ON THE TOPOLOGY OF
MODULI SPACES OF
RIEMANN SURFACES

Part II

Homology Operations

C.-F. Bödigheimer
Mathematisches Institut
Bunsenstraße 3-5
D-3400 Göttingen
Introduction

This is the second article in a longer series, intended to investigate the topology of the moduli spaces of Riemann surfaces. Using the description of the moduli space as a kind of configuration space we will develop operations for the homology of these spaces.

In Part I "Hilbert Uniformization" we studied the moduli space \( \overline{\mathcal{M}}(g) \) of directed Riemann surfaces, i.e. the space of conformal equivalence classes \([F,x]\) of closed Riemann surfaces \( F \) of genus \( g \) and given direction \( x \) at some point \( P \in F \). This space has the homotopy type of the classifying space \( B\tilde{\tau}(g) \) of the corresponding mapping class group \( \tilde{\tau}(g) = \pi_0 \text{Diff}^+(F,x) \); \( \text{Diff}^+(F,x) \) is the group of orientation-preserving diffeomorphisms of \( F \) which keep \( x \) fixed. \( \tilde{\tau}(g) \) is isomorphic to the mapping class group \( \tilde{\tau}(g,1) \) of a genus \( g \) surface with one boundary component.

The main objective of Part I was a new description of \( \overline{\mathcal{M}}(g) \) - based on old ideas of geometric function theory; a homotopy equivalence between \( \overline{\mathcal{M}}(g) \) and a space \( \overline{\mathcal{P}}\mathcal{S}(g) \) of so called parallel slit domains was established. A point in \( \overline{\mathcal{P}}\mathcal{S}(g) \) is an equivalence class \( \mathcal{C} = [L] \) of a configuration \( L = (L_1, \ldots, L_{4g}; \lambda) \) consisting of semi-infinite horizontal slits \( L_i \) in \( \mathcal{C} \), paired by an involution \( \lambda \in \Sigma_{4g} \). Such configurations are subject to some regularity conditions; and the equivalence relation allows certain jumps of slits across longer pairs. The group \( \text{Sim}(\mathcal{C}) \) of dilatations and translations of \( \mathcal{C} \) acts freely on \( \overline{\mathcal{P}}\mathcal{S}(g) \). And the orbit space \( \overline{\mathcal{P}}\mathcal{S}(g)/\text{Sim}(\mathcal{C}) \) was proved to be homeomorphic to \( \overline{\mathcal{M}}(g) \).

This parametrization of \( \overline{\mathcal{M}}(g) \) shows the distinctive characteristics of a configuration space: a point \( \mathcal{C} \) comprises two kinds of finite data, geometric
and combinatoric, namely the position of the slits and their ordering and pairing. From the viewpoint of moduli: the conformal structure of \([F,x]\) is screened against the complex plane, and the picture so obtained is contained in an essentially finite piece of the plane. Whenever such a situation occurs, one is tempted to patch several such pictures together. This method has a long history in homotopy theory under the name Dyer-Lashof operations, where it is used with great success to study iterated loop spaces, function spaces or classifying spaces of certain groups. In this article we will introduce such patchings systematically for the moduli spaces, taking the spaces \(\text{PS\Sigma}(g)\) as substitutes.

The aim is to construct operations

\[ \Theta : \tilde{X}(n,g_0) \times \text{PS\Sigma}(g)^n \to \text{PS\Sigma}(g_0+ng) \]

\(\Sigma_n\)

for various spaces \(\tilde{X}(n,g_0)\) which depend on \(n \geq 1, g_0 \geq 0\) and have a free \(\Sigma_n\)-action. These parameter spaces for the operations will be configuration spaces of points, pairs of points, etc., in the plane, in a surface, or in the universal surface bundle over the moduli space. A homology class \(\alpha \in H_q(\tilde{X}(n,g_0)/\Sigma_n)\) induces a homology operation

\[ \Theta_{\alpha} : H^*_\text{PS\Sigma}(g) \to H^{n+q}_\text{PS\Sigma}(g_0+ng) \]

We call \(g_0\) the genus and \(q\) the degree of \(\Theta_{\alpha}\). Homology operations should be useful first to create new homology classes from old ones, and then also to organize the homology of the moduli spaces.

These are several interesting subspaces of \(\text{PS\Sigma}(g)\), some of which admit operations of a more intricate nature than the Dyer-Lashof operations above. In particular, we use the subspace \(\text{PS\Sigma}_+(g)\) of parallel slit domains which
are decomposable into a part in the upper resp. lower half-plane. Such partitioned parallel slit domains allow the two parts to be implanted into a vertical pair of patches in \( \mathcal{C} \). The new configuration spaces are configuration spaces of vertical pairs of points in the plane, in a surface, or in the universal surface bundle over the moduli space. We call these operations symplectic.

Plan of the chapters.

Part II consists of 8 chapters, which are paired: one chapter develops the geometric side of some kind of operations, the other the homological side. Chapter is confined to the most basic of all operations, the sum operation, which implant two parallel slit domains into \( \mathcal{C} \), the first into the upper, the second into the lower half-plane. This operation

\[
\mu : \text{PSG}(g_1) \times \text{PSG}(g_2) \longrightarrow \text{PSG}(g_1 + g_2)
\]

makes \( \text{PSG} = \bigcup \text{PSG}(g) \) an H-space. A special case is the stabilization map

\[
\sigma : \text{PSG}(g) \longrightarrow \text{PSG}(g+1), \quad \sigma(C^{(1)}) = \mu(C^{(1)}, C^{(1)})
\]

for \( C^{(1)} \) a fixed surface of genus 1. In Chapter 2 the homology \( H_*(\text{PSG}; A) \), \( A \) a commutative ring with unit, is considered as a ring, a so-called Pontrjagin ring, with multiplication induced by \( \mu \). We mention Harer's stability theorem for \( H_*(\text{PSG}; \mathbb{Z}) \) and the polynomial subalgebra in \( H_*(\text{PSG}; \mathbb{Q}) \) found by Miller as the best global results on this ring so far. In Chapter 3 we see the first proper results on this ring so far. In Chapter 3 we see the first proper operations as Dyer-Lashof maps

\[
\bigotimes^n \mathcal{C} \times \text{PSG}(g)^n \longrightarrow \Sigma_n \text{PSG}(ng)
\]
where $\mathcal{C}^n(\mathbb{C})$ is the ordered configuration space of the plane $\mathbb{C}$. The existence of such operations was noticed by several authors, in particular [Miller 1986], [Cohen 1987], [Maginnis 1987], described rather on the group level than on the level of moduli spaces. The definition of $\mathcal{A}$ requires some technical preparations; we give full details here to save work in similar situations to come.

Theorem (3.6.2) summarizes the main properties of these operations. In chapter we study the operations in mod-2 homology $H_2(\text{PS}^2;\mathbb{Z}_2)$ induced by $\mathcal{A} : \mathcal{C}^2(\mathbb{C}) \times \Sigma_2 \to \text{PS}^2(2g)$. The generator in $H_0(\mathcal{C}^2(\mathbb{C});\mathbb{Z}_2)$ induces the squaring operation $Q_0 : H_2(\text{PS}^2;\mathbb{Z}_2) \to H_{2p}(\text{PS}^2(2g);\mathbb{Z}_2)$, $Q_0(x) = x \# x = x^2$; the generator in $H_1(\mathcal{C}^2(\mathbb{C});\mathbb{Z}_2)$ induces a Dyer-Lashof operation $Q_1 : H_p(\text{PS}^2;\mathbb{Z}_2)$ binary operations, the generator in $H_0(\mathcal{C}^2(\mathbb{C});\mathbb{Z}_2)$ induces the ring multiplication, and the generator in $H_1(\mathcal{C}^2(\mathbb{C});\mathbb{Z}_2)$ induces a Browder operation $R_1 : H_p(\text{PS}^2;\mathbb{Z}_2) \to H_{p+q+1}(\text{PS}^2(2g+2);\mathbb{Z}_2)$. The theory of such operations is well-developed. We give some formulas for $Q_0$, $Q_1$ and $R_1$.

In chapter 5 and chapter 6 we introduce the symplectic operations. They are defined for the subspace $\text{PS}^2_-(g)$, but stably $\text{PS}^2_-(\mathbb{C}) \simeq \text{PS}^2(\mathbb{C})$. Since these operations take the finer slit structure of a surface (and not only the handle structure) into account, we obtain two new operations $Q'_1$, $Q''_1$ of degree 1.

In the final chapters 7 and 8 we briefly present two generalizations of the operations introduced so far. First we replace the complex plane $\mathbb{C}$ by any fixed, (punctured) parallel slit domain $\mathcal{L}^0$; i.e. we use configurations of points (resp. vertical pairs of points) in $\mathcal{L}^0$ to parametrize implantations of other (partitioned) parallel slit domains. Since configuration spaces of surfaces have homology classes in dimensions greater than one, we obtain operations of degree greater than one, which are not products of degree one operations. Having gone so far, the next step of generalization consists of varying the parallel slit domain $\mathcal{L}^0$ itself. The new parameter spaces are fibrewise configuration spaces of the universal surface bundle over $\text{PS}^2(g)$. We only give some examples.
References.

References to Part I are denoted by (I.x.y.z), and within Part II by (x.y.z).
All other literature is quoted as [name, year] and listed on the pages 84–85;
Part I appears as [Bödigheimer 1990].

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Chapter 1

Basic Operations

1.1 The sum operation.
1.2 The stabilization map.
1.3 Conjugation and Schottky double.

Replacing the moduli space $\overline{M}(g)$ of conformal equivalence classes of directed
Riemann surfaces by the homotopy-equivalent space $\overline{PSL}(g)$ of parallel slit
domains immediately suggests to imitate certain constructions well-known in
the theory of loop spaces. The most basic is the sum operation which unites to
parallel slit domains to represent the connected sum of two Riemann surfaces.
As a special case we also have a stabilization procedure, adding a standard
elliptic curve. These operations will be pursued and generalized in later chap-
ters.

There are many more such constructions, of which we mention only the conjugation
and the Schottky-doubling.
1.1 The sum operation.

Recall that a parallel slit domain $\mathcal{L} = [L] = [L_1, \ldots, L_{4g}; \lambda]$ in $\text{PS}^\ast(g)$ is an equivalence class of a non-degenerate configuration $L = (L_1, \ldots, L_{4g}; \lambda)$; $L$ consists of slits $L_i \subseteq \mathbb{C}$ and a pairing $\lambda \in \Sigma_{4g}$, satisfying (I.4.1.2); the non-degeneracy condition is given in (I.4.2.4); and the equivalence class $\mathcal{L} = [L]$ is obtained by performing all possible crossings (I.4.3.2).

Let $\mathcal{L}^1$ and $\mathcal{L}^2$ be two parallel slit domains of genus $g_1$ and $g_2$, respectively. Remembering that a conformal equivalence class in $\overline{\mathbb{H}}(g)$ corresponds to a similarity class of a parallel slit domain (see (I.5.5.1)), there is an obvious way to unite $\mathcal{L}^1$ and $\mathcal{L}^2$: one shifts $\mathcal{L}^1$ upwards till all its slits lie in the upper half-plane $\mathbb{H}$, and shifts $\mathcal{L}^2$ down till all its slits lie in the lower half-plane $\mathbb{H}^-$, and then the union is a new parallel slit domain of genus $g_1 + g_2$. To define this operation as a map

$$\mu = \mu_{g_1, g_2} : \text{PS}^\ast(g_1) \times \text{PS}^\ast(g_2) \longrightarrow \text{PS}^\ast(g_1 + g_2)$$

some technical preparations are necessary. Recall the functions $a_+, a_-, b_+$, $t$ defined in (I.4.8). Note that $a_+(\mathcal{L}) = a_-(\mathcal{L})$ is possible, but we always have $b_-(\mathcal{L}) < b_+(\mathcal{L})$.

(1.1.1)

\[ \mu = \mu_{g_1, g_2} : \text{PS}^\ast(g_1) \times \text{PS}^\ast(g_2) \longrightarrow \text{PS}^\ast(g_1 + g_2) \]

The translation of $\mathcal{L}$ by any complex number $c$ is denoted by $\mathcal{L} + c = [L_1 + c, \ldots, L_{4g} + c; \lambda]$, see (I.4.9).
Assume $\mathcal{L}^1 = [L_1^1, \ldots, L_{4g_1}^1 ; \lambda^1]$ and $\mathcal{L}^2 = [L_1^2, \ldots, L_{4g_2}^2 ; \lambda^2]$, and we set $b^1 = b_-(\mathcal{L}^1)$ and $b^2 = b_+(\mathcal{L}^2)$. Then all slits of $\mathcal{L}^1 - ib^1 + \frac{i}{2}$ are contained in $H$, and all slits of $\mathcal{L}^2 - ib^2 - \frac{i}{2}$ are contained in $H^\ast$. Their ordering by imaginary parts is

\[
\begin{align*}
L_1^1 - i(b_1 - \frac{1}{2}), & \ldots, L_{4g_1}^1 - i(b_{4g_1} - \frac{1}{2}), \\
L_1^2 - i(b_2 + \frac{1}{2}), & \ldots, L_{4g_2}^2 - i(b_{4g_2} + \frac{1}{2}), \\
S_1^1 - i(b_1 - \frac{1}{2}) , & S_2^2 - i(b_2 + \frac{1}{2})
\end{align*}
\]

where we denote the sequences of slits by $S^1$ and $S^2$, respectively. We denote the juxtaposition of the two sequences in (1.1.4) by $S^1 \circ S^2$. Likewise is the (Whitney) $\lambda^1 \circ \lambda^2$ sum of the two permutations the image under the inclusion $\Sigma_{4g_1} \times \Sigma_{4g_2} \to \Sigma_{4(g_1+g_2)}$.

All in one formula gives

\[
(1.1.4) \quad \mu : \text{PSO}(g_1) \times \text{PSO}(g_2) \longrightarrow \text{PSO}(g_1+g_2)
\]

\[
\mu([S^1 ; \lambda^1], [S^2 ; \lambda^2]) = [S^1 \circ S^2 ; \lambda^1 \circ \lambda^2] =
\]

with $S^1 = S^1 - i(b_-(\mathcal{L}^1) - \frac{1}{2})$, $S^2 = S^2 - i(b_+(\mathcal{L}^2) + \frac{1}{2})$.

It is obvious that $\mathcal{L}^1 \circ \mathcal{L}^2$ is a non-degenerate equivalence class; this follows directly from the definition (I.4.2); compare also (1.1.5) below. Furthermore $\mathcal{L}^1 \circ \mathcal{L}^2$ is a well-defined equivalence class, for any crossing possible in $\mathcal{L}^1$ or $\mathcal{L}^2$ is still possible in $\mathcal{L}^1 \circ \mathcal{L}^2$. And because $b_+ \circ b_-$ are continuous functions, it follows that $\mu$ is continuous.
Geometrically, \( \mathcal{L}^1 \oplus \mathcal{L}^2 \) is the connected sum of the two Riemann surfaces \( F_1 = F(\mathcal{L}^1) \) and \( F_2 = F(\mathcal{L}^2) \) along the boundary of two discs. On \( F_1 \), we cut out the disc \( D_1 \) given by \( v_1 < b_-(\mathcal{L}^1) - \frac{1}{2} \), and on \( F_2 \) the disc \( D_2 \) given by \( v_2 > b_+(\mathcal{L}^2) + \frac{1}{2} \), where \( v_i = y \) is the harmonic conjugate of the dipole functions \( u_i \). Both are indeed discs, because they contain no stagnation points. The curves \( v_1 = b_-(\mathcal{L}^1) - \frac{1}{2} \) and \( v_2 = b_+(\mathcal{L}^2) + \frac{1}{2} \) are real-analytic curves through the dipoles \( P_1 \) and \( P_2 \), and tangent to the directions \( x_1 \) and \( x_2 \), respectively. If \( F_1 - D_1 \) and \( F_2 - D_2 \) are glued along these curves by identifying points with the same u-value, the result is a closed surface \( F \) of genus \( g_1 + g_2 \). \( F \) has an obvious complex structure; \( P_1 \) and \( P_2 \) are identified to become the new dipole \( P \); and similarly the two directions \( x_1 \), \( x_2 \) become the same in \( F \). The two harmonic functions \( u_1 \) and \( u_2 \) agree by construction on the new curve \( \partial D_1 = \partial D_2 \), and thus determine a dipole function for \( F \). Only the harmonic conjugates \( v_1 \), \( v_2 \) have to be renormalized by adding the imaginary integration constants \( -i(b_-(\mathcal{L}^1) - \frac{1}{2}) \) and \( -i(b_+(\mathcal{L}^2) + \frac{1}{2}) \) respectively; now they are both zero on \( \partial D_1 = \partial D_2 \), and \( v_1 \) is positive on \( F_1 - D_1 \subseteq F \), and \( v_2 \) is negative on \( F_2 - D_2 \subseteq F \).

\[(1.1.5)\]
For $g = 0$ the auxiliary functions $a_t$, $b_t$ are not defined; the formula (1.1.4) makes nevertheless sense if $\mathcal{L}^1$ or $\mathcal{L}^2$ are the only element $[\emptyset]$ in $\text{PSG}(0)$. Recall that we defined basepoints $\mathcal{L}^g \in \text{PSG}(g)$ in (I.4.5.21).

