Seminar S2D3 (Bachelor) / S4D2 (Master) **Summer Term 2025**

Prof. Dr. C.-F. Bödigheimer

Cohomology of Groups

Tuesdays, 14:15 — 16:00, Seminarraum 1.007

Begin: Tuesday, 8. April 2025 — End: Tuesday, 15. July 2025
No talk on Tuesday, 10. June 2025 (due to the pentecost holidays)

This seminar is an introduction to the homology and cohomology theory of discrete groups. The origin of this theory lies more in algebra than in topology; the early pioneers are algebraists, specially representation theoriests, like Frobenius and Schur, the later heroes are topologists like H.Cartan, Eilenberg and MacLane. Nowadays the theory is perhaps better understood as part of algebraic topology.

Consider a discrete group G acting by group homomorphisms on some R-module M, where R is a commutative ring. Given such a G-module, it seems obvious that we should and must be interested in the quotient module $M/G := M/\{x - g.x \mid x \in M, g \in G\}$, the so-called co-invariants, and the submodule $M^G := \{x \in M \mid g.x = x \text{ for all } g \in G\}$, the so-called invariants. Given a short exact sequence of G-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

it would be wonderful, if the two sequences

$$0 \longrightarrow A/G \longrightarrow B/G \longrightarrow C/G \longrightarrow 0$$

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow 0$$

were again exact — but, alas, they are in general not. The first is only right-exact, the second only left-exact.

As an example and exercise, take $G = \mathbb{Z}/2 = \{\pm 1\}$, $R = \mathbb{Z}$, with the action $(\pm 1).x = \pm x$ on any abelian group. Study the exactness of

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

with the second arrow being multiclication by n.

This calls for a systematic investigation into this deficiency, by studying the kernel of $A/G \longrightarrow B/G$ and the cokernel of $B^G \longrightarrow C^G$. Doing this leads to a first derived functor H_1 resp. H^1, but again we find a deficiency in exactness, so we need the next derived functor H_2 resp. H^2. And so on and so on. In the end we define two sequences $H_n(G; M)$ resp. $H^n(G; M)$ of functors, for any group G and any G-module M and obtain a long exact sequence for $M/G = H_0(G, M)$

$$\ldots \to H_n(G;A) \to H_n(G;B) \to H_n(G;C) \to H_{n-1}(G;A) \to H_{n-1}(G;B) \to H_{n-1}(G;C) \to \ldots$$
$$\ldots \to H_1(G;A) \to H_1(G;B) \to H_1(G;C) \to H_0(G;A) \to H_0(G;B) \to H_0(G;C).$$

and for $M^G = H^0(G, M)$ we obtain

$$H^0(G;A) \to H^0(G;B) \to H^0(G;C) \to H^1(G;A) \to H^1(G;B) \to H^1(G;C) \to \ldots$$
$$\ldots \to H^{n-1}(G;A) \to H^{n-1}(G;B) \to H^{n-1}(G;C) \to H^n(G;A) \to H^n(G;B) \to H^n(G;C) \to \ldots$$

Warning: Don't be confused about covariant and contravariant: $H_n(G;M)$ is a covariant functor of G and a covariant functor of M, and $H^n(G,M)$ is a contravariant functor of G, but covariant in M.

What is the connection to topology ? — First, each group G acting on a space X acts also on all of its homology groups $M = H_i(X)$ and cohomology groups $H^i(X)$.

A really important situation is the fundamental group $G = \pi_1(Y)$ of a space Y with its action on the universal covering $X = \widetilde{Y}$.

And we are interested to compare the homology of the quotient $H_n(X/G)$ to the quotient $H_n(X)/G$ of the homology, and the cohomology of the fixed point set $H^n(X^G)$ to the fixed elements $H^n(X)^G$ of the cohomology. Furthermore it turns out that $H_n(G;M)$ is indeed also the singular homology of a certain space, namely $H_n(G;M) \cong H_n(BG;\underline{M}) = H_n(EG \times_G M)$, where BG is the famous classifying space of the group G, and EG is its universal covering space (called the universal G-space), and $\underline{M}$ stands fore a local coefficient system over BG with values in M, and we take singular homology of BG in this system (- whatever that means). This space BG is used to classifiy principal G-bundles, see Talk 2 ; the universal characteristic classes of those bundles are the cohomology classes of BG.

