ON THE TOPOLOGY OF MODULI SPACES OF RIEMANN SURFACES

Part II

Homology Operations

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Introduction

This is the second article in a longer series, intended to investigate the topology of the moduli spaces of Riemann surfaces. Using the description of the moduli space as a kind of configuration space we will develop operations for the homology of these spaces.

In Part I "Hilbert Uniformization" we studied the moduli space $\vec{\mathfrak{M}}(g)$ of directed Riemann surfaces, i.e. the space of conformal equivalence classes $[F,x]$ of closed Riemann surfaces F of genus g and given direction x at some point $P \in F$. This space has the homotopy type of the classifying space $B\vec{\Gamma}(g)$ of the corresponding mapping class group $\vec{\Gamma}(g) = \pi_0 \mathrm{Diff}^+(F,x)$; $\mathrm{Diff}^+(F,x)$ is the group of orientation-preserving diffeomorphisms of F which keep x fixed. $\vec{\Gamma}(g)$ is isomorphic to the mapping class group $\vec{\Gamma}(g,1)$ of a genus g surface with one boundary component.

The main objective of Part I was a new description of $\vec{\mathfrak{M}}(g)$ - based on old ideas of geometric function theory; a homotopy equivalence between $\vec{\mathfrak{M}}(g)$ and a space $PS\mathbb{C}(g)$ of so called parallel slit domains was established. A point in $PS\mathbb{C}(g)$ is an equivalence class $\mathfrak{L} = [L]$ of a configuration $L = (L_1,\dots,L_{4g};\lambda)$ consisting of semi-infinite horizontal slits L_i in $\mathbb{C}$, paired by an involution $\lambda \in \Sigma_{4g}$. Such configurations are subject to some regularity conditions; and the equivalence relation allows certain jumps of slits across longer pairs. The group $\mathrm{Sim}(\mathbb{C})$ of dilatations and translations of $\mathbb{C}$ acts freely on $PS\mathbb{C}(g)$. And the orbit space $PS\mathbb{C}(g)/\mathrm{Sim}(\mathbb{C})$ was proved to be homeomorphic to $\vec{\mathfrak{M}}(g)$.

This parametrization of $\vec{\mathfrak{M}}(g)$ shows the distinctive characteristics of a configuration space: a point $\mathfrak{L}$ comprises two kinds of finite data, geometric and combinatoric, namely the position of the slits and their ordering and pairing. From the viewpoint of moduli: the conformal structure of $[F,x]$ i screened against the complex plane, and the picture so obtained is containe in an essentially finite piece of the plane. Whenever such a situation occur. one is tempted to patch several such pictures together. This method has a l history in homotopy theory under the name Dyer-Lashof operations, where it w used with great success to study iterated loop spaces, function spaces or classifying spaces of certain groups. In this article we will introduce such patchings systematically for the moduli spaces, taking the spaces $PS\mathbb{C}(g)$ a substitutes.

The aim is to construct operations

$$\vartheta : \tilde{X}(n,g_o) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(g_o+ng)$$

for various spaces $\tilde{X}(n,g_o)$ which depend on $n \geq 1$, $g_o \geq 0$ and have a fre Σ_n-action. These parameter spaces for the operations will be configuration spaces of points, pairs of points, etc., in the plane, in a surface, or in the universal surface bundle over the moduli space. A homology class $\alpha \in H_q(\tilde{X}(n,g_o)/\Sigma_n)$ induces a homology operation

$$\Theta_\alpha : H_* PS\mathbb{C}(g) \longrightarrow H_{n*+q} PS\mathbb{C}(g_o+ng) \quad .$$

We call g_o the genus and q the degree of Θ_α . Homology operations should be useful first to create new homology classes from old ones, and then also to organize the homology of the moduli spaces.

These are several interesting subspaces of $PS\mathbb{C}(g)$, some of which admit operations of a more intricate nature then the Dyer-Lashof operations above. In particular, we use the subspace $PS\mathbb{C}_{\bullet}(g)$ of parallel slit domains which are decomposable into a part in the upper resp. lower half-plane. Such partitione parallel slit domains allow the two parts to be implanted into a vertical pair of patches in $\mathbb{C}$. The new configuration spaces are configuration spaces of vertical pairs of points in the plane, in a surface, or in the universal surfac bundle over the moduli space. We call these operations symplectic.

Plan of the chapters.

Part II consists of 8 chapters, which are paired: one chapter develops the geometric side of some kind of operations, the other the homological side. Chapter is confined to the most basic of all operations, the sum operation, which implan two parallel slit domains into $\mathbb{C}$, the first into the upper, the second into the lower half-plane. This operation

$$\mu : PS\mathbb{C}(g_1) \times PS\mathbb{C}(g_2) \longrightarrow PS\mathbb{C}(g_1+g_2)$$

makes $PS\mathbb{C} = \coprod PS\mathbb{C}(g)$ an H-space. A special case is the stabilization map

$$\sigma : PS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g+1) , \quad \sigma(\mathcal{L}) = \mu(\mathcal{L},\mathcal{L}^{(1)}) ,$$

for $\mathcal{L}^{(1)}$ a fixed surface of genus 1 . In chapter 2 the homology $H_*(PS\mathbb{C};A)$, A a commutative ring with unit, is considered as a ring, a so called Pontrjagin ring, with multiplication induced by μ . We mention Harer's stability theorem for $H_*(PS\mathbb{C};\mathbb{Z})$ and the polynomial subalgebra in $H_*(PS\mathbb{C};\mathbb{Q})$ found by Miller as the best global results on this ring so far. In chapter 3 we see the first proper results on this ring so far. In chapter 3 we see the first proper operations as Dyer-Lashof maps

$$\tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(ng)$$

where $\tilde{C}^n(\mathbb{C})$ is the ordered configuration space of the plane $\mathbb{C}$. The existenc of such operations was noticed by several authors, in particular [Miller 1986], [Cohen 1987], [Maginnis 1987], described rather on the group level than on the level of moduli spaces. The definition of ϑ requires some technical preparations; we give full details here to save work in similar situations to come. Theorem (3.6.2) summarizes the main properties of these operations. In chaper we study the operations in mod-2 homology $H_*(PS\mathbb{C};\mathbb{Z}_2)$ induced by $\vartheta : \tilde{C}^2(\mathbb{C}) \times_{\Sigma_2} PS\mathbb{C}(g)^2 \longrightarrow PS\mathbb{C}(2g)$. The generator in $H_o(C^2(\mathbb{C});\mathbb{Z}_2)$ induces the squaring operation $Q_o : H_p(PS\mathbb{C}(g);\mathbb{Z}_2) \longrightarrow H_{2p}(PS\mathbb{C}(2g);\mathbb{Z}_2)$, $Q_o(x) = x \# x = x^2$; the generator in $H_1(C^2(\mathbb{C});\mathbb{Z}_2)$ induces a Dyer-Lashof operation $Q_1 : H_p(PS\mathbb{C}(g);\mathbb{Z}_2)$ binary operations, the generator in $H_o(\tilde{C}^2(\mathbb{C});\mathbb{Z}_2)$ induces the ring multiplication, and the generator in $H_1(\tilde{C}^2(\mathbb{C});\mathbb{Z}_2)$ induces a Browder operation R_1 : $H_p(PS\mathbb{C}(g_1);\mathbb{Z}_2) \otimes H_q(PS\mathbb{C}(g_2);\mathbb{Z}_2) \longrightarrow H_{p+q+1}(PS\mathbb{C}(g_1+g_2);\mathbb{Z}_2)$. The theory of such operations is well-developed. We give some formulas for Q_o, Q_1 and R_1. In chapter 5 and chapter 6 we introduce the symplectic operations. They are defined for the subspace $PS\mathbb{C}_{\cdot}(g)$, but stably $PS\mathbb{C}_{\cdot}(\infty) \simeq PS\mathbb{C}(\infty)$. Since these operations take the finer slit structure of a surface (and not only the handle structure) into account, we obtain two new operations Q_1', Q_1'' of degree 1. In the final chapters 7 and 8 we briefly present two generalizations of the operations introduced so far. First we replace the complex plane $\mathbb{C}$ by any fixed, (punctured) parallel slit domain $\mathfrak{L}^o$; i.e. we use configurations of points (resp. vertical pairs of points) in $\mathfrak{L}^o$ to parametrize implantations of other (partitioned) parallel slit domains. Since configuration spaces of surfaces have homology classes in dimensions greater than one, we obtain operations of degree greater than one, which are not products of degree one operations. Having gone so far, the next step of generalization consists of varying the parallel slit domain $\mathfrak{L}^o$ itself. The new parameter spaces are fibrewise configuration spaces of the universal surface bundle over $PS\mathbb{C}(g)$. We only give some examples.

References.

References to Part I are denoted by (I.x.y.z), and within Part II by (x.y.z). All other literature is quoted as [name, year] and listed on the pages 84-85; Part I appears as [Bödigheimer 1990].

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Chapter 1

Basic Operations

Replacing the moduli space $\vec{\mathfrak{M}}(g)$ of conformal equivalence classes of directed Riemann surfaces by the homotopy-equivalent space $PS\mathbb{C}(g)$ of parallel slit domains immediately suggests to imitate certain constructions well-known in the theory of loop spaces. The most basic is the sum operation which unites to parallel slit domains to represent the connected sum of two Riemann surfaces. As a special case we also have a stabilization procedure, adding a standard elliptic curve. These operations will be persued and generalized in later chapters.

There are many more such constructions, of which we mention only the conjugation and the Schottky-doubling.

1.1 The sum operation.

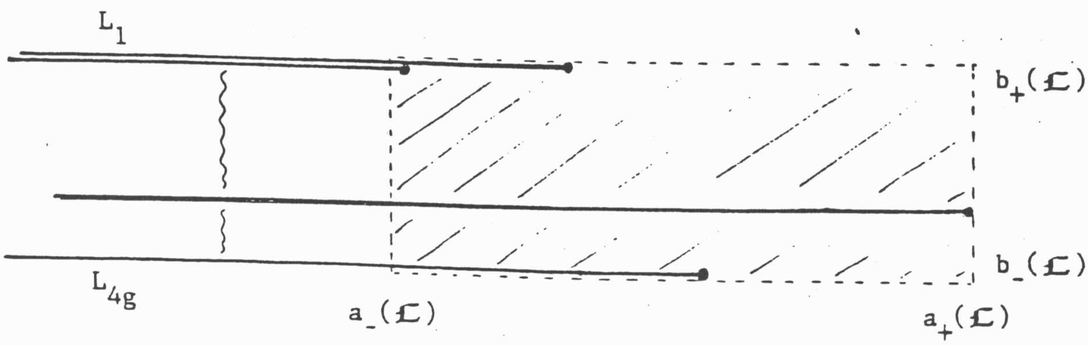
Recall that a parallel slit domain $\mathcal{L} = [L] = [L_1,\ldots,L_{4g};\lambda]$ in $PS\mathbb{C}(g)$ is an equivalence class of a non-degenerate configuration $L = (L_1,\ldots,L_{4g};\lambda)$; L consists of slits $L_i \subseteq \mathbb{C}$ and a pairing $\lambda \in \Sigma_{4g}$, satisfying (I.4.1.2); the non-degeneracy condition is given in (I.4.2.4); and the equivalence class $\mathcal{L} = [L]$ is obtained by performing all possible crossings (I.4.3.2).

Let $\mathcal{L}^1$ and $\mathcal{L}^2$ be two parallel slit domains of genus g_1 and g_2 , respec tively. Remembering that a conformal equivalence class in $\vec{\mathfrak{M}}(g)$ corresponds to a similarity class of a parallel slit domain (see (I.5.5.1), there is an obvious way to unite $\mathcal{L}^1$ and $\mathcal{L}^2$: one shifts $\mathcal{L}^1$ upwards till all its slit lie in the upper half-plane $\mathbb{H}$, and shifts $\mathcal{L}^2$ down till all its slits lie in the lower half-plane $\mathbb{H}^-$, and then the union is a new parallel slit domain of genus g_1+g_2 . To define this operation as a map

$$(1.1.1) \qquad \mu = \mu_{g_1,g_2} : PS\mathbb{C}(g_1) \times PS\mathbb{C}(g_2) \longrightarrow PS\mathbb{C}(g_1+g_2)$$

some technical preparations are necessary. Recall the functions a_+ , a_- , b_+ , b defined in (I.4.8). Note that $a_+(\mathcal{L}) = a_-(\mathcal{L})$ is possible, but we always have $b_-(\mathcal{L}) < b_+(\mathcal{L})$.

(1.1.2)



The translation of $\mathcal{L}$ by any complex number c is denoted by $\mathcal{L}+c = [L_1+c,\ldots,L_{4g}+c\,;\lambda]$, see (I.4.9).

Assume $\mathcal{L}^1 = [L^1] = [L_1^1,\dots,L_{4g_1}^1;\lambda^1]$ and $\mathcal{L}^2 = [L^2] = [L_1^2,\dots,L_{4g_2}^2;\lambda^2]$, and we set $b^1 = b_-(\mathcal{L}^1)$ and $b^2 = b_+(\mathcal{L}^2)$. Then all slits of $\mathcal{L}^1 - ib^1 + \frac{i}{2}$ are contained in $\mathbb{H}$, and all slits of $\mathcal{L}^2 - ib^2 - \frac{i}{2}$ are contained in $\mathbb{H}^-$. Their ordering by imaginary parts is

$$(1.1.3)\qquad \underbrace{L_1^1 - i(b^1-\tfrac{1}{2}),\dots,L_{4g_1}^1 - i(b^1-\tfrac{1}{2})}_{S^1 - i(b^1-\frac{1}{2})}\ ,\ \underbrace{L_1^2 - i(b^2+\tfrac{1}{2}),\dots,L_{4g_2}^2 - i(b^2+\tfrac{1}{2})}_{S^2 - i(b^2+\frac{1}{2})}\ ,$$

where we donote the sequences of slits by S^1 and S^2 , respectively. We denote the juxtaposition of the two sequences in (1.1.4) by $S^1 \oplus S^2$. Likewise is the (Whitney) $\lambda^1 \oplus \lambda^2$ sum of the two permutations the image under the inclusion $\Sigma_{4g_1} \times \Sigma_{4g_2} \longrightarrow \Sigma_{4(g_1+g_2)}$.

All in one formula gives

$$(1.1.4)\qquad \mu : PS\mathbb{C}(g_1) \times PS\mathbb{C}(g_2) \longrightarrow PS\mathbb{C}(g_1+g_2)$$

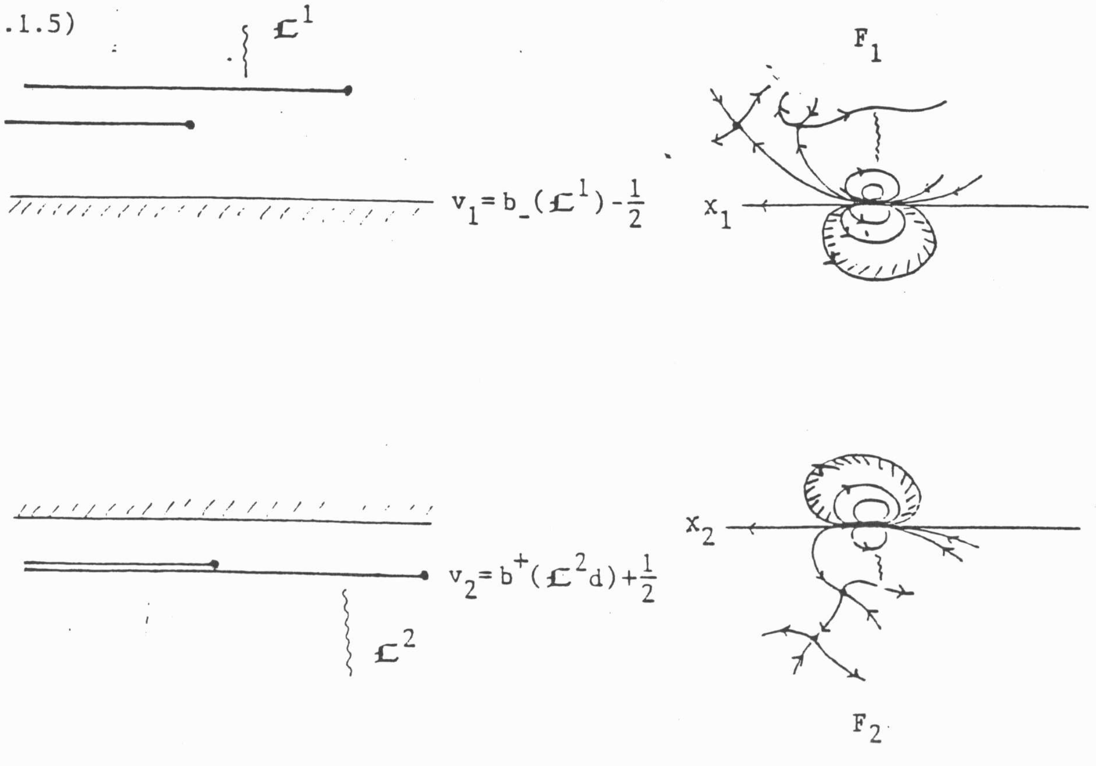
$$\mu([S^1;\lambda^1]\ ,\ [S^2;\lambda^2]) = [\bar{S}^1 \oplus \bar{S}^2;\lambda^1 \oplus \lambda^2] =$$

with $\bar{S}^1 = S^1 - i(b_-(\mathcal{L}^1) - \frac{1}{2})$, $\bar{S}^2 = S^2 - i(b_+(\mathcal{L}^2) + \frac{1}{2})$.

It is obvious that $\mathcal{L}^1 \oplus \mathcal{L}^2$ is a non-degenerate equivalence class; this follows directly from the definition (I.4.2); compare also (1.1.5) below. Furthermore $\mathcal{L}^1 \oplus \mathcal{L}^2$ is a well-defined equivalence class, for any crossing possible in $\mathcal{L}^1$ or $\mathcal{L}^2$ is still possible in $\mathcal{L}^1 \oplus \mathcal{L}^2$. And because b_+ , b_- are continuous functions, it follows that μ is continuous.

Geometrically, $\mathcal{L}^1 \oplus \mathcal{L}^2$ is the connected sum of the two Riemann surfaces $F_1 = F(\mathcal{L}^1)$ and $F_2 = F(\mathcal{L}^2)$ along the boundary of two discs. On F_1, we cut out the disc D_1 given by $v_1 < b_-(\mathcal{L}^1) - \frac{1}{2}$, and on F_2 the disc D_2 given by $v_2 > b_+(\mathcal{L}^2) + \frac{1}{2}$, where $v_i = y$ is the harmonic conjugate of the dipole functions u_i. Both are indeed discs, because they contain no stagnation points. The curves $v_1 = b_-(\mathcal{L}^1) - \frac{1}{2}$ and $v_2 = b_+(\mathcal{L}^2) + \frac{1}{2}$ are real-analytic curves through the dipoles P_1 and P_2, and tangent to the directions x_1 and x_2, respectively. If $F_1 - D_1$ and $F_2 - D_2$ are glued along these curves by identifying points with the same u-value, the result is a closed surace F of genus $g_1 + g_2$. F has an obvious complex structure; P_1 and P_2 are identified to become the new dipole P; and similarly the two directions x_1, x_2 become the same in F. The two harmonic functions u_1 and u_2 agree by construction on the new curve $\partial D_1 = \partial D_2$, and thus determine a dipole function for F. Only the harmonic conjugates v_1, v_2 have to be renormalized by adding the imaginary integration constants $-i(b_-(\mathcal{L}^1) - \frac{1}{2})$ and $-i(b_+(\mathcal{L}^2) + \frac{1}{2})$ respectively; now they are both zero on $\partial D_1 = \partial D_2$, and v_1 is positive on $F_1 - D_1 \subseteq F$, and v_2 is negative on $F_2 - D_2 \subseteq F$.

