Hilbert uniformization of Riemann surfaces : I

SHORT VERSION

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Abstract
We report on a cell decompostion for the moduli space of Riemann surfaces of genus \( g \geq 0 \) with \( n \geq 1 \) boundary curves and \( m \geq 0 \) punctures.

1 Moduli spaces and mapping class groups

Let \( \mathcal{M} = \mathcal{M}_{g,n}^m \) denote the moduli space of conformal equivalence classes of Riemann surfaces \( F = F_{g,h}^m \) of genus \( g \geq 0 \) and with \( n \geq 1 \) boundary curves and \( m \geq 0 \) permutable punctures. Likewise, let \( \Gamma = \Gamma_{g,n}^m \) be the corresponding mapping class group of isotopy classes of orientation-preserving diffeomorphisms fixing the boundary pointwise and permute the punctures.

Since \( n \geq 1 \), the automorphisms of \( F \) are trivial, and thus the action of \( \Gamma \) on the corresponding Teichmüller space is free. Therefore the moduli space \( \mathcal{M} \) is a smooth, non-compact manifold, with the homotopy type of the classifying space \( B\Gamma \). Its dimension is \( d = 6g - 6 + 3n + 2m \). Because we allow punctures to be permuted, it is orientable only in the cases \( m = 0 \) or \( m = 1 \).

2 Results

There is a flat vector bundle \( \mathfrak{D} \to \mathcal{M} \) of dimension \( \bar{d} = m + 3n \), the fibres of which are vector spaces of certain harmonic functions. Following an earlier version [Bödigheimer-90], we construct in [Bödigheimer-05] a finite cell complex \( \mathcal{P} = \mathcal{P}(h,m,n) \) as a compactification of \( \mathfrak{D} \); here \( h = 2g + m + 2n - 2 \) and \( d + \bar{d} = 3h \). A point in \( \mathcal{P} \) is a configuration of \( h \) pairs of horizontal, semi-infinite slits in \( n \) complex planes.

**Theorem.**

There is a subcomplex \( \mathcal{P}' \subset \mathcal{P} \) and a homeomorphism \( \mathcal{H} : \mathfrak{D} \to \mathcal{P} - \mathcal{P}' \).

The inverse homeomorphism is defined and studied in detail [Bödigheimer-05]; the continuity of \( \mathcal{H} \) is discussed in [Ebert-05].

3 Cell structure

To describe the space \( \mathcal{P} \) we concentrate on the case of a single boundary curve. Fix \( g \) and \( m \), and put \( n = 1 \); then \( h = 2g + m \).

The space \( \mathcal{P} = \mathcal{P}(h,m,n) \) is bi-simplicial complex \( \mathcal{P}_{p,q} \), where \( 0 \leq p \leq 2h \) and \( 0 \leq q \leq h \). The cells in \( \mathcal{P}_{p,q} \) are products \( \Delta^q \times \Delta^p \) of simplices, and they are given by \( q \)-tuples \( \Sigma = (\sigma_q, \ldots, \sigma_0) \) of permutations \( \sigma_i \) in the symmetric group \( \mathfrak{S}_{p+1} \), acting on \( 0, 1, \ldots, p \), satisfying the following conditions:
\[ \text{norm}(\sigma) \leq h \]  
\[ \sigma_q \text{ has at most } m + 1 \text{ cycles} \]  
\[ (3.1) \]  
\[ (3.2) \]

Here the \( \text{norm}(\Sigma) \) is the sum of the word lengths of all \( \tau_i = \sigma_i \sigma_{i-1}^{-1} \) for \( i = 1, \ldots, q \), measured with respect to the generating set of all transpositions.

The face operators are given by

\[ d_i'(\Sigma) = (\sigma_q, \ldots, \sigma_i, \ldots, \sigma_0) \]  
\[ (3.3) \]

and

\[ d_j''(\Sigma) = (D_j(\sigma_q), \ldots, D_j(\sigma_0)) \]  
\[ (3.4) \]

where \( D_j : \mathfrak{S}_{p+1} \to \mathfrak{S}_p \) deletes the letter \( j \) from the cycle it occurs in and re-normalizes the indices.

The subcomplex \( P' \) consists of all \( \Sigma \) with norm less than \( h \) or with a \( \sigma_q \) having less than \( m + 1 \) cycles, or where any of the following conditions is violated:

\[ \sigma_i(p) = 0 \text{ for } i = 0, \ldots, q \]  
\[ (3.5) \]

\[ \sigma_0 \text{ is the rotation } 0 \mapsto 1 \mapsto 2 \mapsto \ldots \mapsto p \mapsto 0 \]  
\[ (3.6) \]

\[ \sigma_{i+1} \neq \sigma_i \text{ for } i = 0, \ldots, q - 1 \]  
\[ (3.7) \]

There is no \( k \in \{0, \ldots, p - 1\} \) such that \( \sigma_i(k) = k + 1 \) for all \( i = 0, \ldots q \)  
\[ (3.8) \]

**Remarks.**

The Hilbert uniformization method goes back to work of Hilbert and Courant. It can also be used to parametrize the moduli spaces of surfaces with incoming/outgoing boundary curves, see [Bödigheimer-03]; and it can be used for moduli spaces of conformal equivalence classes of non-orientable surfaces (Kleinian surfaces); see [Ebert-03], [Zaw-04].

**References**


