Seminar S4D2 (Master) **Wintersemester 2017/18**

Cohomology of Groups

Tuesdays, 8:15 — 10:00 Uhr, Seminarraum 0.008

Prof. Dr. C.-F. Bödigheimer

This seminar is an introduction to the homology and cohomology theory of discrete groups. For any group G with an action on a $\mathbb{K}$-module M we define a series of abelian groups, the homology groups $H_n(G)$ and the cohomology groups $H^n(G)$ in a functorial way, namely as the derived functors of the coinvariants M_G resp. the invariants M^G of this action. Thus the homology and cohomology groups of G come into play, whenever G acts on a module, for example, when G acts originally on a space X and therefore on its homology in any degree $M = H_*(X; \mathbb{K})$.

Prerequisites for the seminar are courses Topologie I and Topologie II.

Literature: We will be mostly using the book of K. Brown [B], but for most talks other sources are mentioned as well.

The talks are supposed to be 90 minutes. That means, prepare around 70 minutes and be ready for questions in between. The talks should be consulted with the instructors 2 weeks before the talk date.

Talks

(1) **Definition of $H_*(G)$ and $H^*(G)$** .... F. Abellan Garcia + P. Glas 10.10.2017
Groups rings, G-modules, G-invariants and G-coinvariants, projective resolutions, the functors Tor and Ext. Definition of the homology $H_n(G;M)$ as the n-th derived functor of G-coinvariants $M \mapsto M_G$. Definition of the cohomology $H^n(G;M)$ as the n-th derived functor of G-invariants $M \mapsto M^G$. In particular we obtain a definition for "trivial coefficients", i.e. $M = \mathbb{Z}$ with the trivial operation.
[B, III], [H-S, VI.2], [W, 6.1]

(2) **Classifying spaces of groups** .... J.-P. Lerch 17.10.2017
For any group G there is a well-defined homotopy type BG. Homotopy theoretic definition of BG. Geometric examples: graphs (free groups), tori (finitely generated free abelian groups), infinite lens spaces (finite cyclic groups), some knot complements, nil-manifolds. Classification theorem for principal G-bundles, the bijection $[X, BG] = \mathrm{Hom}(\pi_1(X), G))$.
[H, 4]

(3) **Group homology and classifying spaces** .... T. van Hoof + G. Bosco 24.10.2017
Idenitification $H_*(G;\mathbb{Z}) = H_*(BG;\mathbb{Z})$. Here the left hand side is the group homology from Talk 1 and the right hand side is the singular homology of the classifying space BG. Local coefficients on spaces and covering spaces. Borel construction. Identification of $H_*(G;M) = H_*(BG;\underline{M})$ with $\underline{M}$ the local coefficients system over BG associated to M. Homology of a covering space.

[B, I.4, III.1]

(4) **Homology and cohomology of the cyclic groups** .... M. Bohusch 7.11.2017
Norm-element and periodic resolutions. Calculation of $H_n(\mathbb{Z}/m; M)$ and $H^n(\mathbb{Z}/m; M)$ for trivial and non-trivial coefficients. Application: if BG is finite-dimensional then G is torsionfree.
[E, 2.1], [H-S, VI.7], [W, 6.2]

(5) **Milnor construction and bar resolution** P. Roos Hoefgeest + S. Cnossen 14.11.2017
In this talk a general construction for the space BG should be presented, the Milnor construction. Bar resolution, homogeneous and inhomogeneous version. Normalization. Comparison with the cellular chain complex of EG and BG. Application: rational homology $H_n(G; \mathbb{Q}) = 0$ for finite groups G and $n > 0$.
[B, II.3], [W, 6.5], [E, 2.3]

(6) **Interpretation of $H_1(G)$, $H_2(G)$ and $H^2(G)$** .... M. Fuentes Rumi 21.11.2017
Abelianization of G. Hopf-formula for $H_2(G)$. Extensions and their classification via $H^2(G)$.
[B, II.5, Excs 1], [H-S, VI.4, VI.9+10]