(1.1.5)



For $g = 0$ the auxiliary functions $a_\pm$, $b_\pm$ are not defined; the formula (1.1.4) makes nevertheless sense if $\mathcal{L}^1$ or $\mathcal{L}^2$ are the only element $[\emptyset]$ in $PS\mathbb{C}(0)$. Recall that we defined basepoints $\mathcal{L}^{(g)} \in PS\mathbb{C}(g)$ in (I.4.5.21).

(1.1.6) Proposition.

(i) $\mathcal{L} \oplus [\emptyset] = \mathcal{L} - i(b_+(\mathcal{L}) - \frac{1}{2})$

(ii) $[\emptyset] \oplus \mathcal{L} - i(b_-(\mathcal{L}) + \frac{1}{2})$, for any $\mathcal{L} \in PS\mathbb{C}(g)$,

(iii) $\mathcal{L}^{(g_1)} \oplus \mathcal{L}^{(g_2)} = \mathcal{L}^{(g_1+g_2)} + 4ig_1$. ∎

The proof is obvious. The statements (i) and (ii) say that $[\emptyset]$ is a homotopy-neutral element for the sum operation. There is an associativity law.

(1.1.7) Propositition. For any $g_1, g_2, g_3 \geq 0$ the diagram

$$\begin{array}{ccc} PS\mathbb{C}(g_1) \times PS\mathbb{C}(g_2) \times PS\mathbb{C}(g_3) & \xrightarrow{id \times \mu} & PS\mathbb{C}(g_1) \times PS\mathbb{C}(g_2+g_3) \\ \Big\downarrow {\scriptstyle \mu \times id} & & \Big\downarrow {\scriptstyle \mu} \\ PS\mathbb{C}(g_1+g_2) \times PS\mathbb{C}(g_3) & \longrightarrow & PS\mathbb{C}(g_1+g_2+g_3) \end{array}$$

is homotopy-commutative.

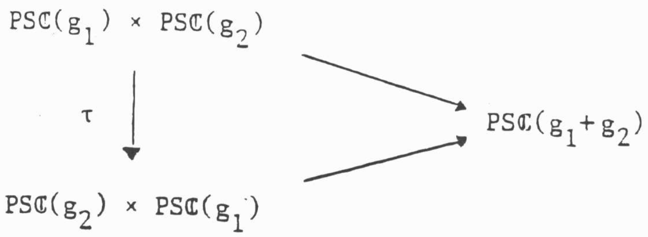
Proof. The required homotopy $\mu \circ (id \times \mu) \simeq \mu \circ (\mu \times id)$ amounts to a homotop of translations. For $t \in [0,1]$ we define

$$m_t : (\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3) \longrightarrow (\mathcal{L}^1 \oplus \mathcal{L}^2) \oplus \mathcal{L}^3 + ith$$

with $h = b_+(\mathcal{L}^2) - b_-(\mathcal{L}^2) + 1$. Clearly, $m_o = \mu(\mu \times id)$, and $m_1 = \mu(id \times \mu)$. ∎

A much more subtle property of μ is the commutativity. Its proof is postpone till we have developed the general framework. In its formulation below τ denotes the twist map interchanging the two factors.

(1.1.8) Proposition. For any $g_1, g_2 \geq 0$ the diagram

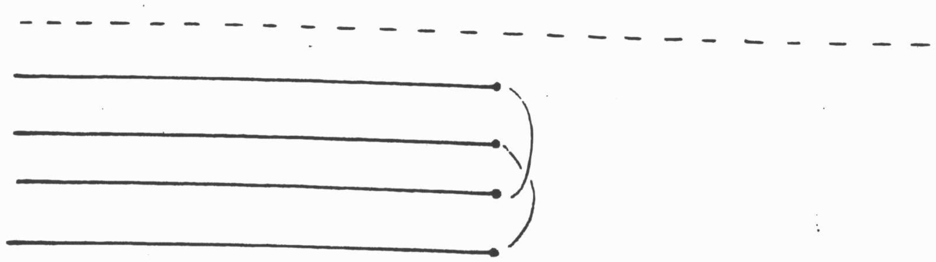


is homotopy-commutative. ∎

1.2 The stabilization map.

Let $\mathfrak{L}^{(1)}$ be the basepoint in $PS\mathbb{C}(1)$, given by the slit end points $s_k = ik$ for $k = 1,\ldots,4$ and the pairing $\lambda = (13)(24)$

(1.2.1)



The stabilization is defined by adding this fixed surface of genus 1 ,

$$(1.2.2) \qquad \sigma : PS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g+1)$$

$$\sigma(\mathfrak{L}) = \mathfrak{L} \circledast \mathfrak{L}^{(1)} .$$

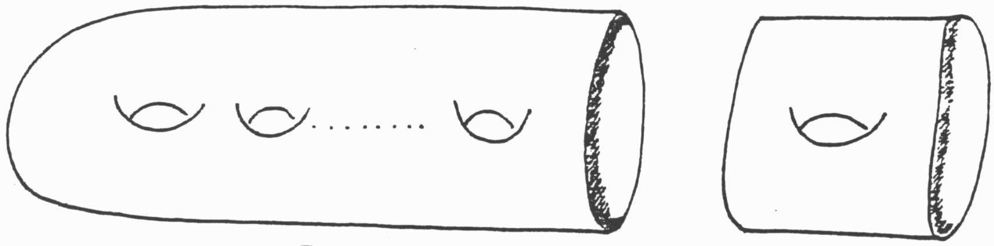
Geometrically, σ inserts a new handle in the surface $F(\mathfrak{L})$, in a small disc near the dipole.

The stabilization map allows to define an infinite object, the space of infinite parallel slit domains

$$(1.2.3) \qquad PS\mathbb{C}(\infty) = \lim_{\overrightarrow{\sigma}} PS\mathbb{C}(g) .$$

From (I.2.4.29, 1.3.7) we conclude, that $PS\mathbb{C}(\infty)$ is the classifying space for the infinite (sometimes called stable) mapping class group $\Gamma(\infty,1) = \lim_{\rightarrow} \Gamma(g,1)$ The inclusions $\Gamma(g,1) \subseteq \Gamma(g+1,1)$ are defined as follows. If $F_{g,1}$ is a surface of genus g with one boundary component, it is contained in $F_{g+1,1} = F_{g,1} \cup T$ with $T = F_{1,2}$ torus with two boundary components.

(1.2.4)



Extending a diffeomorphism of $F_{g,1}$ (which is the identity on $\partial F_{g,1}$) by the identity on T, defines a homomorphism $\mathrm{Diff}^+(F_{g,1}, \partial F_{g,1}) \longrightarrow \mathrm{Diff}^+(F_{g+1,1}, \partial F_{g+1,1})$ inducing $\Gamma(g,1) \longrightarrow \Gamma(g+1,1)$. In other words, let F_∞ be a connecte oriented, smooth, non-compact surface of infinite genus and without boundary; and let $\mathrm{Diff}^+_{cpt}(F_\infty)$ be the group of orientation preserving diffeomorphisms which are the identity outside a compact set; then $\Gamma(\infty,1) = \pi_o \mathrm{Diff}^+_{cpt}(F_\infty)$.

The stabilization is compatible with the sum operation.

(1.2.5) <u>Proposition.</u> (i) $\mu \circ (\sigma \times \mathrm{id}) \simeq \sigma \circ \mu$,

(ii) $\mu \circ (\mathrm{id} \times \sigma) \simeq \sigma \circ \mu$,

(iii) $\mu \circ (\sigma \times \sigma) \simeq \sigma^2 \circ \mu$. ∎

Again the proof is postponed, since it is a special case of general associativity laws.

1.3 Conjugation and Schottky-double.

Given a parallel slit domain $\mathfrak{L} = [L_1,\dots,L_{4g};\lambda]$ we define its conjugate $\overline{\mathfrak{L}}$ to be the image under the conjugation $z = x + iy \longmapsto \overline{z} = x - iy$ of the complex plane. Thus $\overline{\mathfrak{L}} = [\overline{L}_{4g},\dots,\overline{L}_1;\overline{\lambda}]$ with $\overline{\lambda}(k) = 4g+1-\lambda(4g+1-k)$. The new surface $F(\overline{\mathfrak{L}})$ is just $F(\mathfrak{L})$ with the conjugate complex structure.

$D(\mathfrak{L}) = \mathfrak{L} \circledast \overline{\mathfrak{L}}$ is a parallel slit domain of genus $2g$, representing the Schottky-double of $F(\mathfrak{L})$.

The conjugation $\mathfrak{L} \longrightarrow \overline{\mathfrak{L}}$ is an involution on $PS\mathbb{C}(g)$ and only one of several interesting self-maps of $PS\mathbb{C}(g)$ which we will investigate elsewhere.

Chapter 2

The Pontrjagin Ring $H_*(\coprod_g PS\mathbb{C}(g))$

2.1 $PS\mathbb{C} = \coprod_{g \geq 0} PS\mathbb{C}(g)$ as an H-space.

2.2 The Pontrjagin product.

2.3 Homological stability.

The disjoint union $PS\mathbb{C}$ of all $PS\mathbb{C}(g)$ has a multiplication induced by the sum operation. It induces a product on the graded homology module $H_*(PS\mathbb{C})$. This chapter contains only some elementary definitions; and we quote a theorem of E. Miller about a polynomical subalgebra, and J. Harer's stability theorem.

2.1 $PS\mathbb{C} = \coprod PS\mathbb{C}(g)$ as an H-space.

The disjoint union of all spaces $PS\mathbb{C}(g)$, $g \geq 0$, is denoted by $PS\mathbb{C}$. The sum operation μ , defined by (1.1.4) on each component of the product $PS\mathbb{C} \times PS\mathbb{C}$, is a map

$$(2.1.1) \qquad \mu : PS\mathbb{C} \times PS\mathbb{C} \longrightarrow PS\mathbb{C} .$$

The following is a rewording of Propositions (1.1.6 - 8).

(2.1.2) Proposition. $PS\mathbb{C}$ is a h-associative, h-commutative H-space with two-sided h-neutral element $[\emptyset]$. ■

We repeat that the commutativity of μ up to homotopy is still to be proved.

2.2 The Pontrjagin product.

Let A be a commutative ring with unit. The graded homology module $H_*(PS\mathbb{C};A)$ has a multiplication induced by the map μ ; it is defined as the composition

(2.2.1) $$H_*(PS\mathbb{C};A) \otimes H_*(PS\mathbb{C};A) \xrightarrow{\times} H_*(PS\mathbb{C} \times PS\mathbb{C};A) \xrightarrow{\mu_*} H_*(PS\mathbb{C};A)$$

where $\times$ denotes the exterior homology product. This product is called Pontrjagin product and denoted by $a \# b = \mu_*(a \times b)$. It is graded in the sense that $a \# b \in H_{p+q}(PS\mathbb{C};A)$ if $a \in H_p(PS\mathbb{C};A)$, $b \in H_q(PS\mathbb{C};A)$. The associativity $(a \# b) \# c = a \# (b \# c)$ follows from (2.1.2), together with the commutativity $a \# b = (-1)^{|a||b|} b \# a$, where $|\,|$ denotes the dimension of a homology class. Regarded as a homology class, $[\emptyset] \in H_o(PS\mathbb{C}(o);A) \cong A$ is a two-sided unit. Of course $\#$ is A-linear in both variables. Thus $H_*(PS\mathbb{C};A)$ is an algebra over A .

The most important result so far on this algebra is the following.

(2.2.2) Theorem. [Miller 1986] There are classes $\kappa_{2n} \in H_{2n}(PS\mathbb{C};\mathbb{Q})$, $n \geq 1$, which generate a polynomial subalgebra.

A conjecture by Mennford says that these classes generate the entire rational homology of $PS\mathbb{C}$.

2.3 Homological stability.

The following two results by J. Harer on the stabilization map $\sigma : PS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g+1)$ are important, [Harer 1985].

(2.3.1) Theorem. [Harer] The stabilization induces isomorphisms

$$\sigma_* : H_*(PS\mathbb{C}(g);\mathbb{Z}) \longrightarrow H_*(PS\mathbb{C}(g+1);\mathbb{Z})$$

for $* \leq \frac{g+1}{3}$, $g \geq 3$. ∎

Note that the stable range is about $^1/_{18}$ of the dimension of $PS\mathbb{C}(g)$. But by another result in [Harer 1986] the virtual cohomological dimension of $PS\mathbb{C}(g)$ is $4g-2$ if $g \geq 1$. We remark that Ivanov has improved the result above by enlarging the stable range to $* \leq \frac{g+1}{2}$, see [Ivanov 1987].

Also in [Harer 1985] we find the following result about the forgetful maps $B\vec{\Gamma}(g) \simeq B\Gamma(g,1) \longrightarrow B\Gamma(g)$.

(2.3.2) Theorem. [Harer] The epimorphisms $\Gamma(g,1) \longrightarrow \Gamma(g)$ induce isomorphisms

$$H_*(\Gamma(g,1);\mathbb{Z}) \longrightarrow H_*(\Gamma(g);\mathbb{Z})$$

for $* \leq \frac{g+1}{3}$, $g \geq 3$. ∎

Chapter 3

The Geometry of Dyer-Lashof Operations

The sum operation implants one parallel slit domain into the upper half-plane, and a second one into the lower half-plane. The idea of Dyer-Lashof operations - coming from the theory of loop spaces - is to implant several parallel slit domains into small disjoint rectangles moving independently in the plane.

Before we can define these maps we need to recall some technical notions, then introduce configuration spaces; furthermore, we need the concept of interval exchange transformations.

3.1 Normalization of parallel slit domains.

For a parallel slit domain $\mathcal{L} = [L_1,\ldots,L_{4g};\lambda]$ in $PS\mathbb{C}(g)$ we defined in (I.4. the support as the smallest closed rectangle with sides parallel to the x,y-ax containing the endpoints S_i of all slits L_i. Its corners are $a_\pm(\mathcal{L})+ib_\pm($ see 1.1.2 . The rectangle can become a vertical interval, but never a point. W denote by

$$(3.1.1) \qquad a(\mathcal{L}) = a_+(\mathcal{L}) - a_-(\mathcal{L}) \geqq 0 \quad ,$$

$$b(\mathcal{L}) = b_+(\mathcal{L}) - b_-(\mathcal{L}) > 0$$

the <u>length</u> resp. <u>height</u> of $\mathcal{L}$. The center of $\mathcal{L}$ is the center of mass of $\mathrm{supp}(\mathcal{L})$,

$$(3.1.2) \qquad c(\mathcal{L}) = \tfrac{1}{2}(a_+(\mathcal{L}) - a_-(\mathcal{L})) + \tfrac{i}{2}(b_+(\mathcal{L}) + b_-(\mathcal{L})) \ .$$

The <u>normalization</u> $N(\mathcal{L})$ of $\mathcal{L}$ translates this center to the origin of the plane, and then stretches or shrinks the configuration such that the support fits into the unit square,

$$(3.1.3) \qquad N(\mathcal{L}) = \frac{1}{\max\{a(\mathcal{L}),b(\mathcal{L})\}}\,(\mathcal{L} - c(\mathcal{L}))$$

3.2 Configuration spaces.

Several kinds of configuration spaces will serve as parameter spaces of operations. The simplest kind are the configuration spaces of points in the plane. The space

$$(3.2.1) \qquad \tilde{C}^n(\mathbb{C}) = \{(e_1,\dots,e_n) \in \mathbb{C}^n \mid e_i \neq e_j \quad \text{for} \quad i \neq j\}$$

is called the n-th ordered configuration space of $\mathbb{C}$. The symmetric group Σ_n acts freely on $\tilde{C}^n(\mathbb{C})$ and the quotient

$$(3.2.2) \qquad C^n(\mathbb{C}) = \tilde{C}(\mathbb{C}) / \Sigma_n$$

is called the n-th (unordered) configuration space of $\mathbb{C}$. Its fundamental group is the classical braid group $Br(n)$ on n strands, see [Artin 1925], [Birman 1974].

We regard $C^n(\mathbb{C})$ as configuration spaces of rectangles rather than of points. To do this we need a distance function

$$(3.2.3) \qquad \varepsilon : C^n(\mathbb{C}) \longrightarrow]0,\infty[$$

$$\varepsilon(\{e_1,\dots,e_n\}) = \min\{\|e_i - e_j\| \mid 1 \leq i \neq j \leq n\}$$

where $\| \ \|$ is the maximal coordinate norm in $\mathbb{C}$. The rectangles will be square around $e \in \mathbb{C}$, with sides parallel to the x,y-axes and side length $\varepsilon > 0$, i.e

$$(3.2.4) \qquad B_\varepsilon(e) = \{z \in \mathbb{C} \mid \|e - z\| \leq \tfrac{\varepsilon}{2}\} \ .$$

We also need the following extension of $B_\varepsilon(e)$,

$$\hat{B}_{\varepsilon}(e) = \{z \in \mathbb{C} \mid Re(z) \leqq Re(e) + \frac{\varepsilon}{2} , \quad Im(e) - \frac{\varepsilon}{2} \leqq Im(z) \leqq Im(e) + \frac{\varepsilon}{2}\} . \tag{3.2.5}$$

3.3 Dyer-Lashof maps.

Our aim is to define maps

$$(3.3.1) \qquad \vartheta = \vartheta^n_g : \tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(ng) \quad ,$$

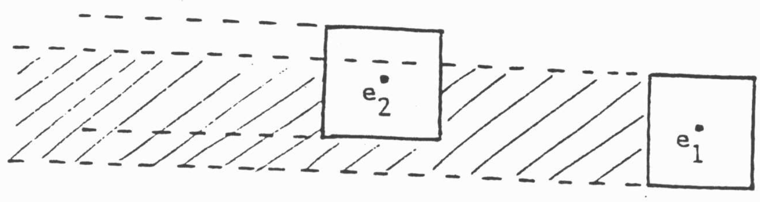
as a family of n-ary multiplications, parametrized by configurations. Naively we would like to set

$$\vartheta((e_1,\ldots,e_n) \underset{\Sigma_n}{\times} (\mathfrak{L}^1,\ldots,\mathfrak{L}^n)) = \bigcup_{k=1}^{n} e_k + \frac{\varepsilon}{2} N(\mathfrak{L}^k)$$

with $\varepsilon = \varepsilon(\{e_1,\ldots,e_n\})$; here $\cup$ is the simple union of parallel slit domains. It would implant $\mathfrak{L}^k$ into a square of length ε centered at e . If the configuration $\{e_1,\ldots,e_n\}$ is such that the extended squares $\hat{B}_\varepsilon(e_k)$ are pairwise disjoint, then this formula is indeed correct. But if for example $Im(e_i) = Im(e_j)$ and $Re(e_i) < Re(e_j)$, then the slits of $\mathfrak{L}^i$ and $\mathfrak{L}^j$ interfere with each other. Thus the implantation needs some extra consideration, which we explain in the following example.