(1.1.6) Proposition.
(i) $\mathcal{L} \oplus [\emptyset] = \mathcal{L} - i(b_+(\mathcal{L}) - \frac{1}{2})$
(ii) $[\emptyset] \oplus \mathcal{L} = \mathcal{L} - i(b_-(\mathcal{L}) + \frac{1}{2}), \text{ for any } \mathcal{L} \in \text{PSG}(g)$,
(iii) $\left(g_1\right) \oplus \left(g_2\right) = \left(g_1 + g_2\right) + 4ig_1$.

The proof is obvious. The statements (i) and (ii) say that $[\emptyset]$ is a homotopy-neutral element for the sum operation. There is an associativity law.

(1.1.7) Proposition. For any $g_1, g_2, g_3 \geq 0$ the diagram

\[
\begin{array}{ccc}
\text{PSG}(g_1) \times \text{PSG}(g_2) \times \text{PSG}(g_3) & \xrightarrow{id \times u} & \text{PSG}(g_1) \times \text{PSG}(g_2 + g_3) \\
\downarrow \mu \times \text{id} & & \downarrow \mu \\
\text{PSG}(g_1 + g_2) \times \text{PSG}(g_3) & \longrightarrow & \text{PSG}(g_1 + g_2 + g_3)
\end{array}
\]

is homotopy-commutative.

Proof. The required homotopy $\mu \circ (id \times \mu) = \mu \circ (\mu \times id)$ amounts to a homotopy of translations. For $t \in [0,1]$ we define

$$m_t : (\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3) \longrightarrow (\mathcal{L}^1 \oplus \mathcal{L}^2) \oplus \mathcal{L}^3 + ith$$

with $h = b_+(\mathcal{L}^2) - b_-(\mathcal{L}^2) + 1$. Clearly, $m_0 = \mu(\mu \times id)$, and $m_1 = \mu(id \times \mu)$.
A much more subtle property of $\mu$ is the commutativity. Its proof is postpone till we have developed the general framework. In its formulation below $\tau$ denotes the twist map interchanging the two factors.

(i.1.8) **Proposition.** For any $g_1, g_2 \geq 0$ the diagram

\[
\begin{array}{ccc}
\text{PSC}(g_1) \times \text{PSC}(g_2) & \xrightarrow{\tau} & \text{PSC}(g_1 + g_2) \\
\downarrow & & \\
\text{PSC}(g_2) \times \text{PSC}(g_1)
\end{array}
\]

is homotopy-commutative.
1.2 The stabilization map.

Let \( \mathcal{L}^{(1)} \) be the basepoint in \( \text{PSG}(1) \), given by the slit end points \( s_k = i^k \) for \( k = 1, \ldots, 4 \) and the pairing \( \lambda = (13)(24) \).

\[
(1.2.1)
\]

The stabilization is defined by adding this fixed surface of genus 1,

\[
(1.2.2) \quad \sigma : \text{PSG}(g) \longrightarrow \text{PSG}(g+1)
\]

\[
\sigma(\mathcal{L}) = \mathcal{L} \ast \mathcal{L}^{(1)}.
\]

Geometrically, \( \sigma \) inserts a new handle in the surface \( F(\mathcal{L}) \), in a small disc near the dipole.

The stabilization map allows to define an infinite object, the space of infinite parallel slit domains

\[
(1.2.3) \quad \text{PSG}(\infty) = \lim_{\sigma} \text{PSG}(g).
\]

From (I.2.4.29, 1.3.7) we conclude, that \( \text{PSG}(\infty) \) is the classifying space for the infinite (sometimes called stable) mapping class group \( \Gamma(\infty,1) = \lim_{\sigma} \Gamma(g,1) \). The inclusions \( \Gamma(g,1) \leq \Gamma(g+1,1) \) are defined as follows. If \( F_{g,1} \) is a
surface of genus $g$ with one boundary component, it is contained in $F_{g+1,1} = F_{g,1} \cup T$ with $T = F_{1,2}$ torus with two boundary components.

(1.2.4)

Extending a diffeomorphism of $F_{g,1}$ (which is the identity on $\partial F_{g,1}$) by the identity on $T$, defines a homomorphism $\mathrm{Diff}^+(F_{g,1}, \partial F_{g,1}) \to \mathrm{Diff}^+(F_{g+1,1}, \partial F_{g+1,1})$ inducing $\Gamma(g,1) \to \Gamma(g+1,1)$. In other words, let $F_\infty$ be a connecte oriented, smooth, non-compact surface of infinite genus and without boundary; and let $\mathrm{Diff}^+_{\text{cpt}}(F_\infty)$ be the group of orientation preserving diffeomorphisms which are the identity outside a compact set; then $\Gamma(\omega,1) = \pi_0 \mathrm{Diff}^+_{\text{cpt}}(F_\infty)$.

The stabilization is compatible with the sum operation.

(1.2.5) Proposition. (i) $\mu \circ (\sigma \times \text{id}) = \sigma \circ \mu$,
(ii) $\mu \circ (\text{id} \times \sigma) = \sigma \circ \mu$,
(iii) $\mu \circ (\sigma \times \sigma) = \sigma^2 \circ \mu$.

Again the proof is postponed, since it is a special case of general associativity laws.
1.3 Conjugation and Schottky-double.

Given a parallel slit domain \( \mathcal{C} = [L_1, \ldots, L_{4g}; \lambda] \) we define its conjugate \( \overline{\mathcal{C}} \) to be the image under the conjugation \( z = x + iy \mapsto \overline{z} = x - iy \) of the complex plane. Thus \( \overline{\mathcal{C}} = [\overline{L_{4g}}, \ldots, \overline{L_1}; \overline{\lambda}] \) with \( \overline{\lambda}(k) = 4g + 1 - \lambda(4g + 1 - k) \). The new surface \( F(\overline{\mathcal{C}}) \) is just \( F(\mathcal{C}) \) with the conjugate complex structure.

\[ D(\mathcal{C}) = \mathcal{C} \ast \overline{\mathcal{C}} \] is a parallel slit domain of genus \( 2g \), representing the Schottky-double of \( F(\mathcal{C}) \).

The conjugation \( \mathcal{C} \rightarrow \overline{\mathcal{C}} \) is an involution on \( \text{PSG}(g) \) and only one of several interesting self-maps of \( \text{PSG}(g) \) which we will investigate elsewhere.
Chapter 2

The Pontrjagin Ring \( H_\ast \left( \bigsqcup \limits_{g \geq 0} \text{PS}(g) \right) \)

2.1 \( \text{PS}(\ast) = \bigsqcup \limits_{g \geq 0} \text{PS}(g) \) as an H-space.

2.2 The Pontrjagin product.

2.3 Homological stability.

The disjoint union \( \text{PS}(\ast) \) of all \( \text{PS}(g) \) has a multiplication induced by the sum operation. It induces a product on the graded homology module \( H_\ast \left( \text{PS}(\ast) \right) \).

This chapter contains only some elementary definitions; and we quote a theorem of E. Miller about a polynomial subalgebra, and J. Harer's stability theorem.
2.1 \( PSC = \coprod_{g \geq 0} PSC(g) \) as an H-space.

The disjoint union of all spaces \( PSC(g), g \geq 0 \), is denoted by \( PSC \). The sum operation \( \mu \), defined by (1.1.4) on each component of the product \( PSC \times PSC \), is a map

\[
(2.1.1) \quad \mu : PSC \times PSC \longrightarrow PSC.
\]

The following is a rewording of Propositions (1.1.6 - 8).

(2.1.2) Proposition. \( PSC \) is a \( h \)-associative, \( h \)-commutative H-space with two-sided \( h \)-neutral element \( [\emptyset] \).

We repeat that the commutativity of \( \mu \) up to homotopy is still to be proved.
2.2 The Pontriagin product.

Let $A$ be a commutative ring with unit. The graded homology module $H_\ast(P\Sigma;A)$ has a multiplication induced by the map $\mu$; it is defined as the composition

\[ H_\ast(P\Sigma;A) \ast H_\ast(P\Sigma;A) \overset{\times}{\longrightarrow} H_\ast(P\Sigma \times P\Sigma;A) \overset{\mu_\ast}{\longrightarrow} H_\ast(P\Sigma;A) \]

where $\times$ denotes the exterior homology product. This product is called the Pontrjagin product and denoted by $a \ast b = \mu_\ast(a \times b)$. It is graded in the sense that $a \ast b \in H_{p+q}(P\Sigma;A)$ if $a \in H_p(P\Sigma;A)$, $b \in H_q(P\Sigma;A)$. The associativity $(a \ast b) \ast c = a \ast (b \ast c)$ follows from (2.1.2), together with the commutativity $a \ast b = (-1)^{|a||b|} b \ast a$, where $||$ denotes the dimension of a homology class.

Regarded as a homology class, $[\emptyset] \in H_0(P\Sigma(o);A) \cong A$ is a two-sided unit. Of course $\ast$ is $A$-linear in both variables. Thus $H_\ast(P\Sigma;A)$ is an algebra over $A$.

The most important result so far on this algebra is the following.

(2.2.2) Theorem. [Miller 1986] There are classes $\chi_n \in H_{2n}(P\Sigma;Q)$, $n \geq 1$, which generate a polynomial subalgebra.

A conjecture by Mennoord says that these classes generate the entire rational homology of $P\Sigma$. 
2.3 **Homological stability.**

The following two results by J. Harer on the stabilization map \( \sigma : \text{PSG}(g) \to \text{PSG}(g+1) \) are important, [Harer 1985].

(2.3.1) **Theorem.** [Harer] \[ \text{The stabilization induces isomorphisms} \]

\[
\sigma_* : H_\alpha(\text{PSG}(g);\mathbb{Z}) \to H_\alpha(\text{PSG}(g+1);\mathbb{Z})
\]

\[
\text{for } \ast \leq \frac{g+1}{3}, \quad g \geq 3.
\]

Note that the stable range is about \( \frac{1}{18} \) of the dimension of \( \text{PSG}(g) \). But by another result in [Harer 1986] the virtual cohomological dimension of \( \text{PSG}(g) \) is \( 4g - 2 \) if \( g \geq 1 \). We remark that Ivanov has improved the result above by enlarging the stable range to \( \ast \leq \frac{g+1}{2} \), see [Ivanov 1987].

Also in [Harer 1985] we find the following result about the forgetful maps \( \text{Br}(g) = \text{Br}(g,1) \to \text{Br}(g) \).

(2.3.2) **Theorem.** [Harer] \[ \text{The epimorphisms } \Gamma(g,1) \to \Gamma(g) \text{ induce isomorphisms} \]

\[
H_\alpha(\Gamma(g,1);\mathbb{Z}) \to H_\alpha(\Gamma(g);\mathbb{Z})
\]

\[
\text{for } \ast \leq \frac{g+1}{3}, \quad g \geq 3.
\]
Chapter 3

The Geometry of

Dyer-Lashof Operations

3.1 Normalization of parallel slit domains.
3.2 Configuration spaces.
3.3 Dyer-Lashof maps.
3.4 Interval exchange transformations.
3.5 Dyer-Lashof maps (continued).
3.6 Main properties.
3.7 Comparison with operations on braid groups and symplectic groups.

The sum operation implants one parallel slit domain into the upper half-plane, and a second one into the lower half-plane. The idea of Dyer-Lashof operations — coming from the theory of loop spaces — is to implant several parallel slit domains into small disjoint rectangles moving independently in the plane.

Before we can define these maps we need to recall some technical notions, then introduce configuration spaces; furthermore, we need the concept of interval exchange transformations.
3.1 Normalization of parallel slit domains.

For a parallel slit domain \( \mathcal{L} = [L_1, \ldots, L_4; \lambda] \) in \( \text{PSL}(g) \) we defined in (I.4.4) the support as the smallest closed rectangle with sides parallel to the \( x, y \)-axes containing the endpoints \( S_i \) of all slits \( L_i \). Its corners are \( a_i(\mathcal{L}) + ib_i(\mathcal{L}) \); see 1.1.2. The rectangle can become a vertical interval, but never a point. We denote by

\[
(3.1.1) \quad a(\mathcal{L}) = a_+(\mathcal{L}) - a_-(\mathcal{L}) \geq 0, \quad b(\mathcal{L}) = b_+(\mathcal{L}) - b_-(\mathcal{L}) > 0
\]

the length resp. height of \( \mathcal{L} \). The center of \( \mathcal{L} \) is the center of mass of \( \text{supp}(\mathcal{L}) \),

\[
(3.1.2) \quad c(\mathcal{L}) = \frac{1}{2}(a_+(\mathcal{L}) - a_-(\mathcal{L})) + \frac{1}{2}(b_+(\mathcal{L}) + b_-(\mathcal{L})).
\]

The normalization \( N(\mathcal{L}) \) of \( \mathcal{L} \) translates this center to the origin of the plane, and then stretches or shrinks the configuration such that the support fits into the unit square,

\[
(3.1.3) \quad N(\mathcal{L}) = \frac{1}{\max(a(\mathcal{L}), b(\mathcal{L}))} (\mathcal{L} - c(\mathcal{L}))
\]
3.2 Configuration spaces.

Several kinds of configuration spaces will serve as parameter spaces of operations. The simplest kind are the configuration spaces of points in the plane. The space

\[ \tilde{C}^n(\mathbb{C}) = \{(e_1, \ldots, e_n) \in \mathbb{C}^n \mid e_i \neq e_j \text{ for } i \neq j\} \]

is called the n-th ordered configuration space of \( \mathbb{C} \). The symmetric group \( \Sigma_n \) acts freely on \( \tilde{C}^n(\mathbb{C}) \) and the quotient

\[ C^n(\mathbb{C}) = \tilde{C}(\mathbb{C}) / \Sigma_n \]

is called the n-th (unordered) configuration space of \( \mathbb{C} \). Its fundamental group is the classical braid group \( \text{Br}(n) \) on \( n \) strands, see [Artin 1925], [Birman 1974].

We regard \( C^n(\mathbb{C}) \) as configuration spaces of rectangles rather than of points. To do this we need a distance function

\[ \varepsilon : C^n(\mathbb{C}) \to ]0, \infty[ \]

\[ \varepsilon((e_1, \ldots, e_n)) = \min\{\|e_i - e_j\| \mid 1 \leq i \neq j \leq n\} \]

where \( \| \| \) is the maximal coordinate norm in \( \mathbb{C} \). The rectangles will be square around \( e \in \mathbb{C} \), with sides parallel to the x,y-axes and side length \( \varepsilon > 0 \), i.e.

\[ B_\varepsilon(e) = \{z \in \mathbb{C} \mid \|e - z\| < \frac{\varepsilon}{2}\} \]

We also need the following extension of \( B_\varepsilon(e) \),
(3.2.5) \[ \hat{B}_e(e) = \{ z \in \mathbb{C} \mid \text{Re}(z) \leq \text{Re}(e) + \frac{\varepsilon}{2}, \]

\[ \text{Im}(e) - \frac{\varepsilon}{2} \leq \text{Im}(z) \leq \text{Im}(e) + \frac{\varepsilon}{2} \} \]
3.3 Dyer-Lashof maps.

