence.

[F-M, 3.6 + 4.2].

(4) **Generators** Jacke McCall — 26.04.2022

Curve complex. Finite generation. Explicite generators of Lickorish or of Humphries.

[F-M, 4.1 + 4.3 + 4.4].

(5) **Finite Presentability** Louis Wüllner — 03.05.2022

Wajnryb's presentation. Proof.

[F-M, 5.2 + 5.3].

(6) **First Homology Group H_1** Quentin Lapie — 10.05.2022

Proof using the Wajnryb presentation. Examples.

[F-M, 5.1].

(7) **Second Homology Group H_2** LEO NAVARRO CHAFLOQUE — 17.05.2022

Sketch of Harer's original approach. Present Pitsch's proof.

[F-M, 5.4], [H-83], [P].

(8) **Euler Class** DANIEL BERMUDEZ — 24.05.2022

Definitions. Examples.

[F-M, 5.5].

(9) **Meyer Signature Cocycle** KONSTANTIN EMMING — 31.05.2022

Definition. Examples.

[F-M, 5.6]

No talk on 07.06.2022, because of the Pentecost Holyday Week

(10) **Outer automorphisms of $\pi_1(S)$** BRUNA DULAR — 14.06.2022

Extended mapping class group. Examples. Preparations: Metric on $\pi_1(S)$, Cayley graph, quasi-isometries. Theorem of Dehn-Nielsen-Baer, and proof.

[F-M, 8].

(11) **Euler characteristic** MORITZ HARTLIEB — 21.06.2022

Definition for a group, using $H_*(\Gamma(S);\mathbb{Q}) \cong H_*(B\Gamma(S);\mathbb{Q})$ and the fact that $B\Gamma(S)$ is rationally the moduli space. Tilings of a surface and gluings of a regular $2n$-gon. Computation of the Euler characteristic.

[L, chap. 8, especially 8.5 + 8.6], [H-Z], [La].

(12) **Homological Stability I** FADI MEZHER — 28.06.2022

Three kinds of stabilizations, in particular $\Gamma_{g,1} \to \Gamma_{g+1,1}$, inducing an isomorphism $H_*(\Gamma_{g,1}) \to H_*(\Gamma_{g+1,1})$ for $* \leq \frac{2}{3}(g-1)$. Ordered curve complex.

[W, §1 + §2]

(13) **Homological Stability II** FLORIAN KRANHOLD — 05.07.2022

Spectral sequence for the arc complex.

[W, §3]

(14) **Homological Stability III** HASAN TUNA & JULIAN BRÜGGEMANN, — 12.07.2022

Connectivity arguments. Closed surfaces via capping.

[W, §4 + §5]

References

[F-M] **Benson Farb - Dan Margalit**: *A Primer on Mapping Class Groups.*
Princeton University Press, Princeton Mathematical Series vol. 49 (2012).

[H-83] **John Harer**: *The second homology group of the mapping class group of an orientable surface.*
Inventiones Math. **72** (1983), 221 - 239.

[H-85] **John Harer**: *Stability of the homology of the mapping class groups of orientable surfaces.*
Ann. Math. **(2) 121** (1985), 215 - 249.

[H-Z] **John Harer - Don Zagier**: *The Euler characteristic of the moduli space of curves.*
Inventiones Math. **86** (1986), 457 - 485.

[L] **Sergei K. Lando**: *Lectures on Generating Functions.*
American Mathematical Society, Student Mathematical Library vol. 23 ((2003).

[La] **Bodo Lass**: *Démonstration combinatoire de la formule de Harer-Zagier.*
Comtes Rendu Acad. Sci. Paris, Série I **333** (2001), 155 - 160.

[P] **Wolfgang Pitsch**: *Un calcul élémentaire de* $H_2(\mathcal{M}_{g,1};\mathbb{Z})$ *pour* $g \geq 4$.
Comtes Rendu Acad. Sci. Paris, Série I **329** (1999), 667 - 670.

[W] **Nathalie Wahl**: *Homological stability for mapping class groups of surfaces* .
Mathematical Association of America, Advanced Lectures in Mathematics vol. 24 (2013), 547 - 583.