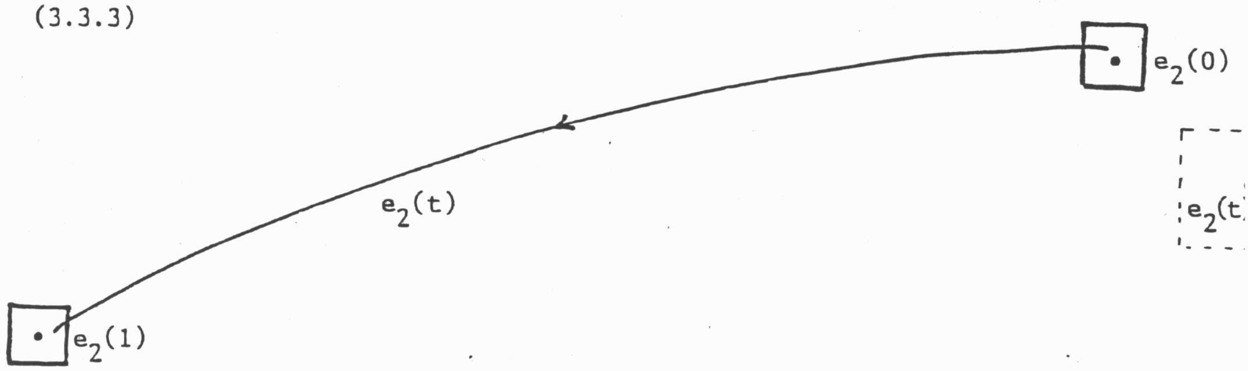
Assume e_1, e_2 are positioned as in the next figure.

(3.3.2)



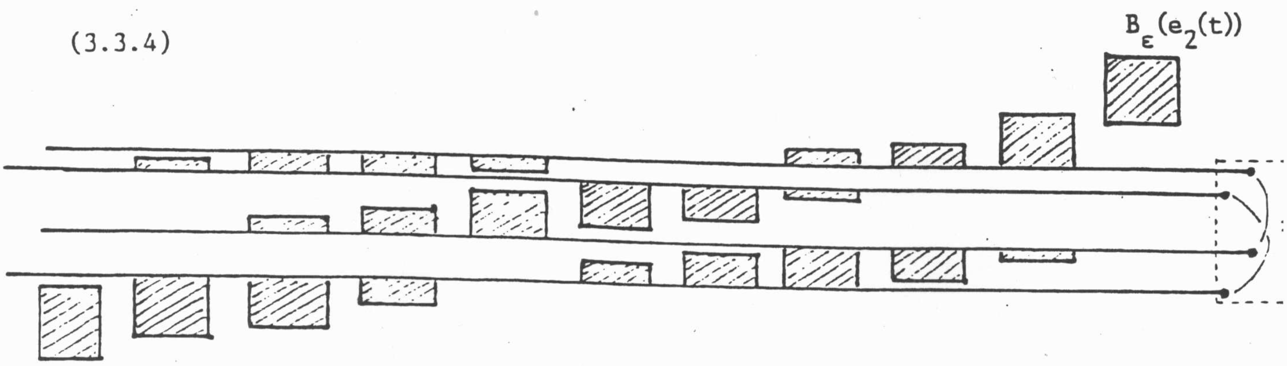
When implanted into $B_\varepsilon(e_1)$ the slits of $\mathfrak{L}^1$ are contained in the shaded region $\hat{B}_\varepsilon(e_1)$. Implanting $\mathfrak{L}^2$ into $B_\varepsilon(e_1)$ might result in a configuration which is not regular, or might mean that this process is not continuous. The correct way to do the implantation of $\mathfrak{L}^2$ is as follows. Recall that the left vertical boundary of $B_\varepsilon(e_1)$ is cut into intervals by the slits of $\mathfrak{L}^1$; and if reglued according to (I.4.2.3), we obtain again a connected interval; this is guaranteed by the regularity, see (I.4.4). In other words, the left side of $B_\varepsilon(e_1)$ is reparametrized by the gluing induced by $\mathfrak{L}^1$. The implantation of $\mathfrak{L}^2$ must therefore be done with respect to this reparametrization of the y-coordinate. The following figure shows the image of the square $B_\varepsilon(e_1)$ under this coordinate change, for various values of $Im(e_2)$. In this way squares further to the left must be woven through the slits of parallel slit domains already implanted in squares further to the right.

(3.3.3)



$e_1(t)$ is a constant curve, the curve $e_2(t)$ is shown.

(3.3.4)



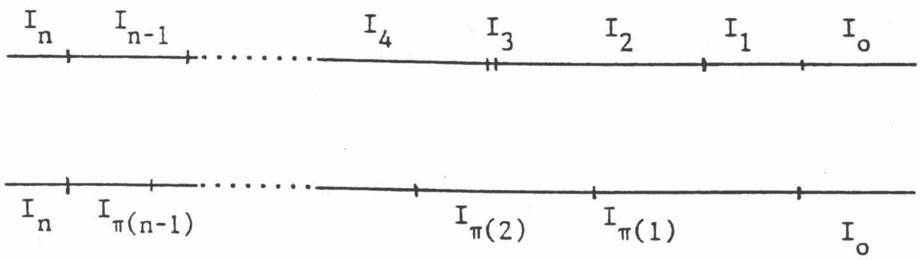
$\mathfrak{L}^1$ is of genus 1, $\mathfrak{L}^2$ is not shown.

The reparametrizations will be introduced in the next section.

3.4 Interval exchange transformations.

In our context an interval exchange transformation is a non-continuous, piecewise affine, orientation preserving self-map of $\mathbb{R}$. It is described by a symbol $v = (v_1,\ldots,v_n;\pi)$, where v_i are real numbers such that $v_i \geq v_{i+1}$, and $\pi \in \Sigma_{n-1}$ is any permutation. The points v_i cut $\mathbb{R}$ into $n+1$ intervals $I_o =]+\infty,v_1]$, $I_1 = [v_2,v_1],\ldots,I_{n-1} = [v_n,v_{n-1}]$ and $I_n =]-\infty,v_n]$. Since $v_{k+1} = v_k$ is not excluded, some intervals may be only points. π permutes the finite intervals $I_1,\ldots,I_{n-1}$, and the new sequence is $I_o,I_{\pi(1)}, I_{\pi(2)},\ldots,I_{\pi(n-1)},I_n$.

(3.4.1)



The associated function $T_v : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as follows. Denote by $|I_k| = |v_{k-1} - v_k|$ the length of I_k and set $h_k = \sum_{i=1}^{k} |I_{\pi(k)}|$ $(k = 1,\ldots,n-1)$, and $h_o = 0$. Let $J_o,J_1,\ldots,J_n$ be the intervals.

(3.4.2) $$J_o = [v_1,\infty[= I_o$$

$$J_k = [v_1 - h_k,\ v_1 - h_{k-1}[\qquad (k = 1,\ldots,n-1)$$

$$J_n =]-\infty,\ v_n[= I_n$$

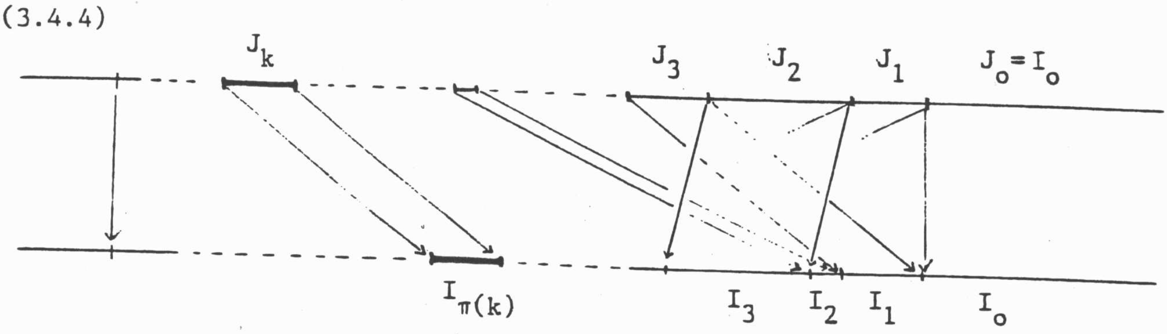
Then $|J_k| = |I_{\pi(k)}|$, and T_v is to map the interior of J_k isometrically and monotonically onto the interior of $I_{\pi(k)}$:

(3.4.3)

$$T_v(y) = \begin{cases} y & \text{for } v_1 < y , \\ v_{\pi^{-1}(k)} + (v_1 - h_{k-1} - y) & \text{for } v_1 - h_k < y < v_1 - h_{k-1} , \\ & (k = 1,\dots,n-1) \\ y & \text{for } y < v_1 - h_{n-1} . \end{cases}$$

To have T_v defined everywhere we extend this definition such that T_v is continuous from the right.

(3.4.4)



Different interval exchange transformations v and v' may give the same reparametrizations $T_v = T_{v'}$. This happens if and only if some intervals are points, i.e. $v_k = v_{k-1}$ for some k . We therefore introduce an equivalence relation; it is generated by the following crossing:

(3.4.5) If $v_{k-1} = v_k$, set $m = \pi^{-1}(k-1)$ and $\ell = \pi(m+1)$, and assume $k < \ell$; then we declare $v = v' = (v_1',\dots,v_n';\pi')$ if

$v_i' = v_i$ for $i = 1,\dots,k-1$,

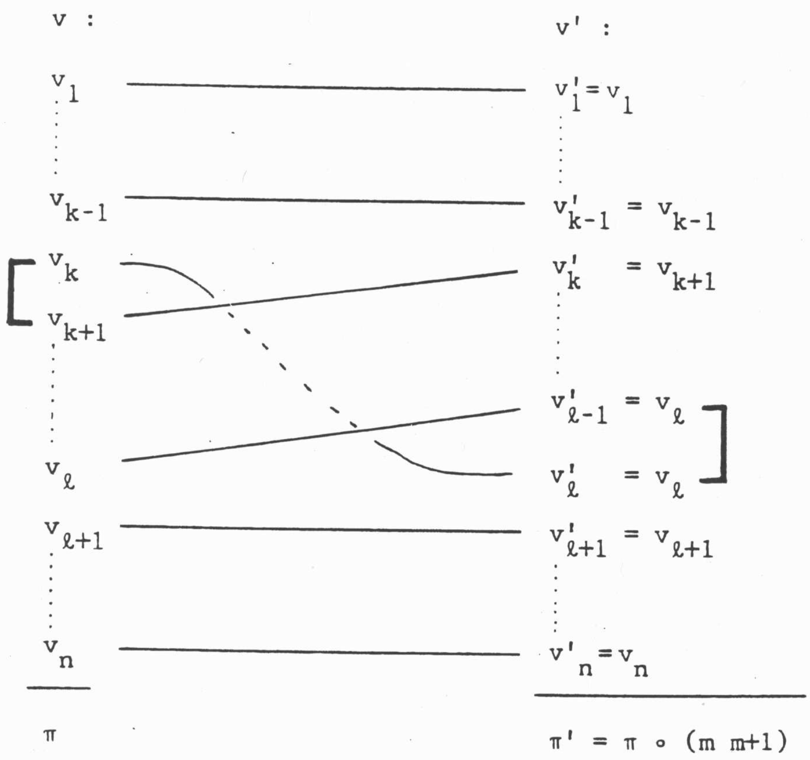
$v_i' = v_{i+1}$ for $i = k,\dots,k-1$

$v_\ell' = v_{\ell-1}$

$v_i' = v_i$ for $i = \ell+1,\ldots,n$, and

$\pi' = \pi \circ (m\ m{+}1)$.

(3.4.6)



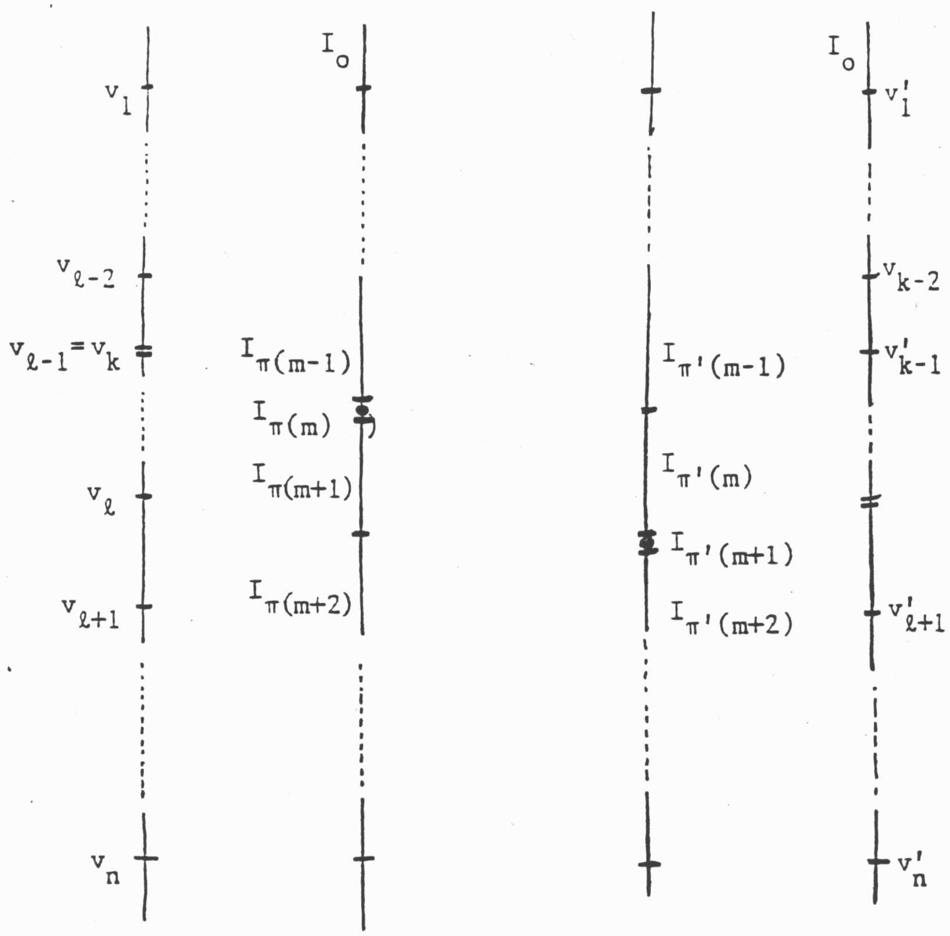
We denote an equivalence class by $\mathfrak{v} = [v] = [v_1,\ldots,v_n;\pi]$.

(3.4.7) Lemma. If $v \approx v'$, then $T_v(y) = T_{v'}(y)$ for all y in the interior of the intervals J_k .

Proof: First note that the intervals I_i, I_i' and J_j, J_j' produced by v, v' coincide up to a renumbering. If y runs from $+\infty$ to $-\infty$, then $T_v(y)$ and $T_{v'}(y)$ runs through these I-intervals in the order $I_o, I_{\pi(1)},\ldots, I_{\pi(n-1)}, I_n$ $I_o, I_{\pi(1)},\ldots, I_{\pi(n-1)}, I_n$. For a crossing (3.4.5) the difference is a transposition of two neighbouring intervals $I_{\pi(m)}$ and $I_{\pi(m+1)}$; since

$I_{\pi(m)} = I_{k-1} = [v_{k-1}, v_k]$ is a point, this gives the same function $T_v = T_{v'}$, except at the bounderies of the J-intervals.

(3.4.8)



∎

As already pointed out in (I.p.VI) a parallel slit domain $\mathcal{L} = [L_1,\dots,L_{4g};\lambda]$ gives rise to an interval exchange transformation. The cut points v_i are given by $v_i = \mathrm{im}(S_i)$, $i = 1,\dots,4g$. To determine the permutation of the intervals I_k recall the sequence of numbers

(3.4.8) $\ell_o = 0$,

$$\ell_{k+1} = \begin{cases} \ell_k + 1 & \text{if } k \text{ is even ,} \\ \lambda(\ell_k) & \text{if } k \text{ is odd ,} \end{cases}$$

which is defined up to a length of $8g$, since $\mathcal{L}$ is regular (I.4.4.6).

In addition we have $\ell_1 = 1$ and $\ell_{8g} = 4g$ for any λ. We extract the evenl indexed numbers and delete the first and last one,

(3.4.9)

$$\ell_1 = 1 \quad \frac{\ell_3 \qquad\qquad \ell_{8g-1}}{\ell_2 \quad \ell_4 \cdots\cdots \ell_{8g-2}} \quad \ell_{8g} = 4g \ .$$

And the remaining sequence of $4g-1$ numbers is the permutation $\pi = (\ell_2 \ell_4 \cdots$ $\pi = (\ell_2\ \ell_4\ \cdots\ \ell_{8g-2})$ in cycle notation. The interval exchange transformatio $v^L = (\mathrm{Im}(S_1),\ldots,\mathrm{Im}(S_{4g});(\ell_2\cdots\ell_{8g-2}))$ depends on the representing configuratio L, but we have

(3.4.10) Lemma. If $L \approx L'$, then $v^L \approx v^{L'}$.

Proof: A crossing of a slit $L_{k-1} \subseteq L_k$ over the pair $L_k, L_{\lambda(k)}$ corresponds to a crossing of $v_{k-1} = \mathrm{Im}(S_{k-1})$ over the "pair" v_k, v_ℓ, where the indices are coupled by $\ell = \pi(\pi^{-1}(k-1) + 1)$. Compare (3.4.6) to (I.4.3.2). ■

The assignment $\mathcal{L} \longrightarrow \mathfrak{v}^{\mathcal{L}}$ is a surjective map from the space of parallel slit domains onto a space of interval exchange transformations. We will investigate this map elsewhere in more detail.

3.5 Dyer-Lashof maps (continued).

We are now prepared to define the Dyer-Lashof maps

(3.5.1) $\tilde{\vartheta} = \vartheta_{g_1,\dots,g_n} : \tilde{C}^n(\mathbb{C}) \times PS\mathbb{C}(g_1) \times \dots \times PS\mathbb{C}(g_n) \longrightarrow PS\mathbb{C}(g_1+\dots+g_n)$

for all $g_i \geq 0$, $n \geq 1$. Let $(e_1,\dots,e_n) \in \tilde{C}^n(\mathbb{C})$ be an ordered configuration, and let $\mathcal{L}^1,\dots,\mathcal{L}^n$ be n parallel slit domains of genus $g_1,\dots,g_n$, resp. . On the configuration we consider the following relation: $e_i < e_j$ if $B_{\varepsilon/2}(e_i) \cap \hat{B}_{\varepsilon/2}(e_j) \neq \emptyset$, where $\varepsilon = \varepsilon(\{e_1,\dots,e_n\})$ is the distance function. The following statements hold.

(3.5.2)

(1) $e_i \nless e_i$

(2) either $e_i < e_j$ or $e_j < e_i$ or none

(3) if $e_i < e_j$ and $e_j < e_k$, then $e_i < e_k$

(4) if $e_i, e_j < e_k$, then either $e_i < e_j$ or $e_j < e_i$ $(i \neq j)$

(5) if $e_k < e_i, e_j$, then either $e_i < e_j$ or $e_j < e_i$ $(i \neq j)$

The last two properties imply that the entire configuration decomposes into maximal chains $e_{i_r} < e_{i_{r-1}} < \dots < e_{i_1}$ (which may contain only one member). The extended squares of any two e_i, e_j in distinct maximal chains have empty intersection. The definition can therefore be reduced to two cases: (a) the configuration consists of only one chain, and (b) the configuration consists of chains with only one member. We begin with the second and easier case (b) .