Our aim is to define maps

\[(3.3.1) \quad \theta = \theta^n(g) : \mathbb{C}^n(\varnothing) \times \operatorname{PSG}(g)^n \rightarrow \operatorname{PSG}(ng),\]

as a family of \(n\)-ary multiplications, parametrized by configurations. Naively we would like to set

\[\theta((e_1, \ldots, e_n) \times (\mathbb{C}^1, \ldots, \mathbb{C}^n)) = \bigcup_{k=1}^{n} e_k + \frac{\varepsilon}{2} N(\mathbb{C}^k)\]

with \(\varepsilon = \varepsilon((e_1, \ldots, e_n))\); here \(U\) is the simple union of parallel slit domains. It would implant \(\mathbb{C}^k\) into a square of length \(\varepsilon\) centered at \(e\). If the configuration \(\{e_1, \ldots, e_n\}\) is such that the extended squares \(\overline{B}_\varepsilon(e_k)\) are pairwise disjoint, then this formula is indeed correct. But if for example \(\operatorname{Im}(e_1) = \operatorname{Im}(e_j)\) and \(\operatorname{Re}(e_1) < \operatorname{Re}(e_j)\), then the slits of \(\mathbb{C}^i\) and \(\mathbb{C}^j\) interfere with each other. Thus the implantation needs some extra consideration, which we explain in the following example.

Assume \(e_1, e_2\) are positioned as in the next figure.

\[(3.3.2)\]

When implanted into \(\overline{B}_\varepsilon(e_1)\) the slits of \(\mathbb{C}^1\) are contained in the shaded region \(\overline{B}_\varepsilon(e_1)\). Implanting \(\mathbb{C}^2\) into \(\overline{B}_\varepsilon(e_1)\) might result in a configuration
which is not regular, or might mean that this process is not continuous. The correct way to do the implantation of \( C^2 \) is as follows. Recall that the left vertical boundary of \( B_\varepsilon(e_1) \) is cut into intervals by the slits of \( C^1 \); and if reglued according to (I.4.2.3), we obtain again a connected interval; this is guaranteed by the regularity, see (I.4.4). In other words, the left side of \( B_\varepsilon(e_1) \) is reparametrized by the gluing induced by \( C^1 \). The implantation of \( C^2 \) must therefore be done with respect to this reparametrization of the \( y \)-coordinate. The following figure shows the image of the square \( B_\varepsilon(e_1) \) under this coordinate change, for various values of \( \text{Im}(e_2) \). In this way squares further to the left must be woven through the slits of parallel slit domains already implanted in squares further to the right.

\[
\text{(3.3.3)}
\]

\[ e_2(t) \]

\[ e_2(0) \]

\[ e_2(1) \]

\[ e_1(t) \text{ is a constant curve, the curve } e_2(t) \text{ is shown.} \]

\[
\text{(3.3.4)}
\]

\[ C^1 \text{ is of genus } 1, \quad C^2 \text{ is not shown.} \]

The reparametrizations will be introduced in the next section.
3.4 Interval exchange transformations.

In our context an interval exchange transformation is a non-continuous, piecewise affine, orientation preserving self-map of $\mathbb{R}$. It is described by a symbol $\tau = (v_1, \ldots, v_n; \pi)$, where $v_i$ are real numbers such that $v_i \geq v_{i+1}$, and $\pi \in \Sigma_{n-1}$ is any permutation. The points $v_i$ cut $\mathbb{R}$ into $n+1$ intervals $I_0 = ]-\infty, v_1]$, $I_1 = [v_2, v_1]$, $\ldots$, $I_{n-1} = [v_n, v_{n-1}]$, and $I_n = ]-\infty, v_n]$.

Since $v_{k+1} = v_k$ is not excluded, some intervals may be only points. $\pi$ permutes the finite intervals $I_1, \ldots, I_{n-1}$, and the new sequence is $I_0, I_{\pi(1)}, I_{\pi(2)}, \ldots, I_{\pi(n-1)}, I_n$.

\[(3.4.1)\]

\[
\begin{array}{ccccccc}
I_n & I_{n-1} & \cdots & I_4 & I_3 & I_2 & I_1 & I_0 \\
\hline
I_n & I_{\pi(n-1)} & I_{\pi(2)} & I_{\pi(1)} & I_0 \\
\end{array}
\]

The associated function $T_\tau : \mathbb{R} \to \mathbb{R}$ is defined as follows. Denote by $|I_k| = |v_{k+1} - v_k|$ the length of $I_k$ and set $h_k = \sum_{i=1}^{k} |I_{\pi(i)}|$ ($k = 1, \ldots, n-1$), and $h_0 = 0$. Let $J_0, J_1, \ldots, J_n$ be the intervals.

\[(3.4.2)\] $J_0 = [v_1, -\infty[ = I_0$, $J_k = [v_1 - h_{k-1}, v_1 - h_k[ = I_k$ ($k = 1, \ldots, n-1$), $J_n = ]-\infty, v_n[ = I_n$.

Then $|J_k| = |I_{\pi(k)}|$, and $T_\tau$ is to map the interior of $J_k$ isometrically and monotonically onto the interior of $I_{\pi(k)}$.\]
(3.4.3) \[
T_v(y) = \begin{cases} 
y & \text{for } v_1 < y, \\
v_{\pi^{-1}(k)} + (v_1 - h_{k-1} - y) & \text{for } v_1 - h_k < y < v_1 - h_{k-1} \quad (k = 1, \ldots, n-1) \\
y & \text{for } y < v_1 - h_{n-1}.
\end{cases}
\]

To have $T_v$ defined everywhere we extend this definition such that $T_v$ is continuous from the right.

(3.4.4)

Different interval exchange transformations $v$ and $v'$ may give the same reparametrizations $T_v = T_v'$. This happens if and only if some intervals are points, i.e. $v_k = v_{k-1}$ for some $k$. We therefore introduce an equivalence relation; it is generated by the following crossing:

(3.4.5) If $v_{k-1} = v_k$, set $m = \pi^{-1}(k-1)$ and $\ell = \pi(m+1)$, and assume $k < \ell$; then we declare $v = v' = (v_1', \ldots, v'_n, v')$ if

$$v'_i = v_i \quad \text{for } i = 1, \ldots, k-1,$$

$$v'_i = v_{i+1} \quad \text{for } i = k, \ldots, k-1,$$

$$v'_i = v_{i-1}.$$
\[ v'_i = v_i \quad \text{for} \quad i = k+1, \ldots, n, \quad \text{and} \]
\[ \pi' = \pi \circ (m \ast n) \]

(3.4.6)

\[
\begin{array}{ll}
\text{v :} & \text{v' :} \\
\vdots & \vdots \\
\vdots & \vdots \\
v_{k-1} & v'_{k-1} = v_{k-1} \\
v_k & v'_{k} = v_{k+1} \\
v_{k+1} & v'_{k+1} = v_{k+1} \\
v_{k+1} & v'_{k+1} = v_{k+1} \\
v_n & v'_{n} = v_{n} \\
\pi & \pi' = \pi \circ (m \ast n) \\
\end{array}
\]

We denote an equivalence class by \( \mathcal{V} = \{v\} = \{v_1, \ldots, v_n; \pi\} \).

(3.4.7) Lemma. If \( v = v' \), then \( T_v(y) = T_{v'}(y) \) for all \( y \) in the interior of the intervals \( J_x \).

Proof: First note that the intervals \( I_j, I'_j \) and \( J_j, J'_j \) produced by \( v, v' \) coincide up to a renumbering. If \( y \) runs from \( +\infty \) to \( -\infty \), then \( T_v(y) \) and \( T_{v'}(y) \) runs through these I-intervals in the order \( I_o, I_{\pi(1)}', \ldots, I_{\pi(n)}', I_{\pi(n-1)}', I_n \). For a crossing (3.4.5) the difference is a transposition of two neighbouring intervals \( I_{\pi(m)} \) and \( I_{\pi(m+1)} \); since
\( I_n(m) = I_{k-1} = [v_{k-1}, v_k] \) is a point, this gives the same function \( T_v = T_{v'} \), except at the boundaries of the \( J \)-intervals.

As already pointed out in (I.p.VI) a parallel slit domain \( \mathcal{L} = [L_1, ..., L_{4g}; \lambda] \) gives rise to an interval exchange transformation. The cut points \( v_i \) are given by \( v_i = \text{im}(S_i) \), \( i = 1, ..., 4g \). To determine the permutation of the intervals \( I_k \) recall the sequence of numbers

\[
(3.4.8) \quad \ell_0 = 0,
\]

\[
\ell_{k+1} = \begin{cases} 
\ell_k + 1 & \text{if } k \text{ is even,} \\
\lambda(\ell_k) & \text{if } k \text{ is odd,}
\end{cases}
\]

which is defined up to a length of \( 8g \), since \( \mathcal{L} \) is regular (I.4.4.6).
In addition we have $l_1 = l$ and $l_{8g} = 4g$ for any $\lambda$. We extract the even indexed numbers and delete the first and last one,

\[
\begin{array}{ccccccc}
  & l_1 = 1 & l_3 & & & l_{8g-1} \\
  l_2 & l_4 & \cdots & l_{8g-2} & & & l_8 = 4g
\end{array}
\]

And the remaining sequence of $4g-1$ numbers is the permutation $\pi = (l_2, l_4, \ldots, l_{8g-2})$ in cycle notation. The interval exchange transformation $v^L = (\text{Im}(S_1), \ldots, \text{Im}(S_{4g}); (l_2, \ldots, l_{8g-2}))$ depends on the representing configuration $L$, but we have

\[
\text{(3.4.10) Lemma. If } L \sim L', \text{ then } v^L \sim v^{L'}.
\]

**Proof:** A crossing of a slit $L_{k-1} \subseteq L_k$ over the pair $L_k, L_{\lambda(k)}$ corresponds to a crossing of $v_{k-1} = \text{Im}(S_{k-1})$ over the "pair" $v_k, v_{l}$, where the indices are coupled by $k = \pi(\pi^{-1}(k-1) + 1)$. Compare (3.4.6) to (I.4.3.2).

The assignment $\mathcal{L} \to \mathcal{I}^{\mathcal{L}}$ is a surjective map from the space of parallel slit domains onto a space of interval exchange transformations. We will investigate this map elsewhere in more detail.
3.5 Dyer-Lashof maps (continued).

We are now prepared to define the Dyer-Lashof maps

\[ \tilde{s} = \tilde{s}_{g_1, \ldots, g_n} : \mathcal{C}^n(\mathcal{C}) \times \text{PS}C(g_1) \times \cdots \times \text{PS}C(g_n) \to \text{PS}C(g_1 + \cdots + g_n) \]

for all \( g_i \geq 0 \), \( n \geq 1 \). Let \((e_1, \ldots, e_n) \in \mathcal{C}^n(\mathcal{C})\) be an ordered configuration, and let \( E^1, \ldots, E^n \) be \( n \) parallel slit domains of genus \( g_1, \ldots, g_n \), resp.. On the configuration we consider the following relation: \( e_i < e_j \) if \( B_{\frac{\varepsilon}{2}}(e_i) \cap B_{\frac{\varepsilon}{2}}(e_j) \neq \emptyset \), where \( \varepsilon = \varepsilon([e_1, \ldots, e_n]) \) is the distance function. The following statements hold.

(3.5.2)

1. \( e_i \not\perp e_i \)

2. either \( e_i < e_j \) or \( e_j < e_i \) or none

3. if \( e_i < e_j \) and \( e_j < e_k \), then \( e_i < e_k \)

4. if \( e_i, e_j < e_k \), then either \( e_i < e_j \) or \( e_j < e_i \) (\( i \neq j \))

5. if \( e_k < e_i, e_j \), then either \( e_i < e_j \) or \( e_j < e_i \) (\( i \neq j \))

The last two properties imply that the entire configuration decomposes into maximal chains \( e_{i_r} < e_{i_{r-1}} < \cdots < e_{i_1} \) (which may contain only one member). The extended squares of any two \( e_i, e_j \) in distinct maximal chains have empty intersection. The definition can therefore be reduced to two cases: (a) the configuration consists of only one chain, and (b) the configuration consists of chains with only one member. We begin with the second and easier case (b).

Let \((e_1, \ldots, e_n)\) be an ordered configuration such that \( e_i \not\perp e_j \) for any two...
members. Thus not only are the numbers $\text{Im}(e_1)$ distinct, their mutual distances are at least $\varepsilon > 0$. Let $\mathcal{L}^1, \ldots, \mathcal{L}^n$ be represented by configurations $L^1, \ldots, L^n$, with $L^k = (L^k_1, \ldots, L^k_4; \lambda^k)$. Their transplants

\begin{equation}
N^k = e_k + \frac{\varepsilon}{2} N(\mathcal{L}^k)
\end{equation}

are represented by configurations $N^k = e_k + \frac{\varepsilon}{2} N(L_k) = (N^k_1, \ldots, N^k_4; \lambda^k)$. The configurations are contained in $n$ disjoint strips $K^k = \mathbb{R} \times [b_-(N^k), b_+(N^k)]$.

Let $\kappa \in \Sigma_n$ the permutation such that the sequence $K^\kappa(1), K^\kappa(2), \ldots, K^\kappa(n)$ is ordered by imaginary parts, i.e. $b_+(N^\kappa(1)) > b_-(N^\kappa(2)) > b_+(N^\kappa(2)) > b_-(N^\kappa(2)) > \ldots > b_+(N^\kappa(n)) > b_-(N^\kappa(n))$. Then

\begin{equation}
N = (N^\kappa_1, \ldots, N^\kappa_1, N^\kappa_1, \ldots, N^\kappa(2), N^\kappa(2), \ldots, N^\kappa(n), N^\kappa(n); \lambda^\kappa(1), \lambda^\kappa(2), \ldots, \lambda^\kappa(n))
\end{equation}

is a configuration, and obviously regular. Its equivalence class $\mathcal{N} = [N]$ depends only on the equivalence classes $\mathcal{N}^k = [N^k]$ since all crossings in some $\mathcal{N}^k$ are still possible in $\mathcal{N}$. We denote this class by

\begin{equation}
\mathcal{N} = [N] = \mathcal{N}^1 \cup \mathcal{N}^2 \cup \ldots \cup \mathcal{N}^n
\end{equation}

and define

\begin{equation}
\bar{\delta}((e_1, \ldots, e_n), (\mathcal{L}^1, \ldots, \mathcal{L}^n)) = \mathcal{N}^1 \cup \mathcal{N}^2 \cup \ldots \cup \mathcal{N}^n.
\end{equation}

If $g_1 = \ldots = g_n$ then $\bar{\delta}$ is $\Sigma_n$-invariant.
In the first and more subtle case (a) let \((e_1, \ldots, e_n)\) be an ordered configuration consisting of one chain, and assume \(e_n < e_{n-1} < \ldots < e_2 < e_1\); otherwise we rename the members \(e_{\pi(n)} < \ldots < e_{\pi(1)}\) for some \(\pi \in \Sigma_n\). Let \(L^1, \ldots, L^n\) be the sequence of parallel slit domains, represented by \(L^1, \ldots, L^n\), respectively. We start by implanting \(L^1\) into the right-most square, the \(\frac{e_1}{2}\) square around \(e_1\), and set

\[
N^1 = e_1 + \frac{\varepsilon}{2} N(L^1) = [N^1_1, \ldots, N^1_{4g_1}; \lambda^1].
\]

Let \(\mathcal{U} = [v_1, \ldots, v_{4g_1}; \pi^1]\) be the interval exchange transformation induced by \(N^1\). The associated reparametrization \(T_\mathcal{U} : \mathbb{R} \to \mathbb{R}\) yields a map \(T : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}\), \(T(x, y) = (x, T_\mathcal{U}(y))\). This non-continuous self-map of the plane will be used to reparametrize the transplant

\[
N^2 = e_2 + \frac{\varepsilon}{2} N(L^2) = [N^2_1, \ldots, N^2_{4g_2}; \lambda^2].
\]

The intervals \(J'_k = [v_{1-k}, v_{1-k}; 1]\) for \(k = 1, \ldots, 4g-1\), and \(J'_0 = [v_1, +\infty]\), \(J'_{4g} = ]-\infty, v_{4g_1}\) are the maximal connected subsets of \(\mathbb{R}\) on which \(T_\mathcal{U}\) is
continuous.