(7) **Mayer-Vietoris sequence for amalgamated products** .... A. Selimi 28.11.2017
For some groups G it is possible to decompose BG into classifying spaces of easier groups and use the Mayer-Vietoris sequence to calculate the homology of a group. Free products and free amalgamated products. Examples: free groups, groups acting on trees, nice gemoetric example $\mathrm{SL}_2(\mathbb{Z})$.
[B, II.7 + Appendix], [W, 6.2], [H-S, VI.8+14], [E, 2.2], [S]

(8) **Products, universal coefficient and Künneth theorem** .... L. Munser 5.12.2017
The cross-product in the homology and the cup-product in the cohomology. The cohomology ring $H^*(G)$. Example: the cohomology ring of the cyclic groups. Example for the Künneth sequence: $G = \mathbb{Z}/m \times \mathbb{Z}/l$.
[B, V.1-4], [E, 3], [B, III.1: Exc 3: V.5], [H-S, VI.15]

(9) **Pontrjagin product** .... J. Quintanilha 12.12.2017
Shuffle-product for the homology of abelian groups. Application: Homology of finitely-generated abelian groups.
[B, V.5+6]

(10) **Restriction, induction and transfer** .... J. Frank + F. Kranhold 19.12.2017
Restriction to subgroups, induction of coefficient modules. Transfer. Cartan-Eilenberg double-coset formula. Application: detection of cohomology by the Sylow subgroups. Example: symmetric groups.
[B, III.10], [H-S, Vi.16], [A-M, II.5+6]

(11) **Spectral sequences I: basics** .... F. Bretz 9.1.2018
Basic definitions. Spectral sequence of a double complex. Spectral sequence of a filtered complex. Application: Gysin seqeunce, Wang sequence. Example: Künneth spectral sequence.
[B, VII], [W, 5.1-5.6], [E, 7]

(12) **Spectral sequences II: L-H-S spectral sequence** .... B. Ruppik 16.1.2018
Spectral sequence of Lyndon-Hochschild-Serre for a group extension $1 \to H \to G \to Q \to 1$, equivalently of a fibration $BH \to BG \to BQ$. Example: homology of dihedral groups and wreath-products (Theorem of Nakaoka).
[B, VII.6], [W, 6.8], [E, 5.3, 7]

(13) **Cohomology theory of finite groups** .... R. Hitzl 23.1.2018

For finite groups G it turns out that the homology and cohomology groups of G have certain similar properties. These enable to organize them together in what is called the Tate cohomology groups $\widehat{H}^*(G)$. Definition and basic properties. Coordinate with the next speaker.
[B, VI.1-5]

(14) **Free actions on spheres** .... J. Davies 30.1.2018

One of the most impressive applications of group homology is related to the question which groups can act freely on finite-dimensional spheres. Why can $\mathbb{Z}/2$ act freely, but $\mathbb{Z}/2 \times \mathbb{Z}/2$ or a symmetric group Σ_n for $n \geq 4$ cannot ? The answer is that such a group must have periodic homology. There exists a characterization of such groups.
[B, I.6,VI.6-9]

Literatur

[A-M] **A. Adem, R. J. Milgram**: *Cohomology of finite Groups.* Grundlehren der Math. Wissenschaften, vol. 309, Springer Verlag (2004^4).

[B] **K. Brown**: *Cohomology of Groups.* Graduate Texts in Mathematics vol 87, Springer Verlag (1982, 1994^2).

[E] **L. Evens**: *The Cohomology of Groups.* Oxford Math. Monographs, Oxford University Press (1991).

[H-S] **P. J. Hilton, U. Stammbach**: *A Course in Homological Algebra.* Graduate Texts in Mathematics vol. 4, Springer Verlag (1997^2).

[H] **D. Husemoller**: *Fibre Bundles.* Graduate Texts in Mathematics vol. 20, Springer Verlag (1994^3).

[S] **J. P. Serre**: *Trees.* Springer Verlag (1980^2).

[W] **C. A. Weibel**: *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics vol. 38, Cambridge University Press (1994).