This shall sufffice to convince you of the importance of this subfield of topology. If you want a striking application in topology, read below the description of the last Talk 14.

Language: You can give your talk in German or in English.

Prerequisites: Courses Topologie I (winter term 2024/25 or earlier) and Topologie II (summer term 2025, parallel to the seminar)

Literature: For most talks we will use the book [B] of K. Brown, but for some talks we need other sources as well.

The talks are supposed to last 90 minutes. That means, prepare around 75 minutes and expect questions during your talk. You should consult me as early as possible, and you should have a complete first version of your talk at least two weeks before the date of your talk.

Talks

(1) **Definition of $H_*(G)$ and $H^*(G)$** .... Ilia Mirkis 08.04.2025
Groups rings, G-modules, G-invariants and G-coinvariants, projective resolutions, the functors Tor and Ext. Definition of the homology $H_n(G; M)$ as the n-th derived functor of G-coinvariants $M \mapsto M_G = M/G$. Definition of the cohomology $H^n(G; M)$ as the n-th derived functor of G-invariants $M \mapsto M^G$. In particular we obtain a definition for "trivial coefficients", i.e. $M = \mathbb{Z}$ with the trivial operation.
[B, III], [H-S, VI.2], [W, 6.1]

(2) **Classifying spaces of groups** .... Paul Hahn 15.04.2025
For any group G there is a well-defined homotopy type BG. Homotopy theoretic definition of BG. Geometric examples: graphs (free groups), tori (finitely generated free abelian groups), infinite lens spaces (finite cyclic groups), some knot complements, nil-manifolds. Classification theorem for principal G-bundles, the bijection $[X, BG] = \mathrm{Hom}(\pi_1(X), G))$.
[H, 4]

(3) **Group homology and classifying spaces** .... Pirmin Kupffer 22.04.2025
Identification $H_*(G; \mathbb{Z}) = H_*(BG; \mathbb{Z})$. Here the left hand side is the group homology from Talk 1 and the right hand side is the singular homology of the classifying space BG. Local coefficients on spaces and covering spaces. Borel construction. Identification of $H_*(G; M) = H_*(BG; \underline{M})$ with $\underline{M}$ the local coefficients system over BG associated to M. Homology of a covering space.
[B, I.4, III.1]

(4) **Homology and cohomology of the cyclic groups** 29.04.2025
Norm-element and periodic resolutions. Calculation of $H_n(\mathbb{Z}/m; M)$ and $H^n(\mathbb{Z}/m; M)$ for trivial and non-trivial coefficients. Application: if BG is finite-dimensional then G is torsionfree.
[E, 2.1], [H-S, VI.7], [W, 6.2]

(5) **Milnor construction and bar resolution** ...... Javier Garrues Apecechea 06.05.2025
In this talk a general construction for the space BG should be presented, the Milnor construction. Bar resolution, homogeneous and inhomogeneous version. Normalization. Comparison with the cellular chain complex of EG and BG. Application: rational homology $H_n(G; \mathbb{Q}) = 0$ for finite groups G and $n > 0$.
[B, II.3], [W, 6.5], [E, 2.3]

(6) **Interpretation of $H_1(G)$, $H_2(G)$ and $H^2(G)$** .... Kasimir Enno Jabben 13.05.2025
Abelianization of G. Hopf-formula for $H_2(G)$. Extensions and their classification via $H^2(G)$.
[B, II.5, Excs 1], [H-S, VI.4, VI.9+10]