The strips \( R \times J^i_h \subseteq C \) partition the slits of \( \mathbb{N}^2 \) by partitioning the index set \( \{1, \ldots, 4g_2\} \) into linearly ordered sets

\[
(3.5.10) \quad A_k = \{ i \mid 1 \leq i \leq 4g_2, N^2_1 \leq R \times J^i_k \}, \quad (k = 0, 1, \ldots, 4g_1).
\]

If \( J^i_k \) is a point, then \( J^i_k \) and \( A_k \) are empty. Reparametrizing \( \mathbb{N}^2 \) by \( T \) means (for the indices) shuffling the indices \( \{1, \ldots, 4g_2\} \) into \( \{1, \ldots, g_1\} \): their new order is

\[
(3.5.11) \quad A_0, 1, A_\pi(1), 2, A_\pi(2), \ldots, 4g_1 - 1, A_\pi(4g_1 - 1), 4g_1, A_4g_1.
\]

The new slits are \( T(N^2_1), i = 1, \ldots, 4g_2 \); and if reordered according to their imaginary parts, their old indices occur in the order of (3.5.11). We denote by \( T(N^2)_{A_k} \) the sequence \( (T(N^2_{a_k}), T(N^2_{a_k + 1}), \ldots, T(N^2_{b_k})) \) if \( A_k \) is the sequence \( a_k, a_k + 1, \ldots, b_k \).

The shuffling (3.5.11) determines also a new pairing function \( \lambda^{2,1} \in \Sigma_{4g_1+4g_2} \) in an obvious way.

We now define the **implantation of** \( \mathcal{L}^2 \) **into a square around** \( e_2 \) **relative to** \( \mathcal{L}^1 \) **implanted into a square around** \( e_1 \) to be the equivalence class

\[
(3.5.12) \quad [e_2; \mathcal{L}^2 | \mathbb{N}^1] := [T(N^2_{A_0}), N^1_1, T(N^2_{A_\pi(1)}), N^1_2, \ldots, T(N^2_{A_\pi(4g_1 - 1)}), N^1_{4g_1}, T(N^2_{A_4g_1}); \lambda^{2,1}]
\]

(Here \( \varepsilon \) depends actually on the entire configuration \( \{e_1, \ldots, e_n\} \) and not only on \( e_2 \).)
To see that this relative implantation is continuous, it is enough to show that it is well-defined. Assume in \( \mathcal{N}^1 \) a slit crosses a pair of slits; by Lemma (3.4.7) this does not change \( T \). The effect on the intervals \( I_k, J_k, J_k' \) and the indices partition \( A_k \) is only a renumbering by a partial cyclic permutation. This comes to a crossing in \( \mathcal{N} \). Assume in \( \mathcal{N}^2 \) a slit crosses a pair. If all the three slits involved are not contained in any slit of \( \mathcal{N}^1 \), this crossing is also possible in \( \mathcal{N} \). If one (and hence two) are contained in some slit of \( \mathcal{N}^1 \), the corresponding index switches from \( A_k \) to \( A_{k+1} \) or \( A_{k-1} \) (being the last in \( A_k \) and becoming the first in \( A_{k+1} \), or being the first in \( A_k \) and becoming the last in \( A_{k-1} \)). Again, this amounts to a crossing in \( \mathcal{N} \).

It is now obvious how to proceed inductively along the chain \( e_n < \ldots < e_3 < e_2 < e_1 \).

In the next step we regard \( [e_2, e_1; \mathcal{L}^2, \mathcal{L}^1] \) as an entity and implant \( \mathcal{L}^3 \) into a square around \( e_3 \) relative to \( [e_2; \mathcal{L}^2 | \mathcal{N}^1] \). And so on. This gives inductively
\[(3.5.14) \quad [e_1; L^1] = e_1 + \frac{\varepsilon}{2} N(L^1) = N^1, \]
\[(e_2, e_1; L^2, L^1) = [e_2; L^2] [e_1; L^1], \]
\[(e_k, e_{k-1}, \ldots, e_1; L^k, L^{k-1}, \ldots, L^1) = [e_k; L^k] [e_{k-1}, \ldots, e_1; L^{k-1}, \ldots, L^1]. \]

Having settled the second case (b) (when the configuration consists of only one chain) we can finally give the definition of the Dyer-Lashof maps, first in non-equivariant form,

\[(3.5.15) \quad \tilde{s} = \tilde{s}_{g_1, \ldots, g_n} : \mathcal{C}^n(\mathcal{E}) \times PSU(g_1) \times \cdots \times PSU(g_n) \rightarrow PSU(g_1 \cdots g_n). \]

The formula reads as

\[\tilde{s}(e_1, \ldots, e_n, L^1, \ldots, L^n) = \bigcup_{e_i \prec \cdots \prec e_1} [e_i; L^i, \ldots, L^1, \ldots] \]

where the union runs over all maximal chains in \( \{e_1, \ldots, e_n\} \). The union is in the sense of \(3.5.5\); note that \( \bigcup \) is strictly associative, and commutative.

If \(g_1 = \ldots = g_n\) in \(3.5.15\) \(\tilde{s}_{g_1, \ldots, g_n}\) is \(\Sigma_n\)-invariant, and we obtain

\[s^n : \mathcal{C}^n(\mathcal{E}) \times PSU(g)^n \rightarrow PSU(ng). \]

3.6 **Main properties.**

The Dyer-Lashof maps satisfy an associativity law. It will imply the commutativity of the sum operation \(\mu\) up to homotopy, and its compatibility with the stabilization \(\sigma\) up to homotopy. To formulate it we first need an operation on the configuration spaces alone.
Let \( r \geq 1 \) and \( n_1, \ldots, n_r \geq 1 \) be given, and set \( n = \sum_{i=1}^{r} n_i \). There is a map
\[
\tilde{t} = \tilde{t}_{n_1, \ldots, n_r}
\]
\[
(3.6.1) \quad \tilde{t} : \tilde{C}^r(\mathcal{C}) \times \tilde{C}^{n_1}(\mathcal{C}) \times \ldots \times \tilde{C}^{n_r}(\mathcal{C}) \longrightarrow \tilde{C}^n(\mathcal{C})
\]
\[
\tilde{t}((e_1, \ldots, e_r), (e_{n_1}^1, \ldots, e_{n_1}^{n_1}), \ldots, (e_{n_r}^r, \ldots, e_{n_r}^{n_r})) =
\]
\[
((e_1 + \frac{\varepsilon}{2} N(e_1^1, \ldots, e_1^{n_1}), e_2 + \frac{\varepsilon}{2} N(e_2^2, \ldots, e_2^{n_2}), \ldots, e_r + \frac{\varepsilon}{2} N(e_r^r, \ldots, e_r^{n_r}))
\]

where \( \varepsilon = \varepsilon((e_1, \ldots, e_r)) \), and \( N(z_1, \ldots, z_m) \) is the normalized sequence \( (z_1', \ldots, z_m') \) with \( z_i' \) defined as follows. If \( c = c(z_1, \ldots, z_m) = \frac{1}{m} \sum z_j \) is the center of maps, and \( d = \text{diam}(h_1, \ldots, z_m) \) the diameter (in the maximal-coordinate norm) of the set \( \{z_1, \ldots, z_m\} \), then \( z_i' = \frac{1}{d}(z_i - c) \). And \( e + \frac{\varepsilon}{2}(z_1', \ldots, z_m') \) is an abbreviation for the sequence \( (e + \frac{\varepsilon}{2} z_1', \ldots, e + \frac{\varepsilon}{2} z_m') \). Geometrically speaking, \( t \) implants the configurations \( (e_1^k, \ldots, e_{n_k}^k) \) into \( \frac{\varepsilon}{2} \) squares around the points \( e_k \).

Note that, if \( n_1 = \ldots = n_r \), then \( \tilde{t} \) is \( \Sigma_r \)-invariant.

We define inclusions \( \tilde{D}_1^1 : \text{PSG}(g) \longrightarrow \tilde{C}^1(\mathcal{C}) \times \text{PSG}(g) \) by \( D_1^1(\mathcal{L}) = (0, \mathcal{L}) \) and \( \tilde{D}_1^2 : \text{PSG}(g) \longrightarrow \tilde{C}^2(\mathcal{C}) \times \text{PSG}(g)^2 \), \( \tilde{D}_1^2(\mathcal{L}) = ((i, -i), \mathcal{L}, \mathcal{L}) \), and \( D_2^1 \) is \( \tilde{D}_1^1 \) composed with the projection to \( \tilde{C}^2(\mathcal{C}) \times \text{PSG}(g)^2 \).

Now we can formulate the main result of this chapter.

(3.6.2) **Theorem.** There are operations
\[
\sigma^n : \tilde{C}^4(\mathcal{C}) \times \text{PSG}(g)^n \longrightarrow \text{PSG}(ng)
\]
\[
\Sigma_n
\]

with the following properties.
(i) **Associativity**

For \( r \geq 1, n \geq 1 \) and \( g \geq 0 \) the diagram

\[
\begin{array}{c}
\Sigma_r \times (\Sigma_n \times \Sigma_g)^r \rightarrow \Sigma_r \times (\Sigma_n \times \Sigma_g)^r \\
\Sigma_n \times (\Sigma_g)^r \rightarrow \Sigma_n \times (\Sigma_g)^r
\end{array}
\]

is commutative.

(ii) **Unit**

For all \( g \geq 0 \) the composition is

\[
\begin{array}{c}
\Sigma_1 \rightarrow \Sigma_1 \\
\Sigma_1 \rightarrow \Sigma_1
\end{array}
\]

is homotopic to the identity.

(iii) **Squaring**

For all \( g \geq 0 \) the composition is

\[
\begin{array}{c}
\Sigma_2 \rightarrow \Sigma_2 \\
\Sigma_2 \rightarrow \Sigma_2
\end{array}
\]

is homotopic to \( \mu_{g,g} \circ \text{diag} \).

**Proof:** To prove (i), we need to know that diagrams of the following kind are commutative, \( n = \Sigma n_i \).
\begin{align*}
(3.6.3) \quad \mathcal{C}^r(\mathcal{C}) \times (\mathcal{C}^1(\mathcal{C}) \times \text{PS\!C}(g_1)^{n_1}) \times \ldots \times (\mathcal{C}^r(\mathcal{C}) \times \text{PS\!C}(g_r)^{n_r}) \\
\Sigma^{n_1} \times \ldots \times \Sigma^{n_r}
\end{align*}

\begin{align*}
\tilde{\tau} \times \text{id} \times \ldots \times \text{id} \\
\text{id} \times \tilde{g}_1 \times \ldots \times \tilde{g}_r
\end{align*}

\begin{align*}
\mathcal{C}^n(\mathcal{C}) \times (\text{PS\!C}(g_1)^{n_1} \times \ldots \times \text{PS\!C}(g_r)^{n_r}) \\
\text{PS\!C}(n_1 g_1 + \ldots + n_r g_r)
\end{align*}

The proof is straightforward, but tedious. The assertion (ii) is clear, since $g^1 \cdot D^1$ is the normalization map $\mathcal{C} \to \text{N}(\mathcal{C})$. And for (iii), we use the homotopy from $\mathcal{C} \to \mathcal{C} \ast \mathcal{C}$ to $\mathcal{C} \to \tilde{g}_1^2(D^2(\mathcal{C}))$ which normalizes the two copies of $\mathcal{C}$ separately in the upper resp. lower half-plane around $+i$ resp. $-i$. 

Form [Boardman-Vogt 1968] and [May 1972] we have a well-developed theory of "little n-cube operads", central to understand n-fold loop spaces. The configuration spaces are such an operad for $n = 2$, see [May 1972, pp.1,30]. This operad, made to act on double loop spaces $\Omega^2 X$, acts by Theorem (3.6.2) on the space $\text{PS\!C} = \bigoplus_{g \geq 0} \text{PS\!C}(g)$ in the sense of [May 1972; pp.4,40]. This will be used in the next chapter.

(3.6.4) Proof of (1.1.8): The homotopy $\mu = \mu \circ \tau$ comes from a curve in $\mathcal{C}^2(\mathcal{C})$ which exchanges the two points $+i$ and $-i$. 
Consider the homotopy \( f_t(\mathcal{C}^1, \mathcal{C}^2) = s^2((ie^{\pi it}, -ie^{\pi it}), (\mathcal{C}^1, \mathcal{C}^2)), 0 \leq t \leq 1 \).

We have \( f_0 = \mu \) by (3.6.2 (iii)). And \( f_1 = \mu \circ \tau \).

(3.6.6) **Proof of (1.2.5):** Consider for \( \mu \circ (\sigma \times id) = \sigma \circ \mu \) the curve \( t \mapsto e(t) \) in \( C^3(\mathcal{C}) \) given by the figure.

(3.6.7)

\[ e_1, \quad e_2, \quad e_3. \]

satisfies \( f_0 = \mu \circ (\sigma \times id) \), and \( f_1 = \sigma \circ \mu \). Similar curves can be found to make homotopies \( \mu \circ (id \times \sigma) = \sigma \circ \mu \) and \( \mu \circ (\sigma \times \sigma) = \sigma \circ \sigma \circ \mu \).

We remark that the operations \( \otimes^n \) do not generally commute with the stabilization, i.e. the diagrams
(3.6.8)\[ \begin{array}{ccc}
\Sigma_n \\
C^n(\mathfrak{g}) \times \text{PSG}(g)^n & \overset{g^n}{\longrightarrow} & \text{PSG}(ng) \\
\downarrow \text{id} \times (\sigma)_n & & \downarrow \sigma^n \\
\Sigma_n \\
C^n(\mathfrak{g}) \times \text{PSG}(g+1)^n & \longrightarrow & \text{PSG}(ng+n)
\end{array} \]

Do not always commute up to homotopy.

3.7 **Comparison with operations on braid groups and symplectic groups.**

On the group level the operations

(3.7.1) \[ a : C^n(\mathfrak{g}) \times \text{PSG}(g)^n \overset{g}{\longrightarrow} \text{PSG}(ng) \]

become homomorphisms from wreath products

(3.7.2) \[ Br(n) \int \tilde{f}(g) \longrightarrow \tilde{f}(ng) . \]

(Here the notation \( G \int H = H \backslash G \) means: \( G \leq \Sigma_n \) is the quotient group, acting as a permutation group on the normal subgroup \( H^n \).) By the \( \Sigma_r \)-invariance of \( \tilde{\tau} \) in case of \( n_1 = \ldots = n_r = g \) in (3.6.1) we obtain maps \( t^r_g : C^r(\mathfrak{g}) \times C^r(\mathfrak{g}) \overset{g}{\longrightarrow} C^r(\mathfrak{g}) \), see [May 1972; p.130]. For the groups, these are homomorphisms.

(3.7.3) \[ Br(r) \int Br(g) \longrightarrow Br(rg) . \]
To connect the two, let \( \tau : \text{pt.} \to \text{PSU}(1) \) the inclusion of the basepoint \( \gamma(1) \). Then the composition

\[
\beta_g : C^G(\mathfrak{g}) \cong \overline{C^G(\mathfrak{g})} \times (\text{pr.})^g \overset{\text{id} \times \tau^g}{\longrightarrow} \overline{C^G(\mathfrak{g})} \times \text{PSU}(1)^g \overset{\Sigma_g}{\longrightarrow} \text{PSU}(g)
\]

establishes a homomorphism \( \text{Br}(g) \to \overline{\Gamma}(g) \). It is the easiest of several such homomorphisms; it associates to a braid the mapping class turning handles around each other according to this braid.

(3.7.5) **Proposition.** The maps \( \beta_m \) commute with the operations \( \tau^g \) and \( \delta^g \).

