Steenrod Operations

Tuesdays, 10:15 — 12:00, room MZ N 0.008
Note the two additional sessions on

Monday 09. July 2018 from 16:15 — 18:00 in room N 0.007 and
Monday 16. July 2018 from 16:15 — 18:00 in room N 0.008.

Prof. Dr. C.-F. Bödigheimer

A natural transformation of cohomology groups
$$\theta\colon h^*(X) \to h^{*+i}(X)$$
is called a cohomology operation of degree i. They are, in addition to a multiplicative structure, an important tool to show the non-existence of maps: If an abstract isomorphism $\psi\colon H^*(X;\mathbb{F}_2) \to H^*(Y;\mathbb{F}_2)$ is induced by a continuous map $f\colon Y \to X$, then $\psi = f^*$ must commute with any cohomology operation, i.e., $\psi \circ \theta = \theta \circ \psi$.

For the singular cohomology theory $h^* = H^*(\_\_;\mathbb{F}_2)$ with coefficients mod 2 there are the famous Steenrod operations
$$\mathrm{Sq}^i\colon H^n(X;\mathbb{F}_2) \longrightarrow H^{n+i}(X;\mathbb{F}_2), \quad a \mapsto \mathrm{Sq}^i(a)$$
with the properties
$$(1)\quad \mathrm{Sq}^0(a) = a, \qquad (2)\quad \mathrm{Sq}^i(a) = a^2 \quad \text{if } i = n, \quad \text{and} \qquad (3)\quad \mathrm{Sq}(a) = 0 \quad \text{if } i > n.$$
They generate an algebra $\mathfrak{A}_2$, the Steenrod algebra, acting on the mod 2 cohomology of any space. There is actually for any prime p such an algebra $\mathfrak{A}_p$ of operations on $H^*(X;\mathbb{F}_p)$.

There are astonishing applications. For example: (A) the non-existence of maps $f\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ of Hopf invariant one, if n is not a power of 2; and (B) the non-existence of 2^m linearly independent vector fields on $\mathbb{S}^{n-1}$, if $n = 2^m(2s+1)$.

In this seminar, we will construct the Steenrod algebra for $p = 2$, verify its properties, and study some interesting applications.

Prerequisites for the seminar are the courses Topologie I & II and Algebraic Topology I.

The talks are supposed to be 90 minutes. That means, you should prepare a talk of approx. 70 minutes and be ready for questions. You should consult with me two weeks before the date of the talk.

Talks

(1) **The cup product via diagonal approximation** .... FREYA BRETZ, 10.4.2018
Eilenberg-Zilber map and Alexander-Whitney map, Eilenberg-Zilber Theorem, cup product, diagonal approximation.
[MacL, chap. VIII, §§7-9], [Br, chap. VI, §4].

(2) **Construction of the cup$_i$-products** .... LEO GRAF, 17.4.2018
Failure of (signed) commutativity of the cup-product on cochain level, definition and properties of cup$_i$-products.
[Mo-Ta, chap. 2, p. 12-16], [Br, chap. VI, §16].

(3) **Construction of the Steenrod squares** Sq^i .... SULEYMAN KARACA, 24.4.2018
Definition of Sq^i as squaring of cup$_i$-product.
[Mo-Ta, chap. 2, pp. 16-21], [Br, chap. VI, §16].

(4) **The quadratic construction** .... JACK DAVIES, 8.5.2018
Quadratic construction $\Gamma_n(X) = \mathbb{S}^n_+ \wedge_{\mathbb{Z}/2} X \wedge X$, the action being the antipodal action on $\mathbb{S}^n$ and the permutation action on $X \wedge X$, i.e., identifying $(\zeta, x, x') \sim (-\zeta, x', x)$, the squaring map $\gamma\colon \Gamma(K(\mathbb{Z}_2, n)) \to K(\mathbb{Z}_2, 2n))$ and the induced transformation in cohomology.
[Gr, chap. 27, pp. 296 and following], [Ha, Sec. 4.L, pp. 501-509].