Let $(e_1,\dots,e_n)$ be an ordered configuratoin such that $e_i \nless e_j$ for any two members. Thus not only are the numbers $\mathrm{Im}(e_i)$ distinct, their mutual distances are at least $\varepsilon > 0$. Let $\mathfrak{L}^1,\dots,\mathfrak{L}^n$ be represented by configurations $L^1,\dots,L^n$, with $L^k = (L_1^k,\dots,L_{4g_k}^k;\lambda_k)$. Their transplants

$$(3.5.3) \qquad \mathfrak{N}^k = e_k + \tfrac{\varepsilon}{2}N(\mathfrak{L}^k)$$

are represented by configurations $N_k = e_k + \frac{\varepsilon}{2}N(L_k) = (N_1^k,\dots,N_{4g_k}^k;\lambda_k)$. The configurations are contained in n disjoint strips $K_k = \mathbf{R} \times [b_-(N^k),b_+(N^k)]$. Let $\kappa \in \Sigma_n$ the permutation such that the sequence $K_{\kappa(1)},K_{\kappa(2)},\dots,K_{\kappa(n)}$ is ordered by imaginary parts, i.e. $b_+(N^{\kappa(1)}) > b_-(N^{\kappa(1)}) > b_+(N^{\kappa(2)}) > b_-(N^{\kappa(2)}) > \dots > b_+(N^{\kappa(n)}) > b_-(N^{\kappa(n)})$. Then

(3.5.4)
$$N = (N_1^{\kappa(1)},\dots,N_{4g_{\kappa(1)}}^{\kappa(1)}, N_1^{\kappa(2)},\dots,N_{4g_{\kappa(2)}}^{\kappa(2)},\dots, N_{4g_{\kappa(n)}}^{\kappa(n)} \; ; \; \lambda^{\kappa(1)} \oplus \lambda^{\kappa(2)} \oplus \dots \oplus \lambda^{\kappa(n)})$$

is a configuration, and obviously regular. Its equivalence class $\mathfrak{N} = [N]$ depends only on the equivalence classes $\mathfrak{N}^k = [N^k]$ since all crossings in some $\mathfrak{N}^k$ are still possible in $\mathfrak{N}$. We denote this class by

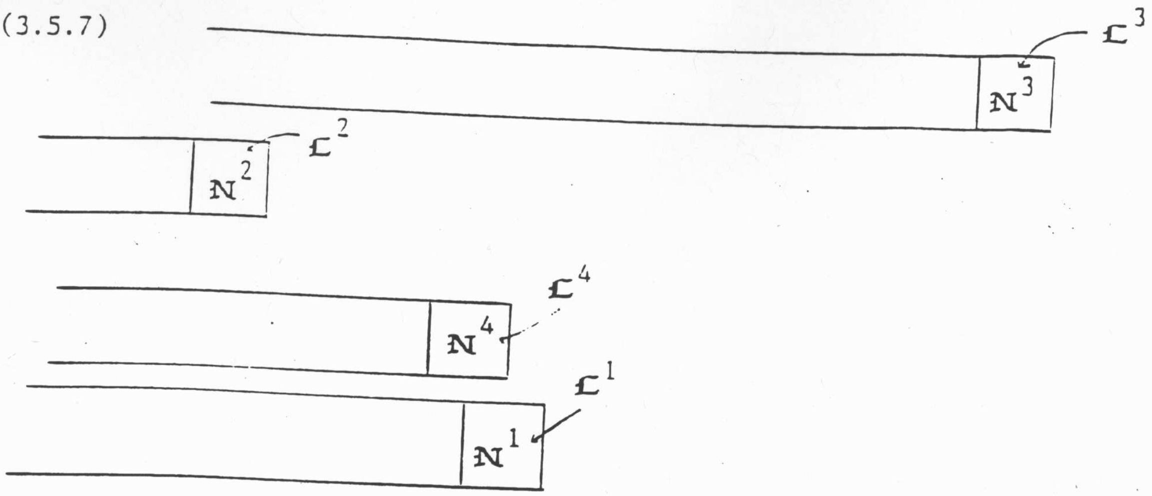
$$(3.5.5) \qquad \mathfrak{N} = [N] = \mathfrak{N}^1 \cup \mathfrak{N}^2 \cup \dots \cup \mathfrak{N}^n$$

and define

$$(3.5.6) \qquad \tilde{\vartheta}((e_1,\dots,e_n),(\mathfrak{L}^1,\dots,\mathfrak{L}^n)) = \mathfrak{N}^1 \cup \mathfrak{N}^2 \cup \dots \cup \mathfrak{N}^n \; .$$

If $g_1 = \dots = g_n$ then $\tilde{\vartheta}$ is Σ_n-invariant.

(3.5.7)



In the first and more subtle case (a) let $(e_1,\dots,e_n)$ be an ordered configurat consisting of one chain, and assume $e_n < e_{n-1} < \dots < e_2 < e_1$; otherwise we rename the members $e_{\kappa(n)} < \dots < e_{\kappa(1)}$ for some $\kappa \in \Sigma_n$. Let $\mathcal{L}^1,\dots,\mathcal{L}^n$ be the sequence of parallel slit domains, represented by $L^1,\dots,L^n$, respectively. We start by implanting $\mathcal{L}^1$ into the right-most square, the $\frac{\varepsilon}{2}$-square around e_1 , and set

$$(3.5.8) \qquad \mathcal{N}^1 = e_1 + \tfrac{\varepsilon}{2} N(\mathcal{L}^1) = [N^1_1,\dots,N^1_{4g_1};\lambda^1] \ .$$

Let $\mathfrak{v} = [v_1,\dots,v_{4g_1};\pi^1]$ be the interval exchange transformation induced by $\mathcal{N}^1$. The associated reparametrization $T_{\mathfrak{v}} : \mathbf{R} \longrightarrow \mathbf{R}$ yields a map $T : \mathbf{R}\times\mathbf{R} \longrightarrow \mathbf{R}\times\mathbf{R}$, $T(x,y) = (x, T_{\mathfrak{v}}(y))$. This non-continuous self-map of the plane wil be used to reparametrize the transplant

$$(3.5.9) \qquad \mathcal{N}^2 = e_2 + \tfrac{\varepsilon}{2} N(\mathcal{L}^2) = [N^2_1,\dots,N^2_{4g_2};\lambda^2] \ .$$

The intervals $J'_k = [v_1 - h_k,\ v_1 - h_{k-1}[$ for $k = 1,\dots,4g-1$, and $J'_o = [v_1,+\infty[$, $J'_{4g} =]-\infty, v_{4g_1}[$ are the maximal connected subsets of $\mathbf{R}$ on which $T_{\mathfrak{v}}$ is continuous.

The strips $\mathbb{R} \times J_n' \subseteq \mathbb{C}$ partition the slits of $\mathfrak{N}^2$ by partitioning the index set $\{1,\dots,4g_2\}$ into linearly ordered sets

$$(3.5.10) \qquad A_k = \{i \mid 1 \leq i \leq 4g_2,\ N_i^2 \subseteq \mathbb{R} \times J_k'\}\ , \qquad (k = 0.1,\dots,4g_1)\ .$$

If J_k is a point, then J_k' and A_k are empty. Reparametrizing $\mathfrak{N}^2$ by T means (for the indices) shuffling the indices $\{1,\dots,4g_2\}$ into $\{1,\dots,g_1\}$: their new order is

$$(3.5.11) \qquad A_o,1,A_{\pi(1)},2,A_{\pi(2)},\dots,4g_1-1,A_{\pi(4g_1-1)},4g_1,A_{4g_1}\ .$$

The new slits are $T(N_i^2)$, $i = 1,\dots,4g_2$; and if re-ordered according to their imaginary parts, their old indices occur in the order of (3.5.11) . We denote by $T(N^2_{A_k})$ the sequence $(T(N^2_{a_k}),T(N^2_{a_k+1}),\dots,T(N^2_{b_k}))$ if A_k is the sequence $a_k,a_k+1,\dots,b_k$.

The shuffling (3.5.11) determines also a new pairing function $\lambda^{2,1} \in \Sigma_{4g_1+4g_2}$ in an obvious way.

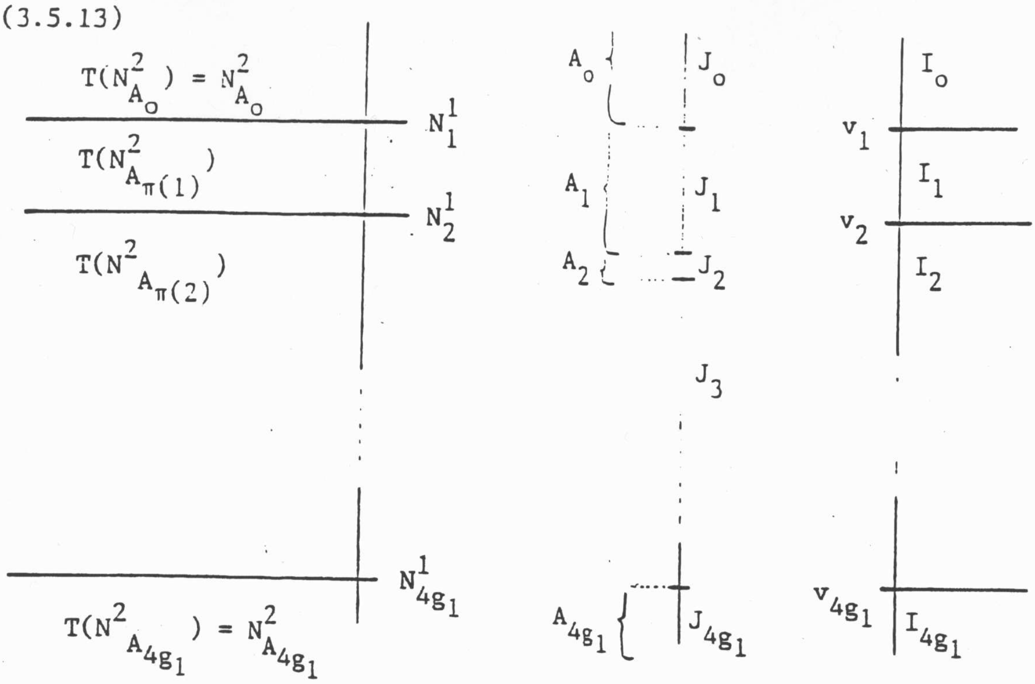
We now define the <u>implantation of</u> $\mathfrak{L}^2$ <u>into a square around</u> e_2 <u>relative to</u> $\mathfrak{L}^1$ <u>implanted into a square around</u> e_1 to be the equivalence class

(3.5.12)

$$[e_2;\mathfrak{L}^2|\mathfrak{N}^1] := [T(N^2_{A_o}),N_1^1,T(N^2_{A_{\pi(1)}})\ N_2^1,\dots,T(N^2_{A_{\pi(4g_1-1)}}),N^1_{4g_1},T(N^2_{A_{4g_1}});\lambda^{2,1}]$$

(Here ε depends actually on the entire configuration $\{e_1,\dots,e_n\}$ and not only on e_2 .)

(3.5.13)



To see that this relative implantation is continuous, it is enough to show that it is well-defined. Assume in $\mathfrak{N}^1$ a slit crosses a pair of slits; by Lemma (3.4.7) this does not change T . The effect on the intervals I_k, J_k, J'_k and the indices partition A_k is only a renumbering by a partial cyclic permutation This comes to a crossing in $\mathfrak{N}$. Assume in $\mathfrak{N}^2$ a slit crosses a pair. If all the three slits involved are not contained in any slit of $\mathfrak{N}^1$, this crossing is also possible in $\mathfrak{N}$. If one (and hence two) are contained in some slit of $\mathfrak{N}^1$, the corresponding index switches from A_k to A_{k+1} or A_{k-1} (being the last in A_k and becoming the first in A_{k+1} , or being the first in A_k and becoming the last in A_{k-1}). Again, this amounts to a crossing in $\mathfrak{N}$.

It is now obvious how to proceed inductively along the chain $e_n < \dots < e_3 < e_2 < e_1$. In the next step we regard $[e_2, e_1; \mathfrak{L}^2, \mathfrak{L}^1]$ as an entity and implant $\mathfrak{L}^3$ into a square around e_3 relative to $[e_2; \mathfrak{L}^2 | \mathfrak{N}^1]$. And so on. This gives inductively

$$(3.5.14)\quad [e_1;\mathcal{L}^1] = e_1 + \tfrac{\varepsilon}{2}N(\mathcal{L}^1) = \mathcal{N}^1 \;,$$

$$[e_2,e_1;\mathcal{L}^2,\mathcal{L}^1] = [e_2;\mathcal{L}^2 \mid [e_1;\mathcal{L}^1]] \;,$$

$$[e_k e_{k-1},\dots,e_1;\mathcal{L}^k,\mathcal{L}^{k-1},\dots,\mathcal{L}^1] = [e_k;\mathcal{L}^k \mid [e_{k-1},\dots,e_1;\mathcal{L}^{k-1},\dots,\mathcal{L}^1]$$

Having settled the second case (b) (when the configuration consists of only one chain) we can finally give the definition of the Dyer-Lashof maps, first in non -equivariant form,

$$(3.5.15)\quad \tilde{\vartheta} = \tilde{\vartheta}_{g_1,\dots,g_n} : \tilde{C}^n(\mathbb{C}) \times PS\mathbb{C}(g_1) \times \dots \times PS\mathbb{C}(g_n) \longrightarrow PS\mathbb{C}(g_1+\dots+g_n) \;.$$

The formula reads as

$$\vartheta((e_1,\dots,e_n),(\mathcal{L}^1,\dots,\mathcal{L}^n)) = \coprod_{e_{i_r}<\dots<e_{i_1}} [e_{i_r},\dots,e_{i_1};\mathcal{L}^{i_r},\dots,\mathcal{L}^{i_1}]$$

where the union runs over all maximal chains in $\{e_1,\dots,e_n\}$. The union is in the sense of (3.5.5) ; note that $\coprod$ is strictly associative, and commutative.

If $g_1 = \dots = g_n$ in (3.5.15) $\tilde{\vartheta}_{g,\dots,g}$ is Σ_n-invariant, and we obtain $\vartheta^n_g : \tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(ng)$.

3.6 Main properties.

The Dyer-Lashof maps satisfy an associativity law. It will imply the commutativity of the sum operation μ up to homotopy, and its compatability with the stabilization σ up to homotopy. To formulate it we first need an operation on the configuration spaces alone.

Let $r \geq 1$ and $n_1,\dots,n_r \geq 1$ be given, and set $n = \sum_{i=1}^{r} n_i$. There is a map $\tilde{t} = \tilde{t}^r_{n_1,\dots,n_r}$

(3.6.1) $$\tilde{t} : \tilde{C}^r(\mathbb{C}) \times \tilde{C}^{n_1}(\mathbb{C}) \times \dots \times \tilde{C}^{n_r}(\mathbb{C}) \longrightarrow \tilde{C}^n(\mathbb{C}) ,$$

$$\tilde{t}((e_1,\dots,e_r),(e^1_1,\dots,e^1_{n_1}),\dots,(e^r_1,\dots,e^r_{n_r})) =$$

$$((e_1 + \tfrac{\varepsilon}{2} N(e^1_1,\dots,e^1_{n_1}), e_2 + \tfrac{\varepsilon}{2} N(e^2_1,\dots,e^2_{n_2}),\dots,e_r + \tfrac{\varepsilon}{2} N(e^r_1,\dots,e^r_{n_r}))$$

where $\varepsilon = \varepsilon(\{e_1,\dots,e_r\})$, and $N(z_1,\dots,z_m)$ is the normalized sequence $(z'_1,\dots,z'_m$ with z'_i defined as follows. If $c = c(z_1,\dots,z_m\} = \frac{1}{m} \Sigma z_j$ is the center of maps, and $d = \operatorname{diam}(h_1,\dots,z_m)$ the diameter (in the maximal-coordinate norm) of the set $\{z_1,\dots,z_m\}$, then $z'_i = \frac{1}{d}(z_i - c)$. And $e + \frac{\varepsilon}{2}(z'_1,\dots,z'_m)$ is an abbreviation for the sequence $(e + \frac{\varepsilon}{2} z'_1,\dots,e + \frac{\varepsilon}{2} z'_m)$. Geometrically speaking, t implants the configurations, $(e^k_1,\dots,e^k_{n_k})$ into $\frac{\varepsilon}{2}$-squares around the points e_k.

Note that, if $n_1 = \dots = n_r$, then $\tilde{t}$ is Σ_r-invariant.

We define inclusions $\tilde{D}^1_g = D^1_g : PS\mathbb{C}(g) \longrightarrow \tilde{C}^1(\mathbb{C}) \times_{\Sigma_1} PS\mathbb{C}(g)$ by $D^1_g(\mathcal{L}) = (0,\mathcal{L})$ and $\tilde{D}^2_g : PS\mathbb{C}(g) \rightarrow \tilde{C}^2(\mathbb{C}) \times PS\mathbb{C}(g)^2$, $\tilde{D}^2_g(\mathcal{L}) = ((+i,-i),\mathcal{L},\mathcal{L})$, and D^2_g is $\tilde{D}^2_g$ composed with the projection to $\tilde{C}^2(\mathbb{C}) \times_{\Sigma_2} PS\mathbb{C}(g)^2$.

Now we can formulate the main result of this chapter.

(3.6.2) Theorem. <u>There are operations</u>

$$\vartheta^n_g : \tilde{C}^4(\mathbb{C}) \times_{\Sigma_n} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(ng)$$

<u>with the following properties.</u>

(i) (associativity)

for $r \geqq 1$, $n \geqq 1$ and $g \geqq 0$ the diagram

$$\begin{array}{ccc}
\tilde{C}^r(\mathbb{C}) \underset{\Sigma_r}{\times} (\tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n)^r & \xrightarrow{\ id \underset{\Sigma_r}{\times} (\vartheta^n_g)^r\ } & \tilde{C}^r(\mathbb{C}) \underset{\Sigma_r}{\times} PS\mathbb{C}(ng)^r \\
\Big\downarrow {\scriptstyle \tilde{t}^r_n \underset{\Sigma_r}{\times} (id^n)^r} & & \Big\downarrow {\scriptstyle \vartheta^r_{ng}} \\
\tilde{C}^r(\mathbb{C}) \underset{\Sigma_r}{\times} PS\mathbb{C}(g)^{rn} & \xrightarrow[\ \vartheta^{rn}_g\ ]{} & PS\mathbb{C}(rng)
\end{array}$$

is commutative.

(ii) (unit) for all $g \geqq 0$ the composition is

$$PS\mathbb{C}(g) \xrightarrow{D^1_g} \tilde{C}^1(\mathbb{C}) \underset{\Sigma_1}{\times} PS\mathbb{C}(g) \xrightarrow{\vartheta^1_g} PS\mathbb{C}(g)$$

homotopic to the identity.