**Proof:** The diagram

\[
\begin{array}{ccc}
\overline{C^r(\mathfrak{g})} \times C^g(\mathfrak{g})^r & \overset{\tau^g}{\longrightarrow} & C^r(\mathfrak{g}) \\
\Sigma_r & \downarrow & \\
\text{id} \times (\beta^r_g) & & \beta_{rg} \\
\Sigma_r & \downarrow & \\
\overline{C^r(\mathfrak{g})} \times \text{PSU}(g)^r & \overset{\delta^r_g}{\longrightarrow} & \text{PSU}(rg) \\
\Sigma_r & & \\
\end{array}
\]

is a special case of the diagram (3.6.3): set \( n_1 = \ldots = n_r = g_1 = \ldots = g_r = 0 \), and use that the diagram is then also \( \Sigma^r \)-equivariant.

In the other direction, mapping out of \( \text{PSU}(g) \), we have the monodromy homomorphism

(3.7.6) \( M_g : \text{PSU}(g) = \overline{\Gamma}(g) \to \Gamma(g) \to \text{BS}_{2g}(\mathbb{Z}) \).
where the last map associates (on the group level) to a mapping class $[\gamma]$ the homomorphism $\gamma_\pi : H^1(F_g;\mathbb{Z}) \rightarrow H^1(F_g;\mathbb{Z})$ induced on the first cohomology group of our reference surface $F_g$. Since the cup-product, being a symplectic form on $H^1(F_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, is preserved, $\gamma_\pi$ can be regarded as a $2g \times 2g$ symplectic matrix over $\mathbb{Z}$. Without using geometry there are operations

\[ \Sigma_n \int \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow \text{Sp}_{2ng}(\mathbb{Z}) \]

defined by letting the normal subgroup $\text{Sp}_{2g}(\mathbb{Z})^n$ act on $(\mathbb{Z}^{2g})^n$ by direct sum matrices, and letting the quotient group $\Sigma_n$ act by permuting the summands.

Using the facts $C^n(\mathbb{R}^m) = E \Sigma_n$ and $C^n(\mathbb{R}^m) = B \Sigma_n$, we have maps

\[ S^n_g : C^n(\mathbb{R}^m) \times B\text{Sp}_{2g}(\mathbb{Z}) \longrightarrow B\text{Sp}_{2ng}(\mathbb{Z}). \]

(3.7.10) Proposition. The monodromy maps $M_m$ commute with the operations $S^n_g$ and $S^n_g$.

Proof: The assertion is the commutativity of the diagrams

\[ C^n(E) \times \text{PSO}(g)^r \xrightarrow{\gamma^n_g} \text{PSO}(rg) \]

\[ C^n(\mathbb{R}^m) \times \text{BSp}_{2g}(Z)^r \xrightarrow{S^n_g} \text{BSp}_{2ng}(Z) \]

Their commutativity follows from the fact that any implantation is geometrically a connected sum; hence the first cohomology group splits (as a symplectic module) into a direct sum.
Chapter 4

The Homology of Dyer-Lashof Operations

4.1 The Dyer-Lashof operations $Q_0$ and $Q_1$.
4.2 The Browder operation $R_1$.
4.3 Formulas for $Q_0$.
4.4 Formulas for $Q_1$.
4.5 Formulas for $R_1$.

The Dyer-Lashof homology operations will be derived from the Dyer-Lashof maps. The Browder operation will measure the non-linearity of $Q_1$. For the definition and the first properties we specialize to coefficients $A = F_2$.

It will become apparent that we can profit from the classical theory of homology operations for loop spaces, as developed in [Araki-Kudo 1956], [Dyer-Lashof 1962], [Browder 1960], [Milgram 1966], [May 1972], [Cohen 1976] and others. Especially [Cohen 1976] contains the whole apparatus for "little n-cube operations".
4.1 The Dyer-Lashof operations \( Q_0 \) and \( Q_1 \).

The operations \( s^n_\Sigma : \Sigma^n(\Sigma) \times \text{PS}(\Sigma) \to \text{PS}(\Sigma) \) can be used to define homology operations. We restrict our attention to homology with coefficients in the field \( \mathbb{F}_2 \) of two elements and write \( H_\Sigma(\_ ; \mathbb{F}_2) \). Let \( n = 2 \), and \( g \geq 0 \) arbitrary. There are two coverings \( \Sigma^2(\Sigma) \to \Sigma^2(\Sigma) \) and \( \Sigma^2(\Sigma) \times \text{PS}(\Sigma)^2 \to \Sigma^2(\Sigma) \times \text{PS}(\Sigma)^2 \). If \( \tilde{w} \) is a (singular) chain in \( \Sigma^2(\Sigma) \) which projects to a cycle \( w \) in \( \Sigma^2(\Sigma) \), then, for any cycle \( x \) in \( \text{PS}(\Sigma) \), \( \tilde{w} \ast x \ast x \) is a cycle in \( \Sigma^2(\Sigma) \times \text{PS}(\Sigma)^2 \). Note that \( \Sigma^2(\Sigma) \) and \( \Sigma^2(\Sigma) \) both are homotopy-equivalent to a circle; indeed the covering \( \Sigma^2(\Sigma) \to \Sigma^2(\Sigma) \) is homotopy-equivalent to \( S^1 \to \mathbb{R}P^1 \). We denote by \( \tilde{w}_0 \) and \( \tilde{w}_1 \) a 0-chain resp. 1-chain which project to the non-zero elements in \( H_0 \Sigma^2(\Sigma) \) and \( H_1 \Sigma^2(\Sigma) \), [ ] denotes homology classes, and \( s^n_\Sigma \) is the induced map in homology. The definition is

\[
(4.1.1) \quad Q_0 = H_q \Sigma \text{PS}(\Sigma) \to H_{2q} \Sigma \text{PS}(2\Sigma),
\]

\[
Q_0(x) := s^2_\Sigma[\tilde{w}_0 \ast x \ast x],
\]

\[
(4.1.2) \quad Q_1 = H_q \Sigma \text{PS}(\Sigma) \to H_{2q+1} \Sigma \text{PS}(2\Sigma),
\]

\[
Q_1(x) := s^2_\Sigma[\tilde{w}_1 \ast x \ast x].
\]

\( Q_0(x) \) and \( Q_1(x) \) are well-defined homology classes.

The chains \( \tilde{w}_0 \) and \( \tilde{w}_1 \) are shown in a figure. \( \tilde{w}_0 \) is, of course, just a point and \( \tilde{w}_1 \) is a curve.
The next figure shows schematically the homology classes $Q_1(x)$, where the homology class $x$ is depicted only symbolically.

The operation $Q_0$ is like a simple multiplication, and $Q_1$ is a closed curve of such multiplications. They are the operations denoted by $Q^q$, resp. $Q^{q+1}$ for $|x| = q$ in [May 1970; 2.2] or [Cohen 1976; p.213,248]; $Q^{q+1}$ is the "top operation", since we use the two-dimensional theory, or double loop spaces.

4.2 The Browder operation.

If $\tilde{v}$ is a cycle in $\tilde{C}^2(\mathbb{C})$, then $\tilde{v} \ast x \ast y$ is a cycle in $\tilde{C}^2(\mathbb{C}) \times \text{PS}\mathbb{C}(g_1) \times \text{PS}\mathbb{C}(g_2)$ for any two cycles $x$, $y$ in $\text{PS}\mathbb{C}(g_1)$ resp. $\text{PS}\mathbb{C}(g_2)$. Since $\tilde{C}^2(\mathbb{C}) = S^1$, there is essentially one such cycle $\tilde{v}$, which projects to twice $\tilde{w}_1$. The
The operation $R_1$ is the Browder operation in [Cohen 1976; pp.245,248].

4.3 Formulas for $Q_o$.

The operation $Q_o : H_{\text{PSG}(g)} \rightarrow H_{2q} \text{PSG}(2g)$ is nothing but the squaring in the Pontrjagin ring:

\begin{equation}
(4.3.1) \quad \text{(squaring)} \quad Q_o(x) = x^2 = x \# x
\end{equation}
for all \( x \in H_3 \mathcal{P} \mathcal{S} \). This is immediate from the definition (4.1.1) and (3.6.2)(iii). Since we work in mod-2 homology this implies

\[
(4.3.2) \quad \text{(linearity)} \quad Q_0(x+y) = Q_0(x) + Q_0(y) \quad \text{and}
\]

\[
(4.3.3) \quad \text{(multiplicativity)} \quad Q_0(xy) = Q_0(x) Q_0(y)
\]

for any \( x, y \in H_3 \mathcal{P} \mathcal{S} \). Recall the stabilization \( \sigma_{q^n} : H_{q^n} \mathcal{P} \mathcal{S}(g) \rightarrow H_{q^n} \mathcal{P} \mathcal{S}(g+1) \).

The formula

\[
(4.3.4) \quad \text{(stability)} \quad Q_0(\sigma_{q^n}(x)) = \sigma_{q^n}^2(Q_0(x))
\]

follows from (1.2.5), proved in (3.6.6). If \( l_g \in H_0 \mathcal{P} \mathcal{S}(g) \) denotes the generator, then we have by (1.1.6(iii))

\[
(4.3.5) \quad \text{(units)} \quad Q_0(l_g) = l_{2g}.
\]

Let \( \text{Sq}_r : H_q(\cdot) \rightarrow H_{q-r}(\cdot) \) be the dual Steenrod squares. In particular, \( \text{Sq}_1 = \beta \), the Bockstein (boundary) homomorphism induced by the exact sequence

\[
0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0.
\]

From [Cohen 1976; p.214] we have

\[
(4.3.6) \quad \text{(Nishida relations)} \quad \text{Sq}_r Q_0(x) = Q_0 \text{Sq}_r(x),
\]

\[
\text{Sq}_{2r+1} Q_0(x) = 0.
\]
4.4 Formulas for $Q_1$.

Apart from the basic squaring operation $Q_0$, we have only one other Dyer-Lashof operation $Q_1$, which is the exceptional "top operation" in [Cohen 1976; p.217]. We first note that $Q_1$ is only a function, and perhaps not a homomorphism; the Browder operation $R_1$ enters most formulas as an error term. But, of course, $Q_1(0) = 0$.

\begin{align*}
(4.4.1) \text{ (linearity)} \quad Q_1(x+y) &= Q_1(x) + R_1(x,y) + Q_1(y), \\
(4.4.2) \text{ (Cartan formula)} \quad Q_1(xy) &= x^2Q_1(y) + xR_1(x,y)y + Q_1(x)y^2,
\end{align*}

for any $x,y \in H_2\mathbb{PSG}$. The formulas follow from [Cohen 1976; Theorem 1.3(5),(2)]. But since $Q_1$ is only of degree one, we can argue geometrically, using for (4.4.1) the following figure:

\begin{align*}
(4.4.3) \\
Q_1(x+y) &
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
x \\
y \\
x \\
y
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\quad Q_1(x) &
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
x \\
y \\
x \\
y
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\quad R_1(x,y) &
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
x \\
y \\
x \\
y
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
Here the "variables" $x, y$ are not independent, but "alternative", since we consider the sum $x+y$. The right figure shows three terms: they result from writing the chain $\tilde{w}_1(t)$ as a sum of four terms (after a homotopy) where $t$ is called $t_1, t_2, t_3, t_4$ on the intervals $[0, 1/4], [1/4, 1/2], [1/2, 3/4]$ resp. $[3/4, 1]$. The two middle terms amount to $Q_1(x)$ and $Q_1(y)$, whereas the sum of the two outer terms is $R_1(x, y)$.

If, in the same figure, $x$ and $y$ are regarded as independent, it provides a proof of (4.4.2).

For the basepoint class $1_o = 1 \in H_0^{PS\mathbb{Z}(0)}$ we have obviously

(4.4.4) $Q_1(1_o) = 0$.

Since $\sigma^2(x) = x \cdot 1_1$, we have $Q_1(\sigma(x)) = \sigma^2(Q_1(x)) + \sigma(xR_1(x, 1_1)) + x^2Q_1(1_1)$ for the behaviour with respect to stability. For the Steenrod squares we find from [Cohen 1976; p.217,(3)]

(4.4.5) (Nishida relations)

\[ Sd_{2r}Q_1(x) = Q_1S_{r-1}(x) + \sum_{i+j=2r}^{i \leq j} R_1(Sq_i(x), Sq_j(x)) \]

\[ Sq_{2r+1}Q_1(x) = Q_0S_{r-1}(x) + \sum_{i+j=2r+1}^{i \leq j} R_1(Sq_i(x), Sq_j(x)) \]
4.5 Formulas for $R_1$.

The binary operation $R_1: H_p \text{PSG}(g_1) \otimes H_q \text{PSG}(g_2) \rightarrow H_{p+q+1} \text{PSG}(g_1 \ast g_2)$ occurs as an error term in other formulas. But in the same sense as $Q_1$ is the degree 1 analog of the squaring operation $Q_0$, $R_1$ is the analog of the sum operation $\mu$. Here are some of the formulas.

(4.5.1) (commutativity) \hspace{1em} R_1(x,y) = R_1(y,x) .

(4.5.2) \hspace{1em} R_1(x,x) = 0

(4.5.3) (unit) \hspace{1em} R_1(1,x) = 0 = R_1(x,1)

for the unit $1 = 1_o \in H_0 \text{PSG}(0)$

(4.5.4) (Cartan-formula)

$R_1(xy,y'y') = xR_1(y,x')y' + R_1(x,x')yy' + x'R_1(y,y') + x'R_1(x,y')y$ .

(4.5.5) (Jacobi identity) \hspace{1em} R_1(x,R_1(y,z)) + R_1(y,R_1(z,x)) + R_1(z,R_1(x,y)) = 0

(4.5.6) (Nishida relations) \hspace{1em} Sq^rR_1(x,y) = \sum_{i+j=r} R_1(Sq^i(x),Sq^j(y))

(4.5.7) (Bockstein relation) \hspace{1em} \beta R_1(x,y) = R_1(\beta x,y) + R_1(x,\beta y)

(4.5.8) (Adam relations) \hspace{1em} R_1(x,Q_0(y)) = 0 = R_1(Q_0(x),y)

$R_1(x,Q_1(y)) = 0 = R_1(R_1(x,y),y)$
The formulas (4.5.1) and (4.5.2) are only true since we work modulo 2, see [Cohen 1976; p.215,(3)]. In the same reference we find (4.5.3) as (4), (4.5.4) as (5), (4.5.5) as (6), and (4.5.6), (4.5.7) as (7), (8) on p.216, and (4.5.8) as (4) on p.218.
Chapter 5

The Geometry of

Symplectic Operations

5.1 Configuration spaces of vertical pairs.
5.2 The subspace of partitioned parallel slit domains.
5.3 The operation.
5.4 Main properties.
5.5 Double braid groups.

Since the Dyer-Lashof operations implant entire parallel slit domains into patches moving in the plane, they make very little use of the more intricate structure given by the slits. The smallest units they operate with are handles. The new sort of operations take several parallel slit domains apart into an upper and lower half and implant the halves into vertical pairs of rectangles moving independently in the plane. The disadvantage is that these operations are only defined on a subspace of $\text{PS}\mathbb{C}(g)$; but the stabilization of these subspaces is homotopy-equivalent to $\text{PS}\mathbb{C}(\infty)$.
5.1 Configuration spaces of vertical pairs.

Let $\tilde{E}^n(\mathcal{C})$ be the ordered configuration space of $n$-tuples $((e_1^+, \overline{e_1^-}), ..., (e_n^+, \overline{e_n^-}))$ of vertical pairs in $\mathcal{C}$, i.e.

(5.1.1) (1) $e_i^+ = e_j^+, e_j^-$ for all $i, j$ with $i \neq j$.

(2) $\text{Re}(e_i^+) = \text{Re}(\overline{e_i^-})$ for all $i$.

(3) $\text{Im}(e_i^+) > \text{Im}(\overline{e_i^-})$ for all $i$.

$\tilde{E}^n(\mathcal{C})$ is a $3n$-dimensional manifold with a free $\Sigma_n$-action; we write $E^n(\mathcal{C}) = \tilde{E}^n(\mathcal{C})/\Sigma_n$.