(7) **Mayer-Vietoris sequence for amalgamated products** .... Elena Ertle 20.05.2025
For some groups G it is possible to decompose BG into classifying spaces of easier groups and use the Mayer-Vietoris sequence to calculate the homology of a group. Free products and free amalgamated products. Examples: free groups, groups acting on trees, nice gemoetric example $\mathrm{SL}_2(\mathbb{Z})$.
[B, II.7 + Appendix], [W, 6.2], [H-S, VI.8+14], [E, 2.2], [S]

(8) **Products, universal coefficient and Künneth theorem** ... Daniil Zabilskyi 27.05.2025
The cross-product in the homology and the cup-product in the cohomology. The cohomology ring $H^*(G)$. Example: the cohomology ring of the cyclic groups. Example for the Künneth sequence:

$G = \mathbb{Z}/m \times \mathbb{Z}/l$.
[B, V.1-4], [E, 3], [B, III.1: Exc 3: V.5], [H-S, VI.15]

(9) **Pontrjagin product** .... Joachim Roscher 03.06.2025
Shuffle-product for the homology of abelian groups. Application: Homology of finitely-generated abelian groups.
[B, V.5+6]

(10) **Restriction, induction and transfer** .... Yordan Toshev 17.06.2025
Restriction to subgroups, induction of coefficient modules. Transfer. Cartan-Eilenberg double-coset formula. Application: detection of cohomology by the Sylow subgroups. Example: symmetric groups.
[B, III.10], [H-S, Vi.16], [A-M, II.5+6]

(11) **Spectral sequences I: basics** .... Yehor Avdieiev 24.06.2025
Basic definitions. Spectral sequence of a double complex. Spectral sequence of a filtered complex. Application: Gysin seqeunce, Wang sequence. Example: Künneth spectral sequence.
[B, VII], [W, 5.1-5.6], [E, 7]

(12) **Spectral sequences II: L-H-S spectral sequence** ...... Christian Ravnikar 01.07.2025
Spectral sequence of Lyndon-Hochschild-Serre for a group extension $1 \to N \to G \to Q \to 1$, equivalently of a fibration $BN \to BG \to BQ$. Example: homology of dihedral groups and wreath-products (Theorem of Nakaoka).
[B, VII.6], [W, 6.8], [E, 5.3, 7]

(13) **Cohomology theory of finite groups** .... Dimitrios Beligiannis 08.07.2025
For finite groups G it turns out that the homology and cohomology groups of G have certain similar properties. These enable to organize them together in what is called the Tate cohomology groups $\widehat{H}^*(G)$. Definition and basic properties. Coordinate with the next speaker.
[B, VI.1-5]

(14) **Free actions on spheres** .... Olumadamisola Makinde 15.07.2025
One of the most impressive applications of group homology is related to the question which groups can act freely on finite-dimensional spheres. Why can $\mathbb{Z}/2$ act freely, but $\mathbb{Z}/2 \times \mathbb{Z}/2$ or a symmetric group Σ_n for $n \geq 4$ cannot ? The answer is that such a group must have periodic homology. There exists a characterization of such groups.
[B, I.6,VI.6-9]

Literatur

[A-M] **A. Adem, R. J. Milgram**: *Cohomology of Finite Groups.* Grundlehren der Math. Wissenschaften, vol. 309, Springer Verlag (2004^4).

[B] **K. Brown**: *Cohomology of Groups.* Graduate Texts in Mathematics vol 87, Springer Verlag (1982, 1994^2).

[E] **L. Evens**: *The Cohomology of Groups.* Oxford Math. Monographs, Oxford University Press (1991).

[H-S] **P. J. Hilton, U. Stammbach**: *A Course in Homological Algebra.* Graduate Texts in Mathematics vol. 4, Springer Verlag (1997^2).

[H] **D. Husemoller**: *Fibre Bundles.* Graduate Texts in Mathematics vol. 20, Springer Verlag (1994^3).

[S] **J. P. Serre**: *Trees.* Springer Verlag (1980^2).

[W] **C. A. Weibel**: *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics vol. 38, Cambridge University Press (1994).