(iii) (squaring) for all $g \geqq 0$ the composition is

$$PS\mathbb{C}(g) \xrightarrow{D^2_g} \tilde{C}^2(\mathbb{C}) \underset{\Sigma_2}{\times} PS\mathbb{C}(g)^2 \xrightarrow{\vartheta^2_g} PS\mathbb{C}(2g)$$

homotopic to $\mu_{g,g} \circ \text{diag}$.

Proof: To prove (i), we need to know that diagrams of the following kind are commutative, $n = \Sigma n_i$.

(3.6.3)

$$\begin{array}{ccc}
 & \tilde{C}^r(\mathbb{C}) \times (\tilde{C}^{n_1}(\mathbb{C}) \underset{\Sigma_{n_1}}{\times} PS\mathbb{C}(g_1)^{n_1}) \times \dots \times (\tilde{C}^{n_r}(\mathbb{C}) \underset{\Sigma_{n_r}}{\times} PS\mathbb{C}(g_n)^{n_r}) & \\
\swarrow {\scriptstyle \tilde{t} \times id \times \dots \times id} & & \searrow {\scriptstyle id \times \tilde{\vartheta}_{g_1}^{n_1} \times \dots \times \tilde{\vartheta}_{g_r}^{n_r}} \\
\tilde{C}^n(\mathbb{C}) \times (PS\mathbb{C}(g_1)^{n_1} \times \dots \times PS\mathbb{C}(g_r)^{n_r}) & & \tilde{C}^r(\mathbb{C}) \times (PS\mathbb{C}(n_1 g_1) \times \dots \times PS\mathbb{C}(n_r g_r)) \\
\searrow {\scriptstyle \tilde{\vartheta} g_1, \dots, g_1,} & & \swarrow {\scriptstyle \tilde{\vartheta} n_1 g_1, \dots, n_r g_r} \\
 & PS\mathbb{C}(n_1 g_1 + \dots + n_r g_r) &
\end{array}$$

The proof is straightforward, but tedious. The assertion (ii) is clear, since $\vartheta_g^1 \circ D_g^1$ is the normalization map $\mathcal{L} \to N(\mathcal{L})$. And for (iii), we use the homotopy from $\mathcal{L} \to \mathcal{L} \circledast \mathcal{L}$ to $\mathcal{L} \to \vartheta_g^2(D_g^2(\mathcal{L}))$ which normalizes the two copies of $\mathcal{L}$ seperately in the upper resp. lower half-plane around $+i$ resp. $-i$. ∎

Form [Boardman-Vogt 1968] and [May 1972] we have a well-developed theory of "little n-cube operads", central to understand n-fold loop spaces. The configuration spaces are such an operad for $n = 2$, see [May 1972, pp.1,30]. This operad, made to act on double loop spaces $\Omega^2 X$, acts by Theorem (3.6.2) on the space $PS\mathbb{C} = \coprod_{g \geq 0} PS\mathbb{C}(g)$ in the sense of [May 1972; pp.4,40]. This will be used in the next chapter.

(3.6.4) <u>Proof of (1.1.8):</u> The homotopy $\mu \simeq \mu \circ \tau$ comes from a curve in $\tilde{C}^2(\mathbb{C})$ which exchanges the two points $+i$ and $-i$:

(3.6.5)

Consider the homotopy $f_t(\mathcal{L}^1,\mathcal{L}^2) = \vartheta_g^2((ie^{\pi it}, -ie^{\pi it}),(\mathcal{L}^1,\mathcal{L}^2))$, $0 \leq t \leq 1$.
We have $f_o \simeq \mu$ by (3.6.2 (iii)). And $f_1 \simeq \mu \circ \tau$. ■

(3.6.6) <u>Proof of (1.2.5):</u> Consider for $\mu \circ (\sigma \times id) \simeq \sigma \circ \mu$ the curve $t \longrightarrow e(t)$ in $\tilde{C}^3(\mathbb{C})$ given by the figure.

(3.6.7)

satisfies $f_o \simeq \mu \circ (\sigma \times id)$, and $f_1 \simeq \sigma \circ \mu$. Similar curves can be found to make homotopies $\mu \circ (id \times \sigma) \simeq \sigma \circ \mu$ and $\mu \circ (\sigma \times \sigma) \simeq \sigma \circ \sigma \circ \mu$. ■

We remark that the operations ϑ_g^n do not generally commute with the stabilization, i.e. the diagrams

(3.6.8)

$$\begin{array}{ccc} \tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n & \xrightarrow{\vartheta^n_g} & PS\mathbb{C}(ng) \\ \Big\downarrow{\scriptstyle id \underset{\Sigma_n}{\times} (\sigma)^n} & & \Big\downarrow{\scriptstyle \sigma^n} \\ \tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g+1)^n & \longrightarrow & PS\mathbb{C}(ng+n) \end{array}$$

do not always commute up to homotopy.

3.7 Comparison with operations on braid groups and symplectic groups.

On the group level the operations

(3.7.1) $$\vartheta : \tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(ng)$$

become homomorphisms from wreath products

(3.7.2) $$Br(n) \int \vec{\Gamma}(g) \longrightarrow \vec{\Gamma}(ng) \quad .$$

(Here the notation $G \int H = H \wr G$ means: $G \leq \Sigma_n$ is the quotient group, acting as a permutation group on the normal subgroup H^n .). By the Σ_r-invariance of $\tilde{t}$ in case of $n_1 = \ldots = n_r = g$ in (3.6.1) we obtain maps $t^r_g : \tilde{C}^r(\mathbb{C}) \underset{\Sigma_r}{\times} C^g(\mathbb{C}) \longrightarrow C^{rg}(\mathbb{C})$, see [May 1972;p.130]. For the groups, these are homomorphisms.

(3.7.3) $$Br(r) \int Br(g) \longrightarrow Br(rg) \quad .$$

To connect the two, let $\tau : \text{pt.} \longrightarrow PS\mathbb{C}(1)$ the inclusion of the basepoint $\mathcal{L}^{(1)}$. Then the composition

(3.7.4)

$$\beta_g : C^g(\mathbb{C}) \cong \tilde{C}^g(\mathbb{C}) \underset{\Sigma g}{\times} (\text{pt.})^g \xrightarrow{\ id \times \tau^g\ } \tilde{C}^g(\mathbb{C}) \underset{\Sigma_g}{\times} PS\mathbb{C}(1)^g \longrightarrow PS\mathbb{C}(g)$$

establishes a homomorphism $Br(g) \longrightarrow \vec{\Gamma}(g)$. It is the easiest of several such homomorphisms; it associates to a braid the mapping class turning handles around each other according to this braid.

(3.7.5) <u>Proposition.</u> <u>The maps</u> β_m <u>commute with the operations</u> t_g^r <u>and</u> ϑ_g^r.

<u>Proof:</u> The diagram

$$\begin{array}{ccc}
\tilde{C}^r(\mathbb{C}) \underset{\Sigma_r}{\times} C^g(\mathbb{C})^r & \xrightarrow{\ tg\ } & C^{rg}(\mathbb{C}) \\
{\scriptstyle id \underset{\Sigma_r}{\times} (\beta_g)^r} \Big\downarrow & & \Big\downarrow {\scriptstyle \beta_{rg}} \\
\tilde{C}^r(\mathbb{C}) \underset{\Sigma_r}{\times} PS\mathbb{C}(g)^r & \xrightarrow[\ \vartheta_g^r\ ]{} & PS\mathbb{C}(rg)
\end{array}$$

is a special case of the diagram (3.6.3): set $n_1 = \dots = n_r = g$, and $g_1 = \dots = g_r = 0$, and use that the diagram is then also Σ_r-equivariant. ∎

In the other direction, mapping out of $PS\mathbb{C}(g)$, we have the monodromy homomorphism

(3.7.6) $$M_g : PS\mathbb{C}(g) \simeq B\vec{\Gamma}(g) \longrightarrow B\Gamma(g) \longrightarrow BSp_{2g}(\mathbb{Z})$$

where the last map associates (on the group level) to a mapping class $[\gamma]$ the homomorphism $\gamma_* : H^1(F_g;\mathbb{Z}) \longrightarrow H^1(F_g;\mathbb{Z})$ induced on the first cohomology group of our reference surface F_g. Since the cup-product, being a symplectic form on $H^1(F_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, is preserved, γ_* can be regarded as a $2g \times 2g$ symplectic matrix over $\mathbb{Z}$. Without using geometry there are operations

$$\text{(3.7.8)} \qquad \Sigma_n \int Sp_{2g}(\mathbb{Z}) \longrightarrow Sp_{2ng}(\mathbb{Z})$$

defined by letting the normal subgroup $Sp_{2g}(\mathbb{Z})^n$ act on $(\mathbb{Z}^{2g})^n$ by direct sum matrices, and letting the quotient group Σ_n act by permuting the summands. Using the facts $\tilde{C}^n(\mathbb{R}^\infty) \simeq E\Sigma_n$ and $C^n(\mathbb{R}^\infty) \simeq B\Sigma_n$, we have maps

$$\text{(3.7.9)} \qquad S_g^n : \tilde{C}^n(\mathbb{R}^\infty) \underset{\Sigma_n}{\times} BSp_{2g}(\mathbb{Z}) \longrightarrow BSp_{2ng}(\mathbb{Z}) .$$

(3.7.10) <u>Proposition.</u> <u>The monodromy maps</u> M_m <u>commute with the operations</u> ϑ_g^r <u>and</u> S_g^n.

<u>Proof:</u> The assertion is the commutativity of the diagrams

$$\begin{array}{ccc}
\tilde{C}^r(\mathbb{C}) \underset{\Sigma_r}{\times} PS\mathbb{C}(g)^r & \xrightarrow{\ \vartheta_g^r\ } & PS\mathbb{C}(rg) \\
\Big\downarrow {\scriptstyle \iota \underset{\Sigma_r}{\times} (M_g)^r} & & \Big\downarrow {\scriptstyle M_{rg}} \\
\tilde{C}^r(\mathbb{R}^\infty) \underset{\Sigma_r}{\times} BSp_{2g}(\mathbb{Z})^r & \xrightarrow[\ S_g^r\ ]{} & BSp_{2rg}(\mathbb{Z})
\end{array} .$$

Their commutativity follows from the fact that any implantation is geometrically a connected sum; hence the first cohomology group splits (as a symplectic module) into a direct sum. ∎

Chapter 4

The Homology of Dyer-Lashof Operations

The Dyer-Lashof homology operations will be derived from the Dyer-Lashof maps. The Browder operation will measure the non-linearity of Q_1 . For the definition and the first properties we specialize to coefficients $A = \mathbb{F}_2$.

It will become apparent that we can profit from the classical theory of homology operations for loop spaces, as developed in [Araki-Kudo 1956], [Dyer-Lashof 1962], [Browder 1960], [Milgram 1966], [May 1972], [Cohen 1976] and others. Especially [Cohen 1976] contains the whole apparatus for "little n-cube operations".

4.1 The Dyer-Lashof operations Q_o and Q_1 .

The operations $\vartheta_g^n = \vartheta : \tilde{C}^n(\mathbb{C}) \times_{\Sigma_n} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(ng)$ can be used to define homology operations. We restrict our attention to homology with coefficients in the field $\mathbb{F}_2$ of two elements and write $H_*(\ ) = H_*(\ ;\mathbb{F}_2)$. Let $n = 2$, and $g \geq 0$ arbitrary. There are two coverings $\tilde{C}^2(\mathbb{C}) \longrightarrow C^2(\mathbb{C})$ and $\tilde{C}^2(\mathbb{C}) \times PS\mathbb{C}(g)^2 \longrightarrow \tilde{C}^2(\mathbb{C}) \times_{\Sigma_2} PS\mathbb{C}(g)^2$. If $\tilde{w}$ is a (singular) chain in $\tilde{C}^2(\mathbb{C})$ which projects to a cycle w in $C^2(\mathbb{C})$, then, for any cycle x in $PS\mathbb{C}(g)$, $\tilde{w} \otimes x \otimes x$ is a cyle in $\tilde{C}^2(\mathbb{C}) \times_{\Sigma_2} PS\mathbb{C}(g)^2$. Note that $\tilde{C}^2(\mathbb{C})$ and $C^2(\mathbb{C})$ both are homotopy-equivalent to a circle; indeed the covering $\tilde{C}^2(\mathbb{C}) \longrightarrow C^2(\mathbb{C})$ is homotopy-equivalent to $S^1 \longrightarrow \mathbb{R}P^1$. We denote by $\tilde{w}_o$ and $\tilde{w}_1$ a 0-chain resp. 1-chain which project to the non-zero elements in $H_oC^2(\mathbb{C})$ and $H_1C^2(\mathbb{C})$, [ ] denotes homology classes, and ϑ_* is the induced map in homology. The definitior is

(4.1.1) $$Q_o = H_qPS\mathbb{C}(g) \longrightarrow H_{2q}PS\mathbb{C}(2g)$$

$$Q_o(x) := \vartheta^2_{g*}[\tilde{w}_o \otimes x \otimes x] \quad ,$$

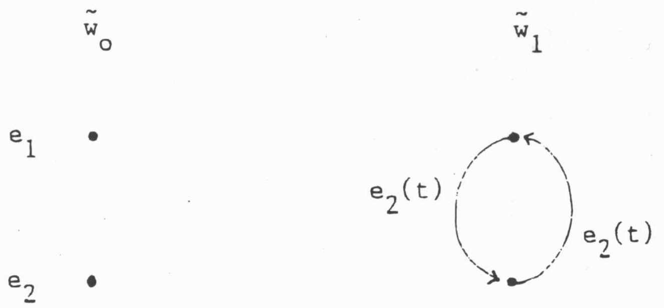
(4.1.2) $$Q_1 : H_qPS\mathbb{C}(g) \longrightarrow H_{2q+1}PS\mathbb{C}(2g)$$

$$Q_1(x) := \vartheta^2_{g*}[\tilde{w}_1 \otimes x \otimes x] \quad .$$

$Q_o(x)$ and $Q_1(x)$ are well-defined homology classes.

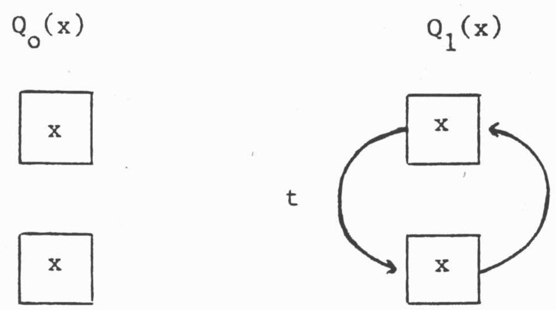
The chains $\tilde{w}_o$ and $\tilde{w}_1$ are shown in a figure. $\tilde{w}_o$ is, of course, just a point and $\tilde{w}_1$ is a curve.

(4.1.3)



The next figure shows schematically the homology classes $Q_i(x)$, where the homology class x is depicted only symbolically.

(4.1.4)



The operation Q_o is like a simple multiplication, and Q_1 is a closed curve of such multiplications. They are the operations denoted by Q^q , resp. Q^{q+1} for $|x| = q$ in [May 1970; 2.2] or [Cohen 1976; p.213,248]; Q^{q+1} is the "top operation", since we use the two-dimensional theory, or double loop spaces.

4.2 The Browder operation.

If $\tilde{v}$ is a cycle in $\tilde{C}^2(\mathbb{C})$, then $\tilde{v} \otimes x \otimes y$ is a cycle in $\tilde{C}^2(\mathbb{C}) \times PS\mathbb{C}(g_1) \times PS\mathbb{C}(g_2)$ for any two cycles x, y in $PS\mathbb{C}(g_1)$ resp. $PS\mathbb{C}(g_2)$. Since $\tilde{C}^2(\mathbb{C}) \simeq S^1$, there is essentially one such cycle $\tilde{v}$, which projects to twice w_1 . The definition is

(4.2.1) $$R_1 : H_p PS\mathbb{C}(g_1) \otimes H_q PS\mathbb{C}(g_2) \longrightarrow H_{p+q+1} PS\mathbb{C}(g_1+g_2) \quad ,$$

$$R_1(x,y) := \tilde{\vartheta}_{g_1,g_2 *}(\tilde{V} \otimes x \otimes y) \quad .$$

The cycle $\tilde{v}$ and $R_1(x,y)$ are shown in the next figure

(4.2.2) $\tilde{v}$ $R_1(x,y)$

The operation R_1 is the Browder operation in [Cohen 1976; pp.245,248].

4.3 Formulas for Q_o .

The operation $Q_o : H_q PS\mathbb{C}(g) \longrightarrow H_{2q} PS\mathbb{C}(2g)$ is nothing but the squaring in the Pontrjagin ring;

(4.3.1) (squaring) $$Q_o(x) = x^2 = x \# x$$

for all $x \in H_*PS\mathbb{C}$. This is immediate from the definition (4.1.1) and (3.6.2)(iii). Since we work in mod-2 homology this implies

(4.3.2) (linearity) $Q_o(x+y) = Q_o(x) + Q_o(y)$ and

(4.3.3) (multiplicativity) $Q_o(xy) = Q_o(x)\,Q_o(y)$

for any $x,y \in H_*PS\mathbb{C}$. Recall the stabilization $\sigma_* : H_qPS\mathbb{C}(g) \longrightarrow H_qPS\mathbb{C}(g+1)$. The formula

(4.3.4) (stability) $Q_o(\sigma_*(x)) = \sigma_*^2(Q_o(x))$

follows from (1.2.5), proved in (3.6.6). If $1_g \in H_oPS\mathbb{C}(g)$ denotes the generator, then we have by (1.1.6(iii))

(4.3.5) (units) $Q_o(1_g) = 1_{2g}$.

Let $Sq_r : H_q(\ ) \longrightarrow H_{q-r}(\ )$ be the dual Steenrod squares. In particular, $Sq_1 = \beta$, the Bockstein (boundary) homomorphism induced by the exact sequence $0 \to \mathbf{Z}_2 \to \mathbf{Z}_4 \to \mathbf{Z}_2 \to 0$.