We obtain subspaces $\tilde{E}_i^n(\mathcal{C}) \subseteq \tilde{E}^n(\mathcal{C})$ and $E_i^n(\mathcal{C}) = \tilde{E}_i^n(\mathcal{C})/\Sigma_n \subseteq E^n(\mathcal{C})$ by replacing (3) above by

(3') $\text{Im}(e_i^+) > 0 > \text{Im}(\overline{e_i^-})$ for all $i$.

These configuration spaces are related to the old ones by inclusions

(5.1.2) $\tilde{c}^n(\mathcal{C}) \hookrightarrow \tilde{E}^n(\mathcal{C}) \subseteq E^n(\mathcal{C}) \subseteq c^{2n}(\mathcal{C})$

where $\tilde{c}((z_1, ..., z_n) = ((e_1^+, \overline{e_1^-}), ..., (e_n^+, \overline{e_n^-}))$ where $e_k^+ = z_k = \text{id}$ with $d = \min \{\text{Im}(e_j^-)\} + 1$.

To study the spaces $\tilde{c}$ one uses the fibrations
(5.1.3) \[ \mathcal{C}^m(\mathcal{C}) \leftarrow \mathcal{C}^{m-1}(\mathcal{C}-1) \leftarrow \mathcal{C}^{m-2}(\mathcal{C}-1,2) \leftarrow \ldots \leftarrow \mathcal{C}^1(\mathcal{C}-1,\ldots,m-1) \]

\[ \mathcal{C} \quad \mathcal{C}-1 \quad \mathcal{C}-1,2 \quad \ldots \ldots \]

(see [Fadell-Neuwirth 1962], [Birman 1974]. The restrictions to \( \mathcal{E}_n^m(\mathcal{C}) \) (and \( \mathcal{E}_n^m(\mathcal{C}) \)) give fibrations

(5.1.4) \[ \mathcal{E}_n^m(\mathcal{C}) \leftarrow \mathcal{E}_{n-1}^m(\mathcal{C}-ti) \leftarrow \mathcal{E}_{n-2}^m(\mathcal{C}-ti,1ti) \leftarrow \ldots \leftarrow \mathcal{E}_1^m(\mathcal{C}-ti,\ldots) \]

\[ \mathcal{E}_n^m(\mathcal{C}) \quad \mathcal{E}_n^m(\mathcal{C}-ti) \quad \mathcal{E}_n^m(\mathcal{C}-ti,1ti) \quad \ldots \ldots \]

(5.1.5) Lemma. \( \mathcal{E}_n^m(\mathcal{C}-ti,\ldots,(k-1)ti) \) resp. \( \mathcal{E}_n^m(\mathcal{C}-ti,\ldots,(k-1)ti) \) is homotopy-equivalent to a bouquet of \( 5k \) resp. \( 3k \) circles.

(5.1.6) Proposition. \( \mathcal{E}_n^m(\mathcal{C}), \mathcal{E}_n^m(\mathcal{C}), \mathcal{E}_n^m(\mathcal{C}) \) and \( \mathcal{E}_n(\mathcal{C}) \) are Eilenberg-MacLane spaces \( K(G,1) \).

Proof: By the lemma, the higher homotopy groups \( \pi_*, \; * \geq 2 \), of all base spaces and the last fibre in (5.1.4) vanish. It follows by induction on \( n \) that the higher homotopy groups of \( \mathcal{E}_n^m(\mathcal{C}) \) vanish. The same follows for the quotients \( \mathcal{E}_n^m(\mathcal{C}) \).

Proof of Lemma (5.1.5): First consider \( \mathcal{E}_n^m(\mathcal{C}-ti) \). The map \( (e^+,e^-) \rightarrow (e^-,e^+,e^+) \) is a homeomorphism onto a subset of \( (\mathcal{C}-\{ti\}) \times \mathbb{R}_+ \), which is the complement of four lines, two of which intersect.
(5.1.7)

This complement is retractable onto a 6-fold punctured sphere. A Mayer-Vietoris argument completes the proof for $k > 1$. For $E_1$, the proof is now similar. 

5.2 The subspace of partitioned parallel slit domains.

For a parallel slit domain $\mathcal{L} = [L_1, \ldots, L_{4g}; \lambda]$ consider the condition

\[(5.2.1) \quad L_i \notin R \quad \text{for all} \quad i = 1, \ldots, 4g.\]

Note that it is satisfied either by all or none of the representing configuration. Assume the index set decomposes into two subsets \(\{1, \ldots, 4m\}\) and \(\{4m+1, \ldots, 4g\}\) which are invariant under $\lambda$. Then the first $4m$ slits and the last $4(g-m)$ slits form a parallel slit domain all by themselves, one in $\mathbb{H}$ and the other in $\mathbb{H}^-$. (5.2.1) and the invariance of the two parts under $\lambda$ prevents moving a slit from $\mathbb{H}$ to $\mathbb{H}^-$, or vice versa. Thus we have $g+1$ components homeomorphic to $\text{PS}_g(m) \times \text{PS}_g(g-m)$, $m = 0, 1, \ldots, g$. But there is one more component: define $\text{PS}_g^+(g)$ to be the subspace of all $\mathcal{L} \in \text{PS}_g(g)$ satisfying (5.2.1) and in addition:
(5.2.2) there is an index \( k \) such that \( k < \lambda(k) \),
\( L_k \subseteq \mathbb{H} \) and \( L_{\lambda(k)} \subseteq \mathbb{H}^- \).

(5.2.3) **Proposition.** \( PSL_2(g) \) is a connected, open submanifold of \( PSL_2(g) \).

**Proof:** We only have to show the connectivity. Pick \( \mathcal{L} \in PSL_2(g) \) and let \( k \)
be a bridging index as in (5.2.2). We will construct a path in \( PSL_2(g) \) from
\( \mathcal{L} \) to \( \mathcal{L}' \) given by \( S_1' = (0,1) \), \( S_k' = (1,-1) \) (\( k = 2, \ldots, 4g \)) and \( \lambda' = (13) \)
\( (24) \ldots (4g-3 4g-1) \) (4g-2 4g).

In a first move we make \( \mathcal{L} \) generic, i.e. all slits disjoint. Then we bring
all slit endpoints to the imaginary axis. There are two groups of slits we have
to move to the lower half-plane: the sequence \( A \) with indices \( 1, \ldots, k-1 \), and
the sequence \( B \) with indices \( k+1, \ldots, m \), where \( m \) is the largest index such
that \( L_m \subseteq \mathbb{H} \).

(5.2.4)
We cross the sequence $A$ downwards over the pair $L_{k}, L_{\lambda(k)}$, starting with $L_{k-1}$, to become a sequence $A'$ between $L_{\lambda(k)}$ and $L_{\lambda(k)+1}$. Then we cross the sequence $B$ upwards under the pair $L_{k}, L_{\lambda(k)}$, starting with $L_{k+1}$, to become a sequence $B'$ between $L_{\lambda(k)-1}$ and $L_{\lambda(k)}$. The remaining way to $L'$ is the same as in the proof of Proposition (I.4.5.6), where the connectivity of $PS\mathcal{E}(g)$ itself was proved.

If $C \in PS\mathcal{E}(g)$ and $L = (L_{1},...,L_{4g};\lambda)$ is a representing configuration, then we write $L = (L_{1},...,L_{m}|L_{m+1},...,L_{4g};\lambda)$ to indicate that $L_{1},...,L_{m} \in \mathbb{H}$ and $L_{m+1},...,L_{4g} \in \mathbb{H}^{-}$. The index $m$ depends on the representative. We denote the upper and lower half of $C$ by $C_{+}$ resp. $C_{-}$.

A partitioned parallel slit domain can be taken apart: the map $[L_{1},...,L_{m}|L_{m+1},...,L_{4g};\lambda] \rightarrow [L_{1}+a,...,L_{m}+a|L_{m+1}+b,...,L_{4g}+b;\lambda]$ is continuous as long as $\text{Re}(a) = \text{Re}(b)$.

The difference between $PS\mathcal{E}(g)$ and $PS\mathcal{E}(g)$ vanishes under stabilization. But first note that $PS\mathcal{E}(g)$ is not invariant under stabilization, in fact, $PS\mathcal{E}(g)$ is mapped by $\sigma$ to the complement of $PS\mathcal{E}(g+1)$. This is overcome by another stabilization map.

(5.2.5) $s : PS\mathcal{E}(g) \longrightarrow PS\mathcal{E}(g+1)$, $s(C) = \sigma(C) + \frac{5}{2}i$.

Since $s(C) \in PS\mathcal{E}(g+1)$, $s$ factors as $j \circ S = s$, where $j$ is the inclusion $PS\mathcal{E}(g) \hookrightarrow PS\mathcal{E}(g)$.

(5.2.6) $s_{*} : PS\mathcal{E}(g) \longrightarrow PS\mathcal{E}(g+1)$ by

$$s_{*}(C) = s_{*}([L_{1},...,L_{m}|L_{m+1},...,L_{4g};\lambda]) =$$
\[ = [L_1 + 2i, \ldots, L_m + 2i, L^1, L^2, L_3, L_4, L_{m+1} - 2i, \ldots, L_{4g} - 2i; \lambda'] \]

With \( S^1 = 2i \), \( S^2 = i \), \( S^3 = -i \), \( S^4 = -2i \), and \( \lambda' \) is obtained from \( \lambda \) by inserting the pairs \((L^1, L^3)\) \((L^2, L^4)\) between \( L_m \) and \( L_{m+1} \).

(5.2.7) \hspace{1cm}

(5.2.8) **Lemma.** In the diagram

\[
\begin{array}{ccc}
\text{PS}(g) & \xrightarrow{s} & \text{PS}(g+1) \\
\downarrow & & \downarrow j \\
\text{PS}_x(g) & \xrightarrow{s_x} & \text{PS}_x(g+1)
\end{array}
\]

(i) \( s = s = j \circ S \) and

(ii) \( s_x = S \circ j \).

**Proof:** (i) is obvious from the definition (5.2.5). A homotopy from \( s \) to \( S \circ j \) consists of several moves. First the two slits \( L^1 \) and \( L^3 \) in \( s(L) \)
are moved horizontally further left than all slits $L_i$ of $\mathcal{L}$.

\[ L^1 \]
\[ L^2 \]
\[ L^3 \]
\[ L^4 \]

Then $\mathcal{L}_-$ is moved upwards over the pair $L^2, L^4$ (starting with $L_{m+1}, \ldots, L_{4g}$); thereby rescaling $\mathcal{L}_-$ to fit between $L^1$ and $L^2$.

\[ L^1 \]
\[ L^2 \]
\[ L^3 \]
\[ L^4 \]

Next note that $\mathcal{L}$ - or better say the slits $L_1, \ldots, L_m, L_{m+1}, \ldots, L_{4g}$ form a subconfiguration; therefore the vertical through $S^1$ is connected after regluing according to the interval exchange transformation induced by $\mathcal{L}$. The path from $S^1$ to the point marked $\times$ in (5.2.10) is connected, and we move $L^1$ or
$S^1$ downwards along this path; during this process $L^1$ will necessarily go through parts of $L_-$ and $L_+$, because $R$ is by (5.2.2) not a separating curve.

\begin{equation}
(5.2.11)
\begin{array}{c}
\{ L_+ \} \\
\{ L \} \\
\{ L_- \} \\
L^1 \times L^2 \\
L^3 \rightarrow L^4 \\
\rightarrow R
\end{array}
\end{equation}

Up to rescaling, this final position is $s(L)$.

We use $s_*$ to stabilize $PS\infty_*(g)$ to an infinite $PS\infty_*(\infty) = \lim_{s \to \infty} PS\infty_*(g)$. As a consequence of the lemma we note

\begin{equation}
(5.2.12) \quad \text{Proposition.} \quad PS\infty_*(\infty) \xrightarrow{\sim} PS\infty_*(\infty)
\end{equation}

is a homotopy-equivalence.

As with the stabilization we use a different sum operation for the spaces $PS\infty_*(g)$.

\begin{equation}
(5.2.13) \quad \mu^* = \mu_{g_1, g_2}^* : PS\infty_*(g_1) \times PS\infty_*(g_2) \rightarrow PS\infty_*(g_1 + g_2).
\end{equation}

The easiest way to give the definition of $\mu^*(L^1, L^2)$ is by the following figure
(5.2.15) **Proposition.** The diagram

\[
\begin{array}{ccc}
\text{PS}_\ast(g_1) \times \text{PS}_\ast(g_2) & \xrightarrow{H} & \text{PS}_\ast(g_1 \oplus g_2) \\
\downarrow j \times j & & \downarrow j \\
\text{PS}_\ast,(g_1) \times \text{PS}_\ast,(g_2) & \xrightarrow{H} & \text{PS}_\ast,(g_1 \oplus g_2) \\
\end{array}
\]

is homotopy-commutative.

**Proof:** The homotopy moves \( L_+^1, L_-^1 \) far to the left, and then weaves \( L_-^1 \) upwards through \( L_-^2 \) and \( L_+^2 \).

5.3 **The operation maps.**

We can now define the new operations

\[
(5.3.1) \quad \Theta : E^n(\mathfrak{s}) \times \text{PS}_\ast,(g)^n \xrightarrow{\Sigma_n} \text{PS}_\ast(ng).
\]
This time we will not give formulas, since they would be similar to those in chapter 3.

Let \((e_1^+, e_1^-), \ldots, (e_n^+, e_n^-)\) be a configuration of pairs, and let \(\mathcal{C}_1, \ldots, \mathcal{C}_n\) be \(n\) partitioned parallel slit domains of genus \(g_1, \ldots, g_n\). The supports of \(\mathcal{C}_k\) are divided by \(R\) into an upper and lower part, of different size in general. Let \(c^*(\mathcal{C}_k)\) denote the point \(\frac{1}{2}(a_+^*(\mathcal{C}_k) - a_-^*(\mathcal{C}_k))\) on \(R\); \(\mathcal{C}_k\) is normalized so that \(c^*(\mathcal{C}_k)\) becomes the origin, and that the support is contained in the unit square around the origin. Around the points \(e_k^+\) we attach upper or lower halves of squares, small enough to be disjoint. Then one starts to implant upper and lower halves \(\mathcal{C}_k^\pm\) of \(\mathcal{C}_k\) into these half-squares: again one must start with the utmost right vertical pair(s), and later implantation must be done relative to the reparametrized \(y\)-axis.

\[
\begin{align*}
&e_1^+ \quad \ast \quad e_3^+ \\
&\ast \quad e_2^+ \quad \ast \quad e_3^- \\
&e_1^- \quad \ast \quad e_2^- 
\end{align*}
\]

This gives first maps

\[
(5.2.3) \quad \tilde{\sigma} : \mathbb{F}^n(\mathcal{C}) \times PS\Sigma_1(g_1) \times PS\Sigma_2(g_2) \times \ldots \times PS\Sigma_n(g_n) \longrightarrow PS\Sigma(g_1 + \ldots + g_n).
\]

With \(g_1 = \ldots = g_n = g\) and using the \(\Sigma_n\)-invariance we get (5.3.1). If we restrict \(\tilde{\sigma}\) or \(\sigma\) to \(\mathbb{E}_J(\mathcal{C})\), we obtain maps.
\[ (5.3.4) \quad \tilde{\sigma} : \tilde{E}_n^\ast(c) \times \text{PSC}_c(g_1) \times \ldots \times \text{PSC}_c(g_n) \rightarrow \text{PSC}_c(g_1 + \ldots + g_n), \]

and

\[ (5.3.5) \quad \sigma : \tilde{E}_n^\ast(c) \times \prod_{\Sigma_n} \text{PSC}_c(g)^n \rightarrow \text{PSC}_c(ng). \]

5.4 Main properties.

Recall the inclusions \( \tilde{E}_n^\ast(c) \subseteq \tilde{E}_n^\ast(c) \rightarrow \tilde{C}_n^\ast(c) \). If we restrict the configurations in (5.2.3) to the subspace \( \tilde{E}_n^\ast(c) \), then the image of \( \sigma \) is contained in \( \text{PSC}_c(c) \). The operations are compatible.