(5) **Properties of the Steenrod squares** .... OLIVER KIENAST, 15.5.2018
Axioms, Cartan formula, (Adem Relations).
[Mo-Ta, chap. 3, pp. 22-28].

(6) **The Bockstein operator** .... KEVIN LI, 29.5.2018.
Coefficient homomorphisms and transformations of cohomology theories.
[Mo-Ta, chap. 3, pp. 22+23; chap. 7, pp. 60-62; chap. 11, pp. 104-108; chap. 17, p. 173].

(7) **The Adem relations** .... DOMENICO MARASCO, 4.6.2018.
Proof of the relations; examples.
[Mo-Ta, chap. 3, pp. 29-31], [Bu-MacD].

(8) **Axiomatic description of Steenrod operations**
.... TIKHON GRITSKEVITCH, 12.6.2018
The axioms determine the operations.
[St-Ep, chap. I].

(9) **The Hopf invariant** .... NIKLAS HELLMER, 19.6.2018.
Definition, properties, application: non-existence of maps with Hopf invariant one.
[Mo-Ta, chap. 4, pp. 33-38].

(10) **Vector fields on spheres** .... XIAOUEN DONG, 26.6.2018.
Stiefel manifolds, cell decomposition, number of vector fields.
[Mo-Ta, chap. 5, pp. 29-44].

(11) **The Steenrod algebra** .... Till Wehrhan, 3.7.2018.
Steenrod algebra, decomposable elements, Hopf algebras.
[Mo-Ta, chap. 5, pp. 45-50].

(12) **The dual of the Steenrod algebra** .... Joao Rocha, **Mo 9.7.2018 (!)**
The dual algebra, Milnor basis, theorem of Milnor-Moore.
[Mo-Ta, chap. 5, pp. 50-57], [Mi].

(13) **Computation of the cohomology ring** $H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2)$
.... Tobias Fleckenstein, 10.7.2018.
Fibrations of Eilenberg-MacLane spaces, spectral sequences, differentials.
[Mo-Ta, chap. 9, pp. 83-88].

(14) **Computation of the cohomology ring** $H^*(K(\mathbb{Z}/2,q);\mathbb{F}_2)$
.... Bastiaan Cnossen, **Mo 16.7.2018 (!)**
Borel's Theorem.
[Mo-Ta, chap. 9, pp. 88-92].

(15) **The Adams spectral sequence**
.... Fernando Abellan, 17.7.2018
Stable homotopy groups, Adams filtration, Ext-groups, applications to homotopy groups.
[Mo-Ta, chap. 18].

References

[Bu-MacD] **S.R. Bullet, I.G. Macdonald**: *On the Adem relations.* Topology 21 (1982), 329-332.

[Br] **G.E. Bredon**: *Topology and Geometry.*
Graduate Texts in Mathematics, vol. 139, Springer-Verlag (1993, correted 3rd print 1997).

[Fo-Fu] **A. Fomenko, D. Fuchs**: *Homotopical Topology.*
Graduate Texts in Mathematics, vo. 273, Springer-Verlag (2016, second edition).

[Gr] **B. Gray**: *Homotopy Theory.*
Academic Press (1975).

[Ha] **A. Hatcher**: *Algebraic Topology.*
Cambridge University Press (2002).

[Mi] **J. Milnor**: *The Steenrod algebra and its dual.* Ann. Math. (2) 67 (1958), 150-171.

[MacL] **S. MacLane**: *Homology.*
Classics in Mathematics (1995), urspr. Grundlehren Bd. 114, Springer-Verlag.

[Mo-Ta] **R.E. Mosher, M.C. Tangora**: *Cohomology Operations and Applications in Homotopy Theory.*
Harper & Row Publishers (1968).

[St-Ep] **N.E. Steenrod. D.B.A. Epstein**: *Cohomology Operations.*
Princeton University Press (1962).