From [Cohen 1976; p.214] we have

(4.3.6) (Nishida relations)

$$Sq_{2r}Q_o(x) = Q_oSq_r(x),$$

$$Sq_{2r+1}Q_o(x) = 0.$$

4.4 Formulas for Q_1 .

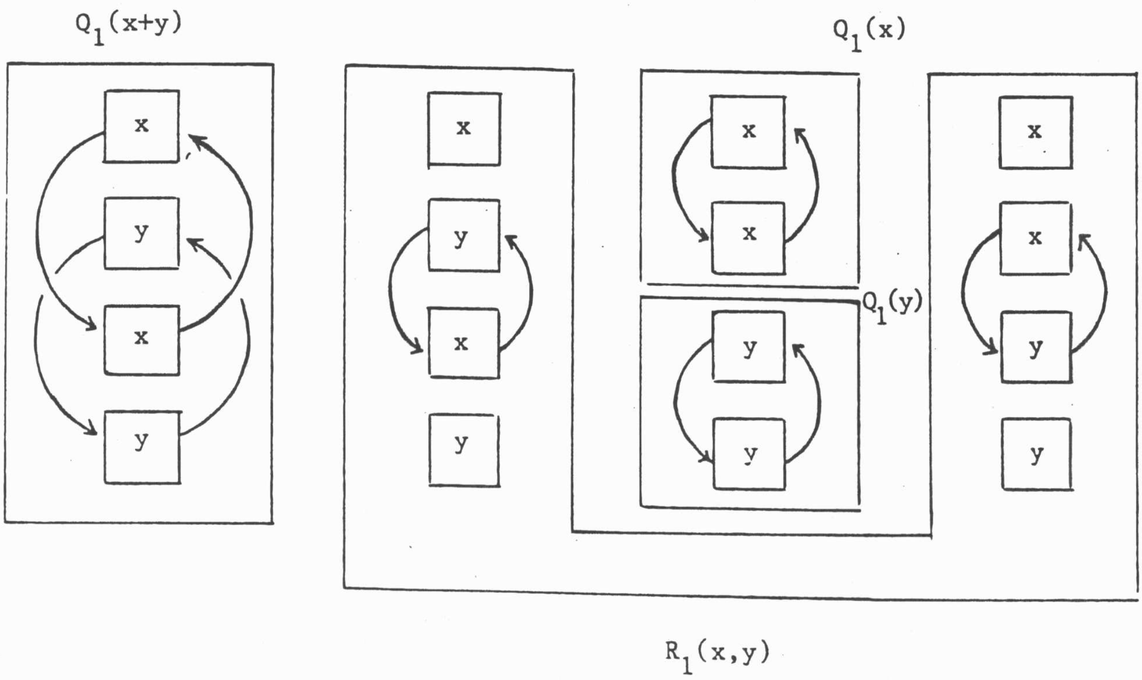
Apart from the basic squaring operation Q_o we have only one other Dyer-Lashof operation Q_1 , which is the exceptional "top operation" in [Cohen 1976; p.217]. We first note that Q_1 is only a function, and perhaps not a homomophism; the Browder operation R_1 enters most formulas as an error term. But, of course, $Q_1(0) = 0$.

(4.4.1) (linearity) $$Q_1(x+y) = Q_1(x) + R_1(x,y) + Q_1(y) ,$$

(4.4.2) (Cartan formula) $$Q_1(xy) = x^2Q_1(y) + xR_1(x,y)y + Q_1(x)y^2 ,$$

for any $x,y \in H_*PS\mathbb{C}$. The formulas follow from [Cohen 1976; Theorem 1.3(5),(2)]. But since Q_1 is only of degree one, we can argue geometrically, using for (4.4.1) the following figure:

(4.4.3)



Here the "variables" x, y are not independent, but "alternative", since we consider the sum $x+y$. The right figure shows three terms: they result from writing the chain $\tilde{w}_1(t)$ as a sum of four terms (after a homotopy) where t is called t_1, t_2, t_3, t_4 on the intervals $[0,1/4], [1/4,1/2], [1/2,3/4]$ resp. $[3/4,1]$. The two middle terms amount to $Q_1(x)$ and $Q_1(y)$, whereas the sum of the two outer terms is $R_1(x,y)$.

If, in the same figure, x and y are regarded as independent, it provides a proof of (4.4.2).

For the basepoint class $1_o = 1 \in H_oPS\mathbb{C}(0)$ we have obviously

(4.4.4) $\quad Q_1(1_o) = 0$.

Since $\sigma_*(x) = x \cdot 1_1$, we have $Q_1(\sigma_*(x)) = \sigma_*^2(Q_1(x)) + \sigma_*(xR_1(x,1_1)) + x^2Q_1(1_1)$ for the behaviour with respect to stability. For the Steenrod squares we find from [Cohen 1976; p.217,(3)]

(4.4.5) (<u>Nishida relations</u>)

$$Sq_{2r}Q_1(x) = Q_1Sq_r(x) + \sum_{\substack{i+j=2r \\ i<j}} R_1(Sq_i(x),Sq_j(x)) ,$$

$$Sq_{2r+1}Q_1(x) = Q_oSq_r(x) + \sum_{\substack{i+j=2r+1 \\ i<j}} R_1(Sq_i(x),Sq_j(x)) .$$

4.5 Formulas for R_1 .

The binary operation $R_1 : H_p PS\mathbb{C}(g_1) \otimes H_q PS\mathbb{C}(g_2) \longrightarrow H_{p+q+1} PS\mathbb{C}(g_1+g_2)$ occurs as an error term in other formulas. But in the same sense as Q_1 is the degree 1 analog of the squaring operation Q_o , R_1 is the analog of the sum operation μ . Here are some of the formulas.

(4.5.1) (commutativity) $R_1(x,y) = R_1(y,x)$.

(4.5.2) $R_1(x,x) = 0$

(4.5.3) (unit) $R_1(1,x) = 0 = R_1(x,1)$

for the unit $1 = 1_o \in H_o PS\mathbb{C}(0)$

(4.5.4) (Cartan-formula)

$$R_1(xy,y'y') = xR_1(y,x')y' + R_1(x,x')yy' + x'xR_1(y,y') + x'R_1(x,y')y \ .$$

(4.5.5) (Jacobi identity)

$$R_1(x,R_1(y,z)) + R_1(y,R_1(z,x)) + R_1(z,R_1(x,y)) = 0$$

(4.5.6) (Nishida relations) $Sq_r R_1(x,y = \sum_{i+j=r} R_1(Sq_i(x),Sq_j(y))$

(4.5.7) (Bockstein relation) $\beta R_1(x,y) = R_1(\beta x,y) + R_1(x,\beta y)$

(4.5.8) (Adem relations) $R_1(x,Q_o(y)) = 0 = R_1(Q_o(x),y)$

$R_1(x,Q_1(y)) = 0 = R_1(R_1(x,y),y)$

The formulas (4.5.1) and (4.5.2) are only true since we work modulo 2, see [Cohen 1976; p.215,(3)]. In the same reference we find (4.5.3) as (4), (4.5.4) as (5), (4.5.5) as (6), and (4.5.6), (4.5.7) as (7), (8) on p.216, and (4.5.8) as (4) on p.218.

Chapter 5

The Geometry of Symplectic Operations

Since the Dyer-Lashof operations implant entire parallel sleit domains into patches moving in the plane, they make very little use of the more intricate structure given by the slits. The smallest units they operate with are handles. The new sort of operations take several parallel slit domains apart into an upper and lower half and implant the halves into vertical pairs of rectangles moving independently in the plane. The disadvantage is that these operations are only defined on a subspace of $PS\mathbb{C}(g)$; but the stabilization of these subspaces is homotopy-equivalent to $PS\mathbb{C}(\infty)$.

5.1 Configuration spaces of vertical pairs.

Let $\tilde{E}^n(\mathbb{C})$ be the ordered configuration space of n-tuples $((e_1^+,e_1^-),\dots,(e_n^+,e_n^-))$ of vertical pairs in $\mathbb{C}$, i.e.

(5.1.1) (1) $e_i^+ \neq e_j^+, e_j^-$ for all i,j with $i \neq j$,

(2) $\mathrm{Re}(e_i^+) = \mathrm{Re}(e_i^-)$ for all i,

(3) $\mathrm{Im}(e_i^+) > \mathrm{Im}(e_i^-)$ for all i.

$\tilde{E}^n(\mathbb{C})$ is a 3n-dimensional manifold with a free Σ_n-action; we write $E^n(\mathbb{C}) = \tilde{E}^n(\mathbb{C})/\Sigma_n$.

We obtain subspaces $\tilde{E}^n_\bullet(\mathbb{C}) \subseteq \tilde{E}^n(\mathbb{C})$ and $E^n_\bullet(\mathbb{C}) = \tilde{E}^n_\bullet(\mathbb{C})/\Sigma_n \subseteq E_n(\mathbb{C})$ by replacing (3) above by

(3') $\mathrm{Im}(e_i^+) > 0 > \mathrm{Im}(e_i^-)$ for all i.

These configuration spaces are related to the old ones by inclusions

(5.1.2)
$$\begin{array}{ccccccc}
\tilde{C}^n(\mathbb{C}) & \xrightarrow{\tilde{\iota}} & \tilde{E}^n_\bullet(\mathbb{C}) & \subseteq & \tilde{E}^n(\mathbb{C}) & \subseteq & \tilde{C}^{2n}(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C^n(\mathbb{C}) & \xrightarrow[\iota]{} & E^n_\bullet(\mathbb{C}) & \subseteq & E^n(\mathbb{C}) & \subseteq & C^{2n}(\mathbb{C})
\end{array}$$

where $\tilde{\iota}(z_1,\dots,z_n) = ((e_1,\bar{e}_1),\dots,(e_n,\bar{e}_n))$ where $e_k = z_k = id$ with $d = \min_j \{\mathrm{Im}(e_j)\} + 1$.

To study the spaces $\tilde{C}$ one uses the fibrations

$$
\text{(5.1.3)}\qquad
\begin{array}{ccccccccc}
\tilde{C}^{m}(\mathbb{C}) & \longleftarrow & \tilde{C}^{m-1}(\mathbb{C}-1) & \longleftarrow & \tilde{C}^{m-2}(\mathbb{C}-1,2) & \longleftarrow & \dots & \longleftarrow & \tilde{C}^{1}(\mathbb{C}-1,\dots,m-1) \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\mathbb{C} & & \mathbb{C}-1 & & \mathbb{C}-1,2 & & \dots\dots & &
\end{array}
$$

(see [Fadell-Neuwirth 1962], [Birmann 1974]. The restrictions to $\tilde{E}^n(\mathbb{C})$ (and $\tilde{E}^n_{\bullet}(\mathbb{C})$) give fibrations

$$
\text{(5.1.4)}\qquad
\begin{array}{ccccccccc}
\tilde{E}^{n}(\mathbb{C}) & \longleftarrow & \tilde{E}^{n-1}(\mathbb{C}-\pm i) & \longleftarrow & \tilde{E}^{n-2}(\mathbb{C}-\pm i,1\pm i) & \longleftarrow & \dots & \longleftarrow & \tilde{E}^{1}(\mathbb{C}-\pm i,\dots) \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\tilde{E}^{1}(\mathbb{C}) & & \tilde{E}^{1}(\mathbb{C}-\pm i) & & \tilde{E}^{1}(\mathbb{C}-\pm i,1\pm i) & & \dots\dots & &
\end{array}
$$

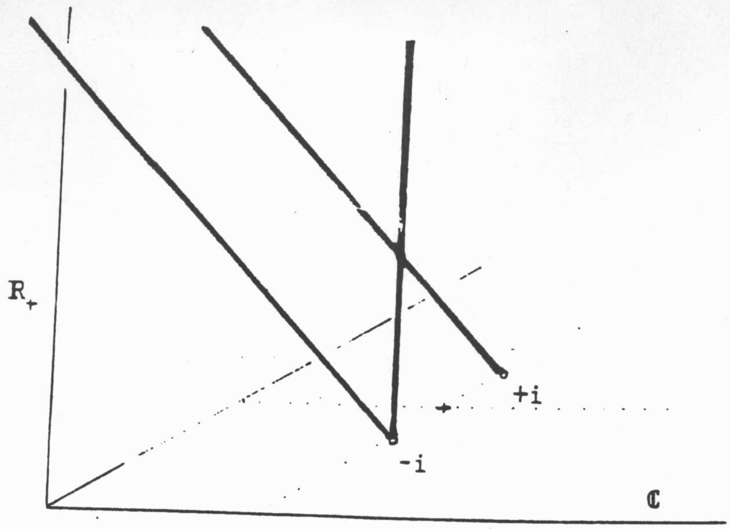
(5.1.5) Lemma. $\tilde{E}^1(\mathbb{C}-\pm i,\dots,(k-1)\pm i)$ resp. $\tilde{E}^1_{\bullet}(\mathbb{C}-\pm i,\dots,(k-1)\pm i)$ is homotopy-equivalent to a bouquet of $5k$ resp. $3k$ circles. □

(5.1.6) Proposition. $\tilde{E}^n(\mathbb{C})$, $\tilde{E}^n_{\bullet}(\mathbb{C})$, $E^n(\mathbb{C})$ and $E_{\bullet}(\mathbb{C})$ are Eilenberg-MacLane spaces $K(G,1)$.

Proof: By the lemma, the higher homotopy groups π_* , $* \geq 2$, of all base spaces and the last fibre in (5.1.4) vanish. It follows by induction on n that the higher homotopy groups of $\tilde{E}^n(\mathbb{C})$ vanish. The same follows for the quotients $E^n(\mathbb{C})$. ■

Proof of Lemma (5.1.5): First consider $\tilde{E}^1(\mathbb{C}-\pm i)$. The map $(e^+,e^-) \longrightarrow (e^-,e^+-e^+)$ is a homeomorphism onto a subset of $(\mathbb{C}-\{\pm i\}) \times \mathbb{R}_+$, which is the complement of four lines, two of which intersect.

(5.1.7)



This complement is retractable onto a 6-fold punctured sphere. A Mayer-Vietoris argument completes the proof for $k > 1$. For $\tilde{E}^1_{\bullet}$ the proof is now similar. ∎

5.2 The subspace of partioned parallel slit domains.

For a parallel slit domain $\mathcal{L} = [L_1,\dots,L_{4g};\lambda]$ consider the condition

(5.2.1) $\quad L_i \not\subseteq \mathbb{R} \qquad$ for all $i = 1,\dots,4g$.

Note that it is satisfied either by all or none of the representing configuration. Assume the index set decomposes into two subsets $\{1,\dots,4m\}$ and $\{4m+1,\dots,4g\}$ which are invariant under λ. Then the first $4m$ slits and the last $4(g-m)$ slits form a parallel slit domain all by themselves, one in $\mathbb{H}$ and the other in $\mathbb{H}^-$. (5.2.1) and the invariance of the two parts under λ prevents moving a slit from $\mathbb{H}$ to $\mathbb{H}^-$, or vice versa. Thus we have $g+1$ components homeomorphic to $PS\mathbb{C}(m) \times PS\mathbb{C}(g-m)$, $m = 0,1,\dots,g$. But there is one more component: define $PS\mathbb{C}_{\bullet}(g)$ to be the subspace of all $\mathcal{L} \in PS\mathbb{C}(g)$ satisfying (5.2.1) and in addition:

(5.2.2) there is an index k such that $k < \lambda(k)$,

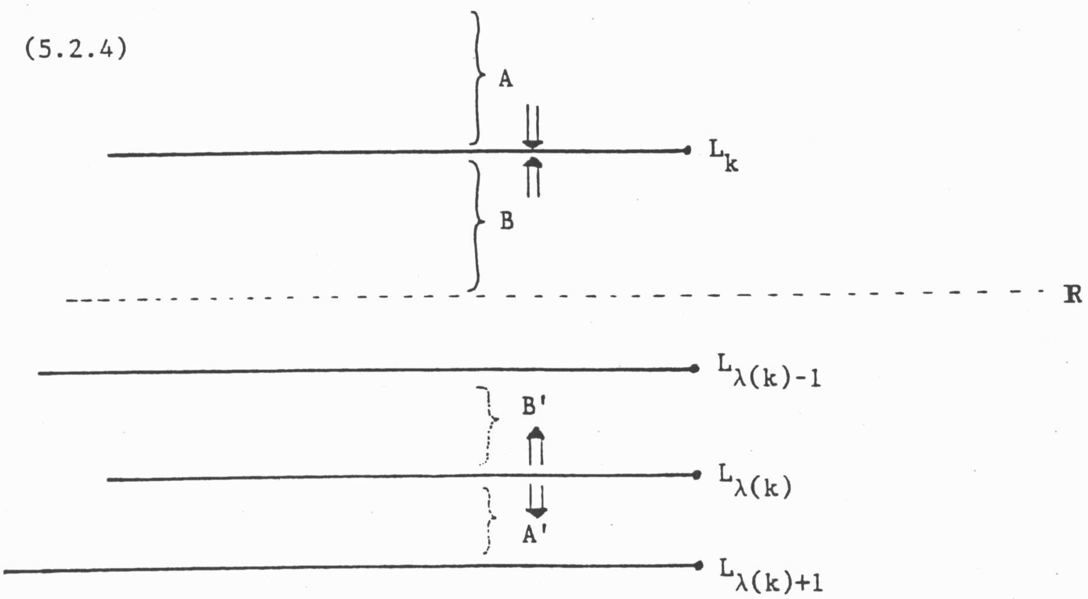
$L_k \subseteq \mathbb{H}$ and $L_{\lambda(k)} \subseteq \mathbb{H}^-$.

(5.2.3) <u>Proposition.</u> $PS\mathbb{C}_\cdot(g)$ <u>is a connected, open submanifold of</u> $PS\mathbb{C}(g)$.

<u>Proof:</u> We only have to show the connectivity. Pick $\mathcal{L} \in PS\mathbb{C}_\cdot(g)$ and let k be a bridging index as in (5.2.2). We will construct a path in $PS\mathbb{C}_\cdot(g)$ from $\mathcal{L}$ to $\mathcal{L}'$ given by $S_1' = (0,1)$, $S_\ell' = (,-\ell)$ $(\ell = 2,\dots,4g)$ and $\lambda' = (1\ 3)$ $(2\ 4)\ \dots\ (4g\text{-}3\ 4g\text{-}1)\ (4g\text{-}2\ 4g)$.

In a first move we make $\mathcal{L}$ generic, i.e. all slits disjoint. Then we bring all slit endpoints to the imaginary axis. There are two groups of slits we have to move to the lower half-plane: the sequence A with indices $1,\dots,k-1$, and the sequence B with indices $k+1,\dots,m$, where m is the largest index such that $L_m \subseteq \mathbb{H}$.

(5.2.4)



We cross the sequence A downwards over the pair L_k , $L_{\lambda(k)}$, starting with L_{k-1} , to become a sequence A' between $L_{\lambda(k)}$ and $L_{\lambda(k)+1}$. Then we cross the sequence B upwards under the pair L_k , $L_{\lambda(k)}$, starting with L_{k+1} , to become a sequence B' between $L_{\lambda(k)-1}$ and $L_{\lambda(k)}$. The remaining way to L' is the same as in the proof of Proposition (I.4.5.6), where the connectivity of $PS\mathbb{C}(g)$ itself was proved. ■

If $\mathcal{L} \in PS\mathbb{C}_{\bullet}(g)$ and $L = (L_1,\dots,L_{4g};\lambda)$ is a representing configuration, then we write $L = (L_1,\dots,L_m | L_{m+1},\dots,L_{4g};\lambda)$ to indicate that $L_1,\dots,L_m \subseteq \mathbb{H}$ and $L_{m+1},\dots,L_{4g} \subseteq \mathbb{H}^-$. The index m depends on the representative. We denote the upper and lower half of $\mathcal{L}$ by $\mathcal{L}_+$ resp. $\mathcal{L}_-$.

A partioned parallel slit domain can be taken apart: the map $[L_1,\dots,L_m | L_{m+1},\dots, L_{4g};\lambda) \longmapsto [L_1+a,\dots,L_m+a | L_{m+1}+b,\dots,L_{4g}+b;\lambda]$ is continuous as long as $Re(a) = Re(b)$.

The difference between $PS\mathbb{C}_{\bullet}(g)$ and $PS\mathbb{C}(g)$ vanishes under stabilization. But first note that $PS\mathbb{C}_{\bullet}(g)$ is not invariant under stabilization, in fact, $PS\mathbb{C}_{\bullet}(g)$ is mapped by σ to the complement of $PS\mathbb{C}_{\bullet}(g+1)$. This is overcome by another stabilization map.