(5.4.1) Proposition. For all \( n \geq 1, g \geq 0 \) the diagram

\[
\begin{array}{ccc}
\tilde{C}_n^\ast(c) \times \text{PSC}_c(g)^n & \xrightarrow{\tilde{\sigma}} & \text{PSC}_c(ng) \\
\uparrow & & \uparrow \\
\tilde{C}_n^\ast(c) \times \text{PSC}_c(g)^n & \xrightarrow{\sigma} & \text{PSC}_c(ng) \\
\downarrow & & \downarrow \\
\tilde{E}_n^\ast(c) \times \text{PSC}_c(g)^n & \xrightarrow{\tilde{\sigma}} & \text{PSC}_c(ng) \\
\downarrow & & \downarrow \\
\tilde{E}_n^\ast(c) \times \text{PSC}_c(g) & \xrightarrow{\sigma} & \text{PSC}_c(ng)
\end{array}
\]

is commutative.
For the associativity we need the following structure maps, which amount to operations themselves,

\[(5.4.2) \quad \tilde{\epsilon} : \tilde{E}_1^r(\mathcal{C}) \times \tilde{E}_1^{n_1}(\mathcal{C}) \times \cdots \times \tilde{E}_1^{n_r}(\mathcal{C}) \to \tilde{E}_1^n(\mathcal{C})\]

\[n = \Sigma n_i.\] The figure below is a better explanation than a formula.

\[(5.4.3)\]

Define inclusions \(D_1^1 : \text{PS}_1(\mathcal{G}) \to \tilde{E}_1^{n_1}(\mathcal{C}) \times \text{PS}_1(\mathcal{G})\), \(D_1^1(\mathcal{C}) = ((+i,-i),\mathcal{C})\), and \(D_2^2 : \text{PS}_2(\mathcal{G}) \to \tilde{E}_2^{n_2}(\mathcal{C}) \times \text{PS}_2(\mathcal{G})^2\), \(D_2^2(\mathcal{C}) = ((+2i,-2i),(+i,-i)), \mathcal{C}\).

After these preparations we can formulate the main result.
Theorem. There are operations

\[ \delta^n : \mathcal{E}^n(g) \times \underset{\Sigma}{\text{PS}_*} (g)^n \rightarrow \text{PS}_* (ng) \]

with the following properties.

(i) (associativity) For all \( r \geq 1, n \geq 1, g \geq 0 \), the diagrams

\[ \mathcal{E}_r^r(g) \times (\mathcal{E}_r^n(g) \times \text{PS}_* (g)^n)^r \rightarrow \mathcal{E}_r^r(g) \times \text{PS}_* (ng)^r \]

\[ \text{id} \times (\delta^n)^r \]

\[ \mathcal{E}_r^{rn}(g) \times \text{PS}_* (g)^{rn} \rightarrow \text{PS}_* (rng) \]

(ii) (unit) The composition

\[ \text{PS}_* (g) \xrightarrow{D^1} \mathcal{E}^1(g) \times \text{PS}_* (g) \xrightarrow{\delta^1} \text{PS}_* (g) \]

is homotopic to the identity.

(iii) (squearing) The composition

\[ \text{PS}_* (g) \xrightarrow{D^2} \mathcal{E}^2(g) \times \text{PS}_* (g) \xrightarrow{\delta^2} \text{PS}_* (2g) \]

is homotopic to \( \mu_{g,g} \circ \text{diag} \).

As with the Dyer-Lashof operations we appeal to the figure (5.4.3) for a proof of (i). The other two assertions are clear.
(5.4.5) **Proposition.** The diagram

\[
\begin{array}{ccc}
\text{PSG}(g_1) \times \text{PSG}(g_2) & \xrightarrow{\tau} & \text{PSG}(g_1 + g_2) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \\
\text{PSG}(g_2) \times \text{PSG}(g_1) & & \\
\end{array}
\]

is homotopy-commutative, where \( \tau(L_1, L_2) = (L_2, L_1) \).

**Proof:** The figure

(5.4.6)

\[
\begin{array}{c}
\text{---}
\end{array}
\]

shows a homotopy from \( g_1^2 \) \( g_1 \cdot g_2 \) \( g_2 \cdot g_1 \) \( \tau \). And \( \mu^1 \) \( g_1, g_2 \) is homotopic to

\[(L_1, L_2) \rightarrow g_1^2 \quad ((+2i, -2i), (+i, -i), L_1, L_2).\]
5.5 Double braid groups.

Recall that $\tilde{E}_n(C)$ and $E_n(C)$ are Eilenberg-MacLane spaces $K(\pi, 1)$; we may call their fundamental groups double braid groups. By (5.4.2) they come equipped with operations, and - as in (3.7.4) - with homomorphisms $\pi_1 E_n(C) \rightarrow \pi_1 \text{PSL}(g)$, making the operations compatible. But we have not investigated the groups involved.
Chapter 6

The Homology of

Symplectic Operations

6.1 Definition of the symplectic operations $Q'_1$ and $Q''_1$.
6.2 The operations $R'_1$ and $R''_1$.
6.3 Some formulas.

The symplectic operation maps will allow to define two more homology operations $Q'_1$, $Q''_1$ of degree 1. There are two binary homology operations $R'_1$, $R''_1$ measuring the deviation from linearity. There are similar formulas as for the Dyer-Lashof operations.
6.1 Definition of the symplectic operations $Q_1'$ and $Q_1''$.

For the symplectic operations we restrict ourselves to mod-2 homology $H_\ast(\mathcal{C}) = H_\ast(\mathcal{C};\mathbb{F}_2)$. To define them we need 1-chains $\tilde{\omega}_k$ in $\tilde{E}^2(\mathcal{C})$ which are already in $\tilde{E}^2(\mathcal{C})$ cycles, or which are cycles $\omega_k$ when projected to $E^2(\mathcal{C}) = \tilde{E}^2(\mathcal{C})/\Sigma_2$.

From Lemma (5.1.5) and (5.1.7) we see that $\tilde{E}^2(\mathcal{C})$ is a bouquet of 5 circles. The best way to see them is as the union of 6 intervals, corresponding to the following 6 paths in $\tilde{E}^2(\mathcal{C})$.

(6.1.1)

\begin{center}
\includegraphics[width=0.8\textwidth]{6.1.1.png}
\end{center}

(6.1.2)

\begin{center}
\includegraphics[width=0.8\textwidth]{6.1.2.png}
\end{center}

The $\Sigma_2$-action on $\tilde{E}^2(\mathcal{C})$ is then the antipodal map, mapping $\tilde{\omega}_k$ to $(\tilde{\omega}_{2-k})^{-1}$ (as an element of the fundamental groupoid) are inverse to each other. Thus the chains $\tilde{\omega}_1$ and $\tilde{\omega}_5$, resp. $\tilde{\omega}_1$ and $\tilde{\omega}_4$, resp. $\tilde{\omega}_2$ and $\tilde{\omega}_3$ project to the same cycle $\omega_1$, resp. $\omega_2$, resp. $\omega_3$ in $E^2(\mathcal{C})$. $\tilde{\omega}_1$ comes from $\tilde{c}^2(\mathcal{C}) \xrightarrow{2} \tilde{E}^2(\mathcal{C})$, i.e. $\iota_4(\tilde{\omega}_1) = \tilde{\omega}_1$ (see (4.1), (3. .)). And $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are in
Therefore we only define with \( \tilde{w}_2 \) and \( \tilde{w}_3 \) two new operations of degree one:

\[
Q'_1, Q''_1 : H_{PSG(g)} \rightarrow H_{2q+1}PSG(2g),
\]

\[
Q'_1(x) = g^{a} \tilde{w}_2 \ast x \ast x \quad \text{and}
\]

\[
Q''_1(x) = g^{a} \tilde{w}_3 \ast x \ast x .
\]

The schematic picture is as follows.

\[
(6.1.4)
\]
6.2 The operations $R'_1$ and $R''_1$.

The chains $\tilde{w}_{ij} = \tilde{w}_i + \tilde{w}_j$ are cycles in $E^2(\xi)$; their homology classes are subject to relations $[\tilde{w}_{ii}] = 0$, $[\tilde{w}_{ij}] = [\tilde{w}_{ji}]$, $[\tilde{w}_{ii}] + [\tilde{w}_{jk}] = [\tilde{w}_{ik}]$. As a basis one can choose $\tilde{w}_{ik}$ for $i = 2, \ldots, 6$. The class $\tilde{w}_{i6}$ comes from $C^2(\xi) \to E^2(\xi)$, and was denoted by $\tilde{v}$ in 4.2. We define binary operations

$$R_{ij} : H^{p, q}_{PSC}(g_1) \otimes H^{p, q}_{PSC}(g_2) \to H^{p+q+1}_{PSC}(g_1 + g_2),$$

$$R_{ij}(x, y) = [\tilde{g}_1, \tilde{g}_2 \cdot [\tilde{w}_{ij} \cdot x \cdot y]] \quad (1 \leq i, j \leq 6).$$

There are relations

$$R_{ii} = 0, \quad R_{ij} = R_{ji}, \quad R_{ij} + R_{jk} = R_{ik}.$$  

We need especially $R'_1 := R_{25}$ and $R''_1 := R_{34}$; they will play the role of symplectic Browder operations.

$$R'_1(x, y) \quad R''_1(x, y)$$
6.3 Some formulas.

The operations $Q'_1$, $Q''_1$ are merely functions on homology classes in $H_2 \mathbb{PS} \mathbb{C}$. The formulas

\begin{align*}
(6.3.1) \text{ (zero)} & \quad Q'_1(0) = Q''_1(0) = 0 \\
(6.3.2) \text{ (unit)} & \quad Q'_1(1) = Q''_1(1) = 0 \quad , \quad 1 \in H_0 \mathbb{PS} \mathbb{C}(0) ,
\end{align*}

follow directly from the definitions. For

\begin{align*}
(6.3.3) \text{ (linearity)} & \quad Q'_1(x+y) = Q'_1(x) + R'_1(x,y) + Q'_1(y) \\
& \quad Q''_1(x+y) = Q''_1(x) + R''_1(x,y) + Q''_1(y) \\
(6.3.4) \text{ (Cartan formula)} & \quad Q'_1(xy) = x^2 Q'_1(y) + x R'_1(x,y) y + Q'_1(x) y^2 \\
& \quad Q''_1(xy) = x^2 Q''_1(y) + x R''_1(x,y) y + Q''_1(x) y^2
\end{align*}

one considers the following figure, analogous to (4.4.3).
There are many more formulas of this kind, involving one or several of the operations $Q'_1$, $Q''_1$, $R'_1$ and $R''_1$. To connect them to the operation $Q_1$, we mention

\begin{equation}
Q_1(x) = Q'_1(x) + R_{12}(x, x)
\end{equation}

\begin{equation}
= Q''_1(x) + R_{13}(x, x) \quad \text{in } H_\times PSG(g),
\end{equation}

which are derived from $[\bar{w}_1] = [\bar{w}_2] + [\bar{w}_1] = [\bar{w}_3] + [\bar{w}_1]$. 

(6.3.6)
Chapter 7

The Geometry of

Higher Genus Operations

7.1 Operations from a single surface.
7.2 The universal surface bundle over $\mathbb{PS}(g)$.
7.3 Operations from families of surfaces.

The operations considered so far had as parameter spaces certain configuration spaces of the complex plane. We will now describe some operations parametrized by configuration spaces of surfaces. This enables us to use the homology of these spaces to induce homology operations; e.g. operations of degree 2.

In a second step we generalize this and the conformal class of the surface. Then the parameter space for operations is a fibrewise configuration space of the universal surface bundle over the moduli space itself.
7.1 Dyer-Lashof operations from a single surface.

Let \( \mathcal{L}^0 = [L^0] = [L_1^0, ..., L_{4g_0}^0 ; \lambda^0] \) be a fixed parallel slit domain of genus \( g_0 \). Away from the dipole \( P = \infty \) and the stagnation points \( S_1^0, ..., S_{4g_0}^0 \) the associated surface \( F(\mathcal{L}^0) \) has canonical \( x,y \)-coordinate charts. The deleted surface

\[
F^-(\mathcal{L}^0) = F(\mathcal{L}^0) - \{s_1^0, ..., s_{4g_0}^0 \}
\]

where \( s_i \) denotes the image of \( S_i^0 \) in \( F(\mathcal{L}) \), has \( m = m(\mathcal{L}^0) \) punctures; \( m \) can vary from 2 to \( 2g_0 + 1 \). The configuration spaces

\[
\begin{align*}
\overline{c^n}(\mathcal{L}^0) &= \overline{c^n}(F^-(\mathcal{L}^0)) \\
c^n(\mathcal{L}^0) &= \overline{c^n}(F^-(\mathcal{L}^0)) / \Sigma_n
\end{align*}
\]

depend on \( g_0 \) and \( m \). They are Eilenberg-MacLane-spaces by [Fadell-Neuwirth 1962; p.113], since the underlying surface is never closed. Their fundamental groups are sometimes called braid groups of the underlying manifold; presentations are given in [Scott 1970].

There is a new distance function \( \epsilon : c^n(\mathcal{L}^0) \rightarrow [0, \infty[ \) induced by the maximal-coordinate norm where this time we take the punctures into account,

\[
\epsilon(\{e_1, ..., e_n\}) = \min \{ \|z - z'\| \mid z, z' \in \{e_1, ..., e_n, s_1^0, ..., s_{4g_0}^0\}, z \neq z' \}
\]

It is now clear what an \( \epsilon \)-square \( B_\epsilon(e_i) \) around \( e_i \) means. Also the notion of an extended \( \epsilon \)-square \( \hat{B}_\epsilon(e_i) \) and the order relation \( < \) of (3.5.2) extends to the new situation.
To define the Dyer-Lashof maps in this situation

\[(7.1.4) \quad \tilde{g} = g^n : C^n(L^0) \times PS\Sigma(g)^n \rightarrow PS\Sigma(g_0 + ng) \]

we proceed exactly as in (3.5): only we assume that \( L^0 \) has been implanted already (in its original position and size) as a first implantation. This defines maps

\[(7.1.5) \quad \tilde{g} = \tilde{g}_{g_1, \ldots, g_n} : C^n(L^0) \times PS\Sigma(g_1) \times \cdots \times PS\Sigma(g_n) \rightarrow PS\Sigma(g_0 + g_1 + \ldots + g_n) \]

which give by \( \Sigma_n \)-invariance (7.1.4) when \( g_1 = \ldots g_n = g \).

\[(7.1.6) \]

If \( g_0 = 0 \) and therefore \( L^0 = [\emptyset] \), then \( F^-([\emptyset]) = L \), and we are in the old situation.