(5.2.5) $$s : PS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g+1) , \quad s(\mathcal{L}) = \sigma(\mathcal{L}) + \frac{5}{2} i .$$

Since $s(\mathcal{L}) \in PS\mathbb{C}_{\bullet}(g+1)$, s factors as $j \circ S = s$, where j is the inclusion $PS\mathbb{C}_{\bullet}(g) \longrightarrow PS\mathbb{C}(g)$.

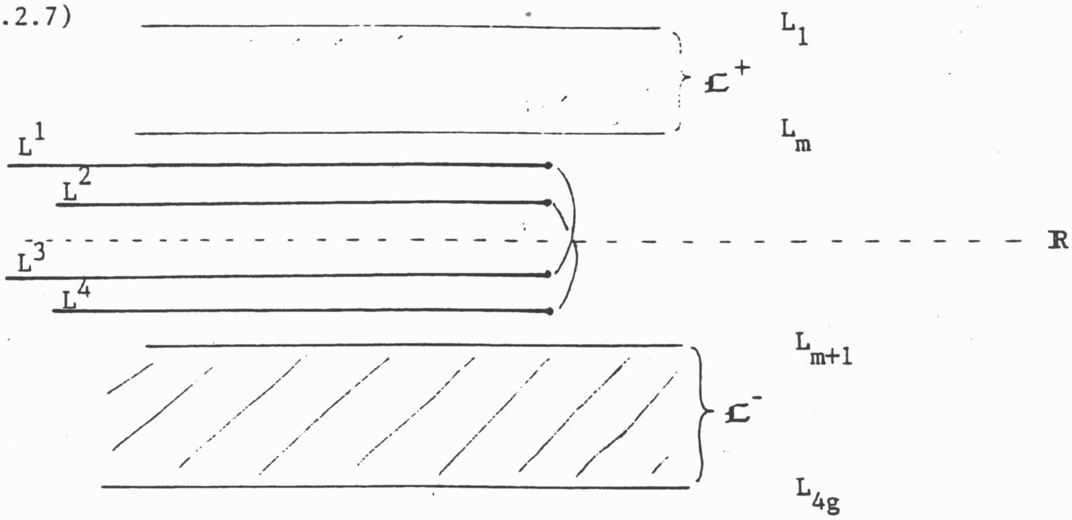
(5.2.6) $$s_{\bullet} : PS\mathbb{C}_{\bullet}(g) \longrightarrow PS\mathbb{C}_{\bullet}(g+1) \quad \text{by}$$

$$s_{\bullet}(\mathcal{L}) = s_{\bullet}([L_1,\dots,L_m | L_{m+1},\dots,L_{4g};\lambda]) =$$

$$= [L_1+2i,\dots,L_m+2i,L^1,L^2|L^3,L^4,L_{m+1}-2i,\dots,L_{4g}-2i;\lambda']$$

With $S^1 = 2i$, $S^2 = i$, $S^3 = -i$, $S^4 = -2i$, and λ' is obtained from λ by inserting the pairs $(L^1\ L^3)\ (L^2\ L^4)$ between L_m and L_{m+1} .

(5.2.7)



(5.2.8) <u>Lemma.</u> <u>In the diagram</u>

$$\begin{array}{ccc} PS\mathbb{C}(g) & \xrightarrow{s} & PS\mathbb{C}(g+1) \\ \downarrow & & \downarrow j \\ PS\mathbb{C}_\bullet(g) & \xrightarrow{s_\bullet} & PS\mathbb{C}_\bullet(g+1) \end{array}$$

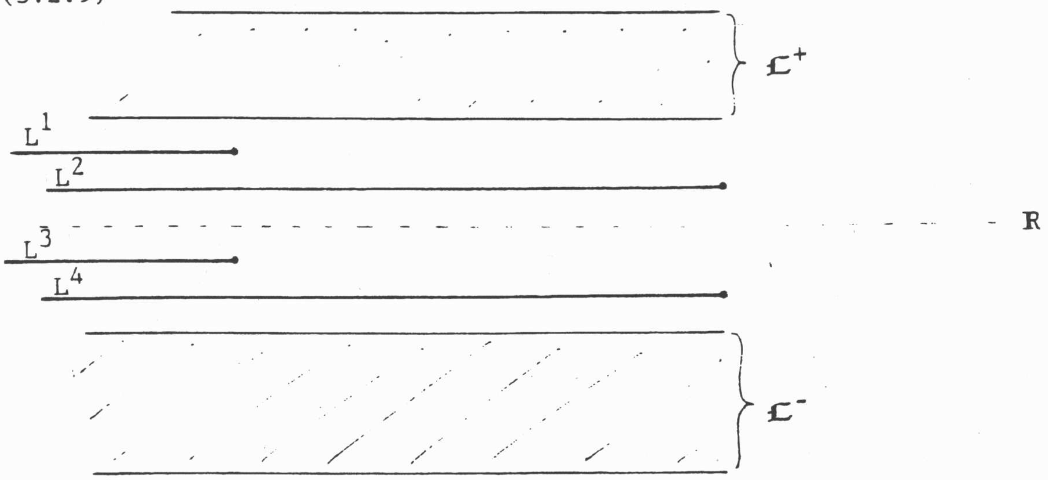
(i) $\sigma \simeq s = j \circ S$ <u>and</u>

(ii) $s_\bullet \simeq S \circ j$.

<u>Proof:</u> (i) is obvious from the definiton (5.2.5). A homotopy from $s_\bullet$ to $S \circ j$ consists of several moves. First the two slits L^1 and L^3 in $s_\bullet(\mathfrak{L})$

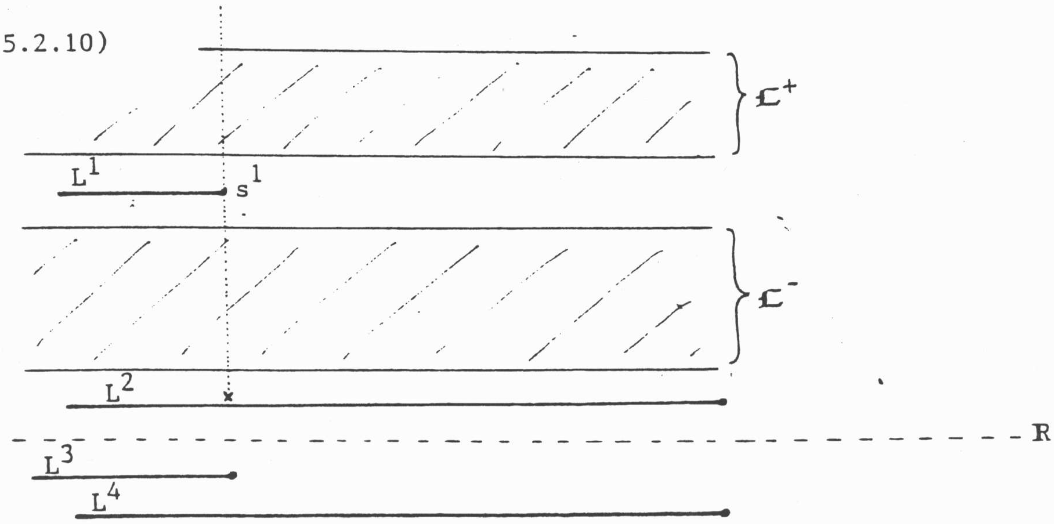
are moved horizontally further left than all slits L_i of $\mathcal{L}$.

(5.2.9)



Then $\mathcal{L}_-$ is moved upwards over the pair L^2, L^4 (starting with $L_{m+1}, \ldots, L_{4g}$) ; thereby rescaling $\mathcal{L}_-$ to fit between L^1 and L^2 .

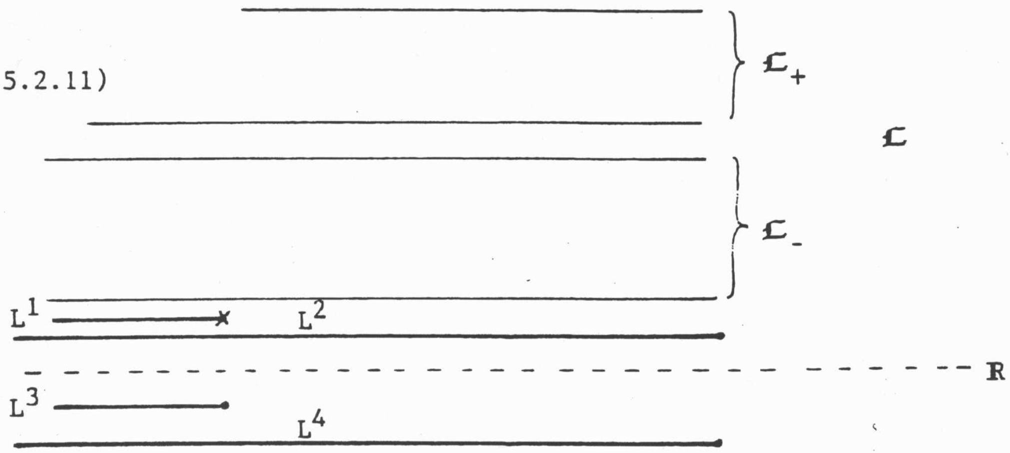
(5.2.10)



Next note that $\mathcal{L}$ - or better say the slits $L_1, \ldots, L_m, L_{m+1}, \ldots, L_{4g}$ form a sub-configuration; therefore the vertical through S^1 is connected after regluing according to the interval exchange transformation induced by $\mathcal{L}$. The path from S^1 to the point marked × in (5.2.10) is connected, and we move L^1 or

S^1 downwards along this path; during this process L^1 will necessarily go through parts of $\mathcal{L}_-$ and $\mathcal{L}_+$, because $\mathbb{R}$ is by (5.2.2) not a separating curve.

(5.2.11)



Up to rescaling, this final position is $s(\mathcal{L})$. ■

We use $s_\bullet$ to stabilize $PS\mathbb{C}_\bullet(g)$ to an infinite $PS\mathbb{C}_\bullet(\infty) = \varinjlim_{s_\bullet} PS\mathbb{C}_\bullet(g)$. As a consequence of the lemma we note

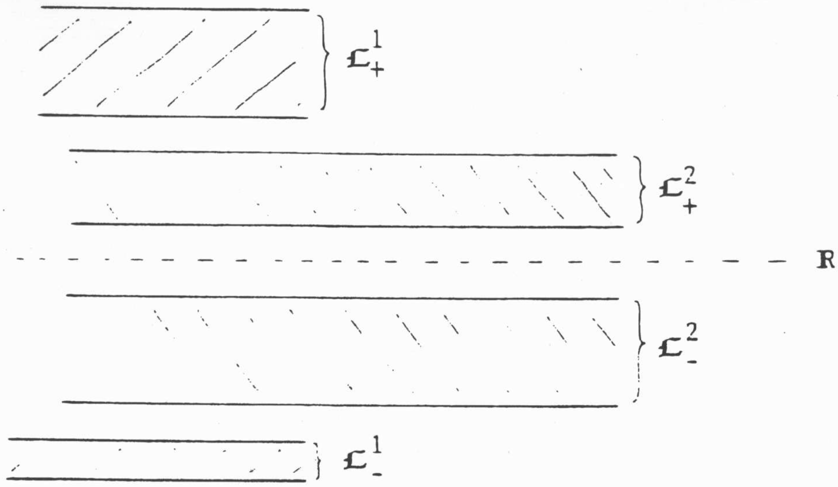
(5.2.12) <u>Proposition.</u> $PS\mathbb{C}_\bullet(\infty) \xrightarrow{j} PS\mathbb{C}(\infty)$ <u>is a homotopy-equivalence.</u> ■

As with the stabilization we use a different sum operation for the spaces $PS\mathbb{C}_\bullet(g)$.

(5.2.13) $\mu^\bullet = \mu^\bullet_{g_1,g_2} : PS\mathbb{C}(g_1) \times PS\mathbb{C}_\bullet(g_2) \longrightarrow PS\mathbb{C}_\bullet(g_1+g_2)$.

The easiest way to give the definition of $\mu^\bullet(\mathcal{L}^1,\mathcal{L}^2,)$ is by the following figure

(5.2.14)



(5.2.15) Proposition. The diagram

$$\begin{array}{ccc} \mathrm{PS}\mathbb{C}(g_1) \times \mathrm{PS}\mathbb{C}(g_2) & \xrightarrow{\mu} & \mathrm{PS}\mathbb{C}(g_1+g_2) \\ \downarrow{\scriptstyle j\times j} & & \downarrow{\scriptstyle j} \\ \mathrm{PS}\mathbb{C}_{\bullet}(g_1) \times \mathrm{PS}\mathbb{C}_{\bullet}(g_2) & \xrightarrow{\mu^{\bullet}} & \mathrm{PS}\mathbb{C}_{\bullet}(g_1+g_2) \end{array}$$

is homotopy-commutative.

Proof: The homotopy moves $\mathcal{L}^1_+$, $\mathcal{L}^1_-$ far to the left, and then weaves $\mathcal{L}^1_-$ upwards through $\mathcal{L}^2_-$ and $\mathcal{L}^2_+$ ∎

5.3 The operation maps.

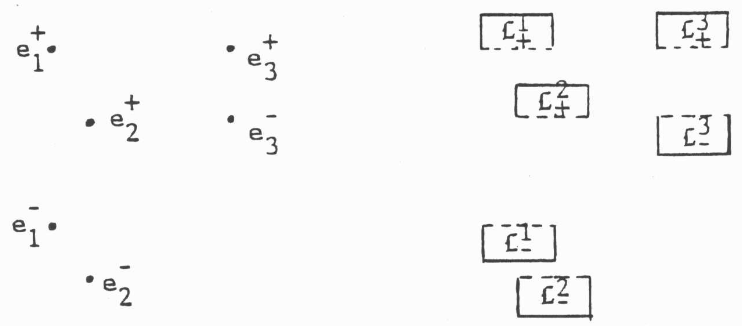
We can now define the new operations

$$(5.3.1) \qquad \vartheta : \tilde{E}^n(\mathbb{C}) \underset{\Sigma_n}{\times} \mathrm{PS}\mathbb{C}_{\bullet}(g)^n \longrightarrow \mathrm{PS}\mathbb{C}(ng) .$$

This time we will not give formulas, since they would be similar to those in chapter 3.

Let $((e_1^+,e_1^-),\dots,(e_n^+,e_n^-))$ be a configuration of pairs, and let $\mathcal{L}^1,\dots,\mathcal{L}^n$ be n partitioned parallel slit domains of genus $g_1,\dots,g_n$. The supports of $\mathcal{L}^k$ are divided by $\mathbb{R}$ into an upper and lower part, of different size in general. Let $c^{\cdot}(\mathcal{L}^k)$ denote the point $\frac{1}{2}(a_+(\mathcal{L}^k)-a_-(\mathcal{L}^k))$ on $\mathbb{R}$; $\mathcal{L}^k$ is normalized so that $c^{\cdot}(\mathcal{L}^k)$ becomes the origin, and that the support is contained in the unit square around the origin. Around the points $e_k^{\pm}$ we attach upper or lower halves of squares, small enough to be disjoint. Then one starts to implant upper and lower halves $\mathcal{L}_{\pm}^k$ of $\mathcal{L}^k$ into these half-squares: again one must start with the utmost right vertical pair(s), and later implantation must be done relative to the reparametrized y-axis.

(5.3.2)



This gives first maps

$$(5.2.3) \qquad \tilde{\vartheta} : \tilde{E}^n(\mathbb{C}) \times PS\mathbb{C}_{\cdot}(g_1) \times \dots \times PS\mathbb{C}_{\cdot}(g_n) \longrightarrow PS\mathbb{C}(g_1+\dots+g_n) \ .$$

With $g_1 = \dots = g_n = g$ and using the Σ_n-invariance we get (5.3.1). If we restrict $\tilde{\vartheta}$ or ϑ to $\tilde{E}_{\cdot}(\mathbb{C})$, we obtain maps

(5.3.4) $$\tilde{\vartheta}^{\cdot} : \tilde{E}^n_{\cdot}(\mathbb{C}) \times PS\mathbb{C}_{\cdot}(g_1) \times \ldots \times PS\mathbb{C}_{\cdot}(g_n) \longrightarrow PS\mathbb{C}_{\cdot}(g_1 + \ldots + g_n) ,$$

and

(5.3.5) $$\vartheta^{\cdot}_g : \tilde{E}^n_{\cdot}(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}_{\cdot}(g)^n \longrightarrow PS\mathbb{C}_{\cdot}(ng) .$$

5.4 Main properties.

Recall the inclusions $\tilde{E}^n_{\cdot}(\mathbb{C}) \subseteq \tilde{E}^n(\mathbb{C}) \xrightarrow{\iota} \tilde{C}^n(\mathbb{C})$. If we restrict the configurations in (5.2.3) to the subspace $\tilde{E}^n_{\cdot}(\mathbb{C})$, then the image of ϑ is contained in $PS\mathbb{C}_{\cdot}(\mathbb{C})$. The operations are compatible.

(5.4.1) Proposition. For all $n \geq 1$, $g \geq 0$ the diagram

$$\begin{array}{ccc}
\tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n & \xrightarrow{\vartheta} & PS\mathbb{C}(ng) \\
\uparrow & & \| \\
\tilde{C}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}_{\cdot}(g)^n & \longrightarrow & PS\mathbb{C}(ng) \\
\downarrow & & \| \\
\tilde{E}^n(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}_{\cdot}(g^n) & \longrightarrow & PS\mathbb{C}(ng) \\
\uparrow & & \uparrow \\
\tilde{E}^n_{\cdot}(\mathbb{C}) \underset{\Sigma_n}{\times} PS\mathbb{C}_{\cdot}(g) & \xrightarrow[\vartheta]{} & PS\mathbb{C}_{\cdot}(ng)
\end{array}$$

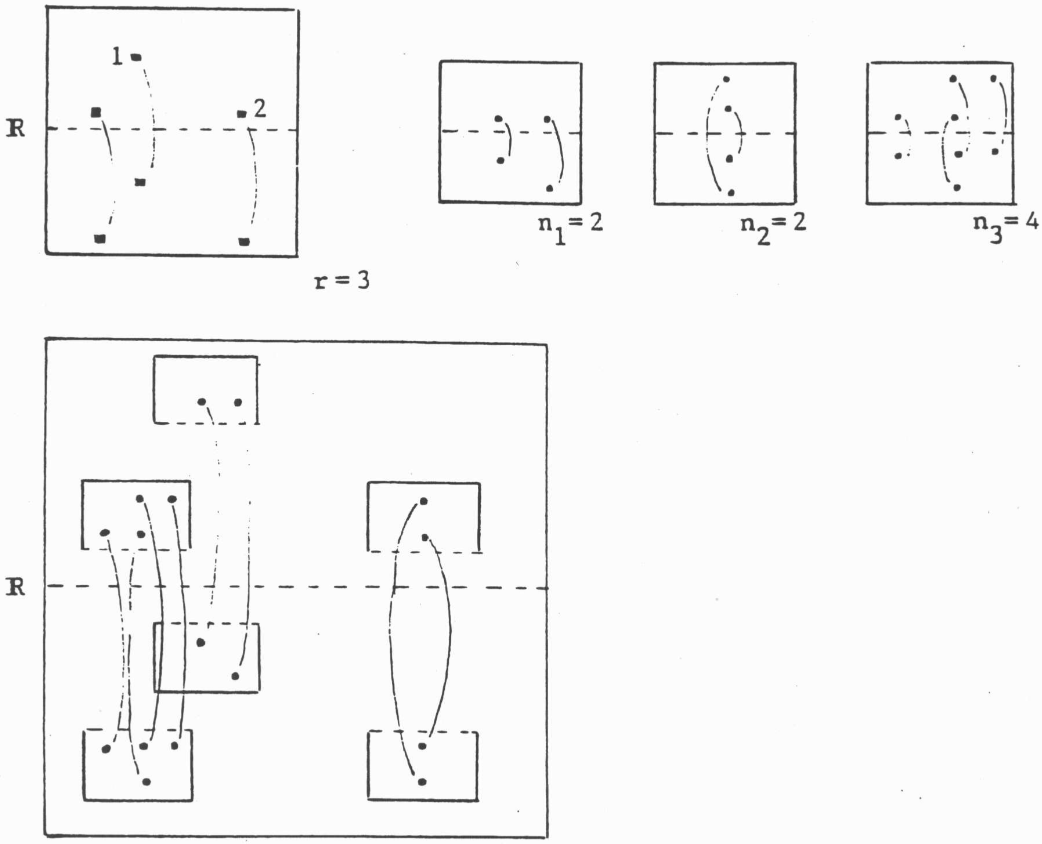
is commutative. ∎

For the associativity we need the following structure maps, which amount to operations themselves,

$$(5.4.2) \qquad \tilde{t} : \tilde{E}^r_\cdot(\mathbb{C}) \times \tilde{E}^{n_1}_\cdot(\mathbb{C}) \times \ldots \times \tilde{E}^{n_r}_\cdot(\mathbb{C}) \longrightarrow \tilde{E}^n_\cdot(\mathbb{C})$$

$n = \Sigma n_i$. The figure below is a better explanation than a formula.