The associativity relations require to let the conformal structure of \( L^0 \) vary as an additional parameter. Before doing so we briefly treat the symplectic operations. For the symplectic operations we used the notion of a vertical pair. This is still possible on any surface \( F^-(L^0) \) by using the harmonic function \( h_L \) (which is just the \( x \)-coordinate). Thus we can
define $\overline{E}^n(\mathcal{L}^0)$ for $\mathcal{L}^0 \in PSC_\ast(g)$. The condition (5.1.2) (3'), defining the subspace $\overline{E}^n(\mathcal{L}^0)$, is now to be interpreted that no point $e^+_i$ lies on $\mathbb{R}$; that does not exclude a point moving from one half-plane to the other by crossing a bridging pair of slits. The distance function and the notion of an (extended) square also carries over. Proceeding as usual we find

operation maps

$$\begin{align*}
\mathcal{L}^0 \in PSC_\ast(g) & \quad \overline{E}^n(\mathcal{L}^0) \times PSC_\ast(g)^n \quad \overline{E}^n(\mathcal{L}^0) \times PSC_\ast(g)^n \quad PSC_\ast(g_0 + ng).
\end{align*}$$

7.2 The universal surface bundle over $PSC(g)$.

The space $PSC(g)$ supports - as the moduli space of directed Riemann surfaces - a universal surface bundle $FSC(g_0)$. The easiest definition is

$$\begin{align*}
EDiff^+(F_g, x) \times F_g & \quad BDiff^+(F_g, x) = BDiff^+(F_g, x),
\end{align*}$$

useful for our purposes. As a set,

$$FSC(g) = \bigcup_{\mathcal{L} \in PSC(g)} F(\mathcal{L}).$$

To give the topology, let $\tilde{\mathcal{C}}_0, \ldots, \tilde{\mathcal{C}}_4 g$ be $4g+1$ disjoint copies of $\tilde{\mathcal{C}} = \mathbb{C} \cup \infty$. A point in $\tilde{\mathcal{C}}_m$ will be denoted by $(z, m)$. Recall the space $RegConf(g)$ from (I.4.5.1) the gluing rules (I.4.2.3) and the crossings (I.4.3.2). In $RegConf(g) \times (\tilde{\mathcal{C}}_0 \cup \ldots \cup \tilde{\mathcal{C}}_4 g)$ we consider the subspace given by all $(L_1 z, m) = (L_1, \ldots, L_{4g}; \lambda; z, m)$ such that

$$\text{Im}(S_m) \ni \text{Im}(z) \ni \text{Im}(S_{m+1}).$$
In terms of the gluing process for $F(L)$, $z$ lies in the $m$-th strip $F^m$ of
$L$; see (I.4.2.1). For $m = 0$ resp. $4g$ the left resp. right condition is
to be disregarded. On this space we consider the equivalence relation $\equiv$ gen-
erated simultaneously by crossings and gluings:

(7.2.3)

$(1)$ if $(z,m) \sim (z',m')$ in $L$, then $(L;z,m) \equiv (L;z',m')$;

$(2)$ assume $L = L'$ by a crossing of $L_{k-1}$ over the pair $L_k, L_{\lambda(k)}$
with $k < \lambda(k)$; set $h_k = \text{Im}(S_{\lambda(k)} - S_k)$;

$(2a)$ if $z \notin L_{j-1}, L_k, L_{\lambda(k)}$, then $(L;z,m) \equiv (L';z,\rho(m))$ where $\rho$
is the partial cyclic permutation $\rho = (\lambda(k) \ldots k - 1)$;

$(2b)$ if $z \in L_{k-1} \subseteq L_k$ and $m = k - 2$, then $(L;z,k-2) \equiv (L';z,k-2) \equiv$
$(L';z+h_k;\lambda(k)-1)$;

$(2c)$ if $z \in L_{k-1} \subseteq L_k$ and $m = k - 1$, then $(L;z,k-1) \equiv (L';z+h_k,\lambda(k))$

$(2d)$ if $z \in L_{\lambda(k)}$ and $m = \lambda(k)$, then $(L;z,\lambda(k)) \equiv (L';z,\lambda(k))$.

(7.2.4)
The map \( \pi : \text{PS}(g) \to \text{PS}(g) \), \( \pi[L;z,m] = [L] \) is well-defined (and therefore continuous), because the crossings influence the gluings, but not vice versa.

(7.2.6) Proposition. \( \pi \) is a locally trivial fibre bundle with fibres homeomorphic to a surface of genus \( g \).

Proof: We use the components of \( \text{RegConf}(g) \) (or rather their image in \( \text{PS}(g) \)) as parts over which \( \pi \) certainly is a product. Each such component is determined by a pairing function \( \lambda \in \Sigma_{4g} \) which is connected in the sense of (I:4.4.6). Note that these components \( R^\lambda \) are not open (in \( \text{PS}(g) \)), the boundary consists of such configurations \( L = (L_1, \ldots, L_{4g}; \lambda) \) with \( L_{k-1} \leq L_k \), or \( L_{k+1} \leq L_k \) for some \( k \). In \( R^\lambda \) choose as a basepoint the configuration \( L' \) with \( S'_k = -ki \). We contract \( R^\lambda \) to \( L' \) by the following homotopy: a configuration \( L \) is translated downwards below the horizontal \( \text{Im}(h) = -(4g+1) \); then the slit endpoints \( S_k \) are moved along a straight line to \( S'_k = -ki \), one after the other, starting with \( S_1 \). Note that \( L \) stays in the component \( R^\lambda \) all the time, and that \( L \) stays regular (since no forbidden subconfigurations are created). Let \( F' = F(L') \) be the fibre over \( L' \). The contraction just constructed lifts to a (tautological) deformation of \( \pi^{-1}(R^\lambda) \): a point \( (L;z,m) \) over \( L \) first follows the translation downwards, then remains constant while the first \( m \) slits move to their final position, and finally moves upwards with \( L_{m+1} \), then remains constant again. From fibre to fibre, these are piecewise affine homeomorphisms. If we make the provision that \( z \) remains for all time a point on \( L_{m+1} \) if it was so at the beginning, then these homotopies extend over a neighbourhood of the boundary of \( R^\lambda \).
7.3 Operations from families of surfaces.

In each fibre $F(L) \subseteq \text{FSG}(g)$ the set $\mathcal{S}(L)$ of all stagnation points is the image of the set of slit endpoints $S_i$ ($i = 1, \ldots, 4g$) under the gluing identifications. If counted with multiplicities, $\mathcal{S}(L)$ has $2g$ elements. Furthermore, $\mathcal{S}(L)$ varies continuously with $L$, i.e., there is a section $\mathcal{S} : \text{FSG}(g) \to \text{SP}_{2g}(\pi)$ into the $2g$-fold fibrewise symmetric product of $\text{FSG}(g)$. More precisely, define $\overline{\text{SP}}_{2g}(\pi) = \{ (W_1, \ldots, W_{2g}) \in \text{FSG}(g)^{2g} \mid \pi(W_1) = \cdots = \pi(W_{2g}) \}$, and set $\text{SP}_{2g}(\pi) = \overline{\text{SP}}(\pi)/\Sigma_{2g}$. The cardinality of $\mathcal{S}(L)$ varies between 1 and $2^g$; thus the image of $\mathcal{S}$ - taken as a union of the sets $\mathcal{S}(L)$ - is a finite, branched covering over $\text{FSG}(g)$.

Similarly, let $\overline{\mathcal{S}}(L) = \mathcal{S}(L) \cup \infty_L$ define an element in $\text{SP}_{2g+1}(F(L)) \subseteq \text{SP}_{2g+1}(\pi)$. We are interested in the complement $F^{-}(L) = \overline{\mathcal{S}}(L)$. Denote by

$$F^{-}(g) = FSG(g) - \bigcup_{L} \mathcal{S}(L) = \bigcup_{L} F^{-}(L)$$

the complement of all stagnation points, and the restriction of $\pi$ to $F^{-}(g)$ is denoted by $\pi^{-}$. Although it is not a fibration, we can form the fibrewise configuration space

$$F^{-}(g) = \{ (W_1, \ldots, W_n) \in \text{FSG}(g)^{n} \mid W_i \neq W_j \text{ for } i \neq j , \pi^{-}(W_1) = \cdots = \pi^{-}(W_n) \}$$

The $W_i = [L_i, z^i, m^i]$ in such a configuration must have the same $[L_i] = [L]$; so we can write $[L,(z^1,m^1),\ldots,(z^n,m^n)]$. We set

$$\mathcal{C}^n(g) = \mathcal{C}^n(g)/\Sigma_n$$
Both spaces are spaces over $\underline{\text{PSG}}(g)$,

\[
\begin{array}{ccc}
\mathring{\mathcal{C}}^n(g) & \to & \mathcal{C}^n(g) \\
\text{PSG}(g) & \downarrow & \text{PSG}(g) \\
\end{array}
\]

with configuration spaces $\mathring{\mathcal{C}}^n(F^-(\mathcal{L}))$ of deleted surfaces as fibres. For each $n$ there are sections

\[
(7.3.5) \quad \mathring{\mathcal{C}}^n : \text{PSG}(g) \to \mathring{\mathcal{C}}^n(g),
\]

with $z^k = (a_+(L)) + i(b_+(L)+1)$.

The distance function $\varepsilon : \mathcal{C}^n(g) \to ]0,\infty[ \cup \{0\}$ is determined by the distance functions in the fibres $\mathcal{C}^n(F^-(\mathcal{L}))$.

Our aim are maps

\[
(7.3.6) \quad \tilde{s} : \mathring{\mathcal{C}}^n(g_0) \times \text{PSG}(g_1) \times \ldots \times \text{PSG}(g_n) \to \text{PSG}(g_0 + g_1 + \ldots + g_n),
\]

and

\[
\tilde{\theta} : \mathring{\mathcal{C}}^n(g_0) \times \text{PSG}(g)^n \to \text{PSG}(g_0 + ng),
\]

for any $n \geq 1$ and $g_1, g \geq 0$. Again we will not give all details. The left sides are spaces over $\text{PSG}(g_0)$; the fibre over $L^0$ is $\mathring{\mathcal{C}}^n(L^0) \times \text{PSG}(g_n)$. On this space the map $\tilde{s}$ was given in (7.1.4). So $\tilde{s}$ is defined on the entire space. The continuity follows if $\tilde{s}$ is well-defined with respect to
the equivalence classes \( L^0 \); we omit this here.

To formulate the associativity let

\[
(7.3.7) \quad \tilde{c} : \tilde{C}(h_0) \times \tilde{C}(h_1) \times \ldots \times \tilde{C}(h_r) \longrightarrow \tilde{C}(h)
\]

with \( n = n_1 + \ldots + n_r \) and \( h = h_0 + h_1 + \ldots + h_r \), implant \( r \) configurations of respective lengths \( n_i \), lying in some parallel slit domains \( \mathcal{L}^i \) of genus \( h_i \), into squares around \( r \) points, lying in a parallel slit domain of genus \( h_0 \).

\[
(7.3.8) \quad \text{associativity} \quad \text{For } r, n \geq 1, h_0, g_0, g \geq 0 \quad \text{the following diagram is commutative.}
\]

\[
\begin{array}{ccccccc}
\tilde{C}(h_0) \times (\tilde{C}(g_0))^r & \xrightarrow{\Sigma_r} & \tilde{C}(h_0) \times \text{PSL}(g_0 + ng) \\
\Sigma_n & \xrightarrow{\Sigma_r} & \tilde{C}(h_0 + r g_0) \times \text{PSL}(g_0 + r n g) \\
\end{array}
\]

All Dyer-Lashof operations constructed are special cases of these last ones.

For the symplectic operations we proceed similarly. One defines the parameter spaces \( \tilde{F}^n(g) \) as subspaces of \( \tilde{C}^{2n}(g) \), \( \tilde{F}^n(g) = \bigcup \tilde{F}^n(\mathcal{L}) \), \( \mathcal{L} \in \text{PSL}(g) \).

then there are operations

\[
(7.3.9) \quad \tilde{F}^n(g_0) \times \text{PSL}(g_0)^n \longrightarrow \text{PSL}(g_0 + ng).
\]
Chapter 8

The Homology of Higher Genus Operations

8.1 Some examples of higher operations.

The parameter spaces $C^n(g)$ for the higher operations have a rich homological structure. For example, it allows to define operations of degree 2 and higher. We confine ourselves to some examples, since we have so far no systematic treatment for these homology operations.
8.1 Some examples of higher operations.

About the homology of the higher genus operations we do have very little systematic knowledge so far. But we can give some justification and examples.

We remark first that the homology $H_2 C^2(\text{C})$ of configuration spaces of (deleted) surfaces $F^-(\text{C})$ is known by [Bödigheimer-Cohen-Taylor 1989], [Bödigheimer-Cohen 1988] and [Bödigheimer-Cohen-Milgram 1989]. The coefficients are in a finite field $\xi$ or in $\mathbb{Q}$. For operations in mod-2 homology one uses $C^2(\text{C})$ as a parameter space. If the genus $g_0$ of $\text{C}$ is positive, then there are several two-dimensional classes, in contrast to $C^2(\xi)$ and $E^2(\xi)$. In fact the rank of $H_2 C^2(\text{C})$ grows quadratically in $g_0$. These classes induce operations of degree two.

Note furthermore, that the parameter spaces $C^2(g_0)$ have non-trivial homology; since there are sections to the map $C^2(g_0) \to \text{PSG}(g_0^*)$, the homology of $\text{PSG}(g_0^*)$ is a direct summand of the homology of $C^2(g_0^*)$.

We finish with some examples.

(1) Let $\text{C} \in \text{PSG}(g_0^*)$ be fixed and consider the curve $w$ in $C^1(\text{C})$ given in figure (8.1.1). If $\theta$ is the operation $C^1(\text{C}) \times \text{PSG}(g) \to \text{PSG}(g_0 + g)$, the induced homology operation $\theta_1 : H_q \text{PSG}(g) \to H_{q+1} \text{PSG}(g_0 + g)$, $\theta_1(x) = \theta_\alpha[w \otimes x]$, is depicted in the same figure. We clearly have $\theta_1(x) = R_1(1_{g_0}, x)$ for $1_{g_0} \in H_0 \text{PSG}(g_0^*)$.

(8.1.1)
(2) For the same $\mathcal{L}$, take $w'$ to be the curve in $\mathcal{C}^2(\mathcal{L})$ shown in (8.1.2). Then $\theta : \mathcal{C}^2(\mathcal{L}) \times \text{PSG}(g) \rightarrow \Sigma_2 \text{PSG}(g_0 \times 2g)$ induces $\theta_1 : H_\text{PSG}(g) \rightarrow H_{2q+1}\text{PSG}(g_0 + g)$, $\theta_2(x) = \theta_1[w' \otimes x \otimes x]$. This time we find $\theta_2(x) = Q_1(x) + R_1(1_{g_0}, x)$.

(8.1.2)

(3) Assume $u \in H_p\text{PSG}(g_0)$ is some homology class. (8.1.3) gives a $(p+1)$-dimensional class $u'$ in $\mathcal{C}^1(g_0)$. The operation $\theta_3(x) = \theta_2[u' \otimes x]$ induced by $\theta : \mathcal{C}^1(g_0) \times \text{PSG}(g_0) \rightarrow \Sigma_2 \text{PSG}(g_0 + g)$ has degree $p+1$, $\theta_3(x) \in H_{q+p+1}\text{PSG}(g_0 + g)$ for $x \in H_q\text{PSG}(g)$. $\theta_3$ is not new, since $\theta_3(x) = R_1(u, x)$.

(8.1.3)
(4) The next figure then shows an operation $\Theta_4(x) = x\Theta_3(x) + u\Theta_1(x)$.

(8.1.4)

(5) Let $L \in \text{PS\ell}(g_o)$ be fixed, and let $v$ be a closed, non-trivial curve in $F(L)$, as e.g. in (8.1.5). The induced operation $\Theta_5 : H_q\text{PS\ell}(g) \to H_{q+1}\text{PS\ell}(g_o+g)$ is the first new one.

(8.1.5)
(6) With everything as above, we can also have a \( \Theta_6 : H_{q} \text{PSG}(g) \rightarrow H_{2q+1} \text{PSG}(2g) \).

(8.1.6)

(7) An operation of degree 2 can be constructed as follows. Let \( a \) be the curve in \( \text{PSG}(1) \) given in (8.1.7). It represents a homology class \([a] \in H_1 \text{PSG}(1)\). Together with the points \( e_1, e_2 \) moving in the varying parallel slit domain, we have a torus embedded in \( C^2(1) \). The operation \( \theta : C^2(1) \times \Sigma_2 \text{PSG}(g)^2 \rightarrow \text{PSG}(2g+1) \) induces \( \theta_7(x) = a'[a \circ x \circ x] \), \( \theta_7 : H_q \text{PSG}(g) \rightarrow H_{2q+2} \text{PSG}(2g+1) \), where \( a' \) is the chain in \( C^2(1) \) given geometrically by \((a(t), (e_1(t'), e_2(t')))\).

(8.1.7)
(8) The last figure shows a degree 3 operation \( \theta_3 : H_{\Psi G(q)} \rightarrow H_{2q+3} \Psi G(2g+2) \), parametrized by an embedded 3-manifold in \( C^2(2) \), which projects to a Klein-bottle embedded in \( \Psi G(2) \).

\[ (8.1.8) \]
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