(5.4.3)



Define inclusions $D^1_\cdot : PS\mathbb{C}_\cdot(g) \longrightarrow \tilde{E}^{n_1}(\mathbb{C}) \times_{\Sigma_1} PS\mathbb{C}_\cdot(g)$, $D^1_\cdot(\mathcal{L}) = ((+i,-i),\mathcal{L})$,
and $D^2_\cdot : PS\mathbb{C}_\cdot(g) \longrightarrow \tilde{E}^2(\mathbb{C}) \times_{\Sigma_2} PS\mathbb{C}_\cdot(g)^2$, $D^2_\cdot(\mathcal{L}) = (((+2i,-2i),(+i,-i)),\mathcal{L},\mathcal{L})$.
After these preparations we can formulate the main result.

(5.4.4) Theorem. There are operations

$$\vartheta^n_g : \tilde{E}^n_\bullet(\mathbb{C}) \times_{\Sigma_n} PS\mathbb{C}_\bullet(g)^n \longrightarrow PS\mathbb{C}_\bullet(ng)$$

with the following properties.

(i) (associativity) For all $r \geq 1$, $n \geq 1$, $g \geq 0$ the diagrams

$$\begin{array}{ccc}
\tilde{E}^r_\bullet(\mathbb{C}) \times_{\Sigma_r} (\tilde{E}^n_\bullet(\mathbb{C}) \times PS\mathbb{C}_\bullet(g)^n)^r & \xrightarrow{\ \mathrm{id} \times_{\Sigma_r} (\vartheta^n_g)^r\ } & \tilde{E}^r_\bullet(\mathbb{C}) \times_{\Sigma_r} PS\mathbb{C}_\bullet(ng)^r \\
{\scriptstyle t \times_{\Sigma_r} (\mathrm{id})^r} \big\downarrow & & \big\downarrow {\scriptstyle \vartheta^r_{ng}} \\
\tilde{E}^{rn}_\bullet(\mathbb{C}) \times_{\Sigma_{rn}} PS\mathbb{C}_\bullet(g)^{rn} & \longrightarrow & PS\mathbb{C}_\bullet(rng)
\end{array}$$

(ii) (unit) The composition

$$PS\mathbb{C}_\bullet(g) \xrightarrow{D^1_\bullet} \tilde{E}^1_\bullet(\mathbb{C}) \times_{\Sigma_1} PS\mathbb{C}_\bullet(g) \xrightarrow{\vartheta^1_g} PS\mathbb{C}_\bullet(g)$$

is homotopic to the identity.

(iii) (squaring) The composition

$$PS\mathbb{C}_\bullet(g) \xrightarrow{D^2_\bullet} \tilde{E}^2_\bullet(\mathbb{C}) \times_{\Sigma_2} PS\mathbb{C}_\bullet(g) \xrightarrow{\vartheta^2_g} PS\mathbb{C}_\bullet(2g)$$

is homotopic to $\dot{\mu}_{g,g} \circ \mathrm{diag}$. ∎

As with the Dyer-Lashof operations we appeal to the figure (5.4.3) for a proof of (i). The other two assertions are clear.

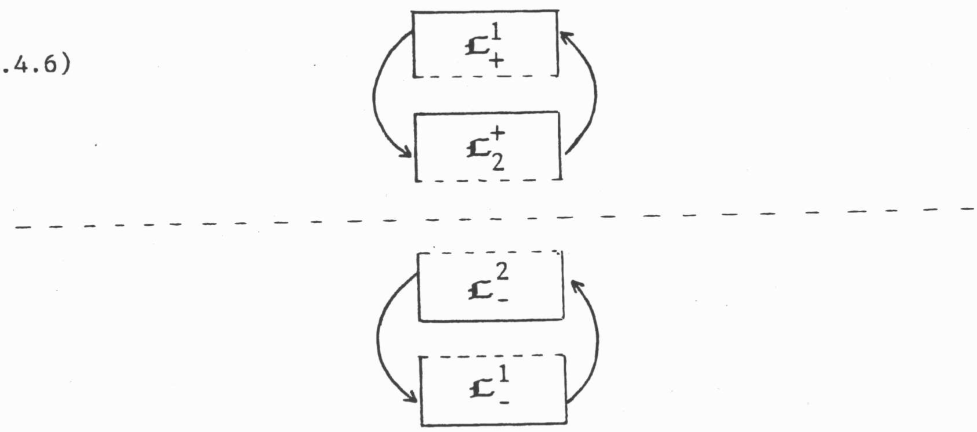
(5.4.5) <u>Proposition.</u> <u>The diagram</u>

$$\begin{array}{ccc} PS\mathbb{C}_{\cdot}(g_1) \times PS\mathbb{C}_{\cdot}(g_2) & \xrightarrow{\dot{\mu}_{g_1,g_2}} & \\ \downarrow \tau & & PS\mathbb{C}_{\cdot}(g_1+g_2) \\ PS\mathbb{C}_{\cdot}(g_2) \times PS\mathbb{C}_{\cdot}(g_1) & \xrightarrow{\dot{\mu}_{g_2,g_1}} & \end{array}$$

<u>is homotopy-commutative, where</u> $\tau(\mathcal{L}_1,\mathcal{L}_2) = (\mathcal{L}_2,\mathcal{L}_1)$.

<u>Proof:</u> The figure

(5.4.6)



shows a homotopy from $\vartheta^2_{g_1,g_2}$ to $\vartheta^2_{g_2,g_1} \circ \tau$. And $\dot{\mu}_{g_1,g_2}$ is homotopic to

$(\mathcal{L}_1,\mathcal{L}_2) \longrightarrow \vartheta^2_{g_1,g_2}((+2i,-2i),(+i,-i),\mathcal{L}_1,\mathcal{L}_2)$. ∎

5.5 Double braid groups.

Recall that $\tilde{E}^n(\mathbb{C})$ and $E^n_{\bullet}(\mathbb{C})$ are Eilenberg-MacLane spaces $K(\pi,1)$; we may call their fundamental groups double braid groups. By (5.4.2) they come equipped with operations, and - as in (3.7.4) - with homomorphisms $\pi_1 E^n_{\bullet}(\mathbb{C}) \longrightarrow \pi_1 PS\mathbb{C}_{\bullet}(g)$, making the operations compatible. But we have not investigated the groups involved.

Chapter 6

The Homology of Symplectic Operations

6.1 Definition of the symplectic operations Q_1' and Q_1'' .

6.2 The operations R_1' and R_1'' .

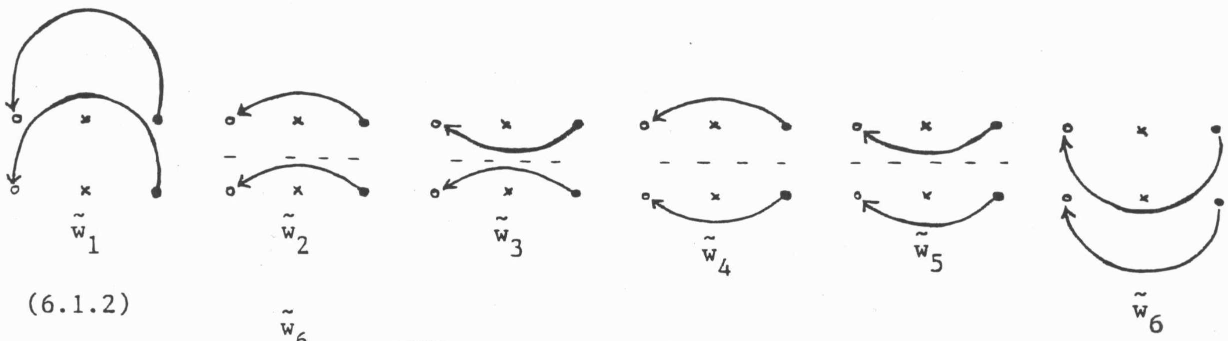
6.3 Some formulas.

The symplectic operation maps will allow to define two more homology operations Q_1' , Q_1'' of degree 1 . There are two binary homology operations R_1' , R_1'' measuring the deviation from linearity. There are similar formulas as for the Dyer-Lashof operations.

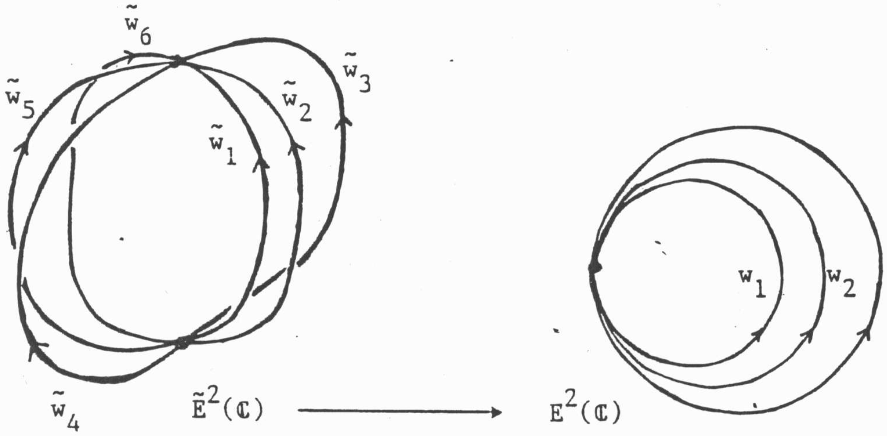
6.1 Definition of the symplectic operations Q_1' and Q_1'' .

For the symplectic operations we restrict ourselves to mod-2 homology $H_*(\ \ ) = H_*(\ \ ;\mathbb{F}_2)$. To define them we need 1-chains $\tilde{w}_\ell$ in $\tilde{E}^2(\mathbb{C})$ which are already in $\tilde{E}^2(\mathbb{C})$ cycles, or which are cycles w_ℓ when projected to $E^2(\mathbb{C}) = \tilde{E}^2(\mathbb{C})/\Sigma_2$. From Lemma (5.1.5) and (5.1.7) we see that $\tilde{E}^2(\mathbb{C})$ is a bouquet of 5 circles. The best way to see them is as the union of 6 intervals, corresponding to the following 6 paths in $\tilde{E}^2(\mathbb{C})$.

(6.1.1)



(6.1.2)



The Σ_2-action o $\tilde{E}^2(\mathbb{C})$ is then the antipodal map, mapping $\tilde{w}_\ell$ to $(\tilde{w}_{z-\ell})^{-1}$ (as an element of the fundamental gruppoid) are inverse to each other. Thus the chains $\tilde{w}_1$ and $\tilde{w}_5$, resp. $\tilde{w}_1$ and $\tilde{w}_4$, resp. $\tilde{w}_2$ and $\tilde{w}_3$ project to the same cycle w_1 , resp. w_2 , resp. w_3 in $E^2(\mathbb{C})$. $\tilde{w}_1$ comes from $\tilde{C}^2(\mathbb{C}) \xrightarrow{2} \tilde{E}^2(\mathbb{C})$, i.e. $\iota_*(\tilde{w}_1) = \tilde{w}_1$ (see (4.1), (3. .)). And $\tilde{w}_2$ and $\tilde{w}_3$ are in

$\tilde{E}^2_{\bullet}(\mathbb{C})$. Therefore we only define with $\tilde{w}_2$ and $\tilde{w}_3$ two new operations of degree one:

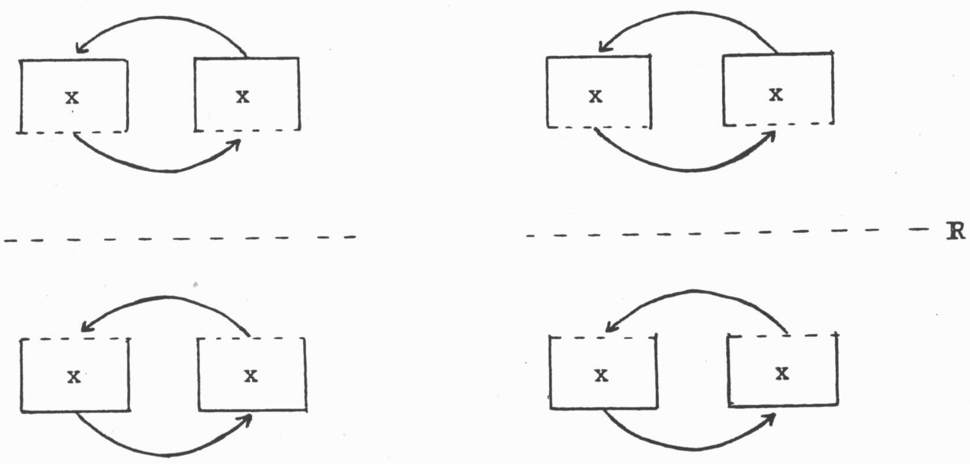
(6.1.3) $$Q_1', Q_1'' : H_q PS\mathbb{C}(g) \longrightarrow H_{2q+1} PS\mathbb{C}(2g) ,$$

$$Q_1'(x) = \vartheta^2_{g*} [\tilde{w}_2 \otimes x \otimes x] \quad \text{and}$$

$$Q_1''(x) = \vartheta^2_{g*} [\tilde{w}_3 \otimes x \otimes x] .$$

The schematic picture is as follows.

(6.1.4)



$Q_1'(x)$ $\qquad\qquad$ $Q_1''(x)$

6.2 The operations R_1' and R_i'' .

The chains $\tilde{w}_{ij} = \tilde{w}_i + \tilde{w}_j^{-1}$ are cycles in $\tilde{E}^2(\mathbb{C})$; their homology classes are subject to relations $[\tilde{w}_{ii}] = 0$, $[\tilde{w}_{ij}] = [\tilde{w}_{ji}]$, $[\tilde{w}_{ij}] + [\tilde{w}_{jk}] = [\tilde{w}_{ik}]$. As a basis one can choose $\tilde{w}_{1\ell}$ for $\ell = 2,\dots,6$. The class $\tilde{w}_{16}$ comes from $\tilde{C}^2(\mathbb{C}) \xrightarrow{\iota} \tilde{E}^2(\mathbb{C})$, and was denoted by $\tilde{v}$ in 4.2 . We define binary operations

(6.2.1) $$R_{ij} : H_p PS\mathbb{C}_\cdot(g_1) \otimes H_q PS\mathbb{C}_\cdot(g_2) \longrightarrow H_{p+q+1} PS\mathbb{C}_\cdot(g_1+g_2) ,$$

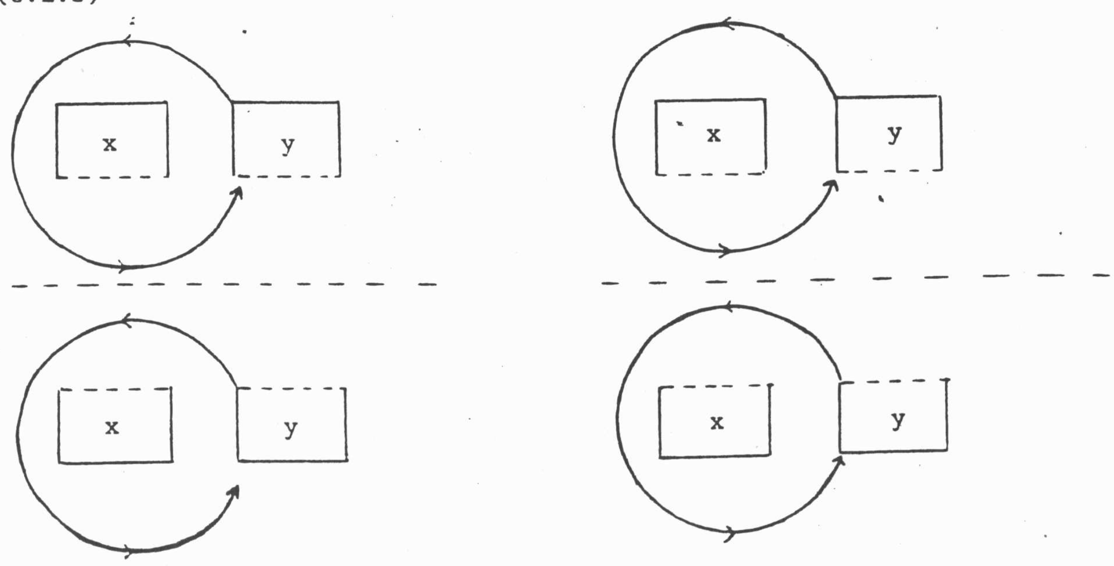
$$R_{ij}(x,y) = \tilde{\vartheta}^\bullet_{g_1,g_2 *}[\tilde{w}_{ij} \otimes x \otimes y] \qquad (1 \leq i,j \leq 6) .$$

There are relations

(6.2.2) $$R_{ii} = 0 , \quad R_{ij} = R_{ji} , \quad R_{ij} + R_{jk} = R_{ik} .$$

We need especially $R_1' := R_{25}$ and $R_1'' := R_{34}$; they will play the role of symplectic Browder operations.

(6.2.3)



$R_1'(x,y)$ $\qquad$ $R_1''(x,y)$

6.3 Some formulas.

The operations Q_1' , Q_1'' are merely functions on homology classes in $H_*PS\mathbb{C}$. The formulas

(6.3.1) (zero) $Q_1'(0) = Q_1''(0) = 0$

(6.3.2) (unit) $Q_1'(1) = Q_1''(1) = 0$, $1 \in H_oPS\mathbb{C}(0)$,

follow directly from the definitions. For

(6.3.3) (linearity) $Q_1'(x+y) = Q_1'(x) + R_1'(x,y) + Q_1'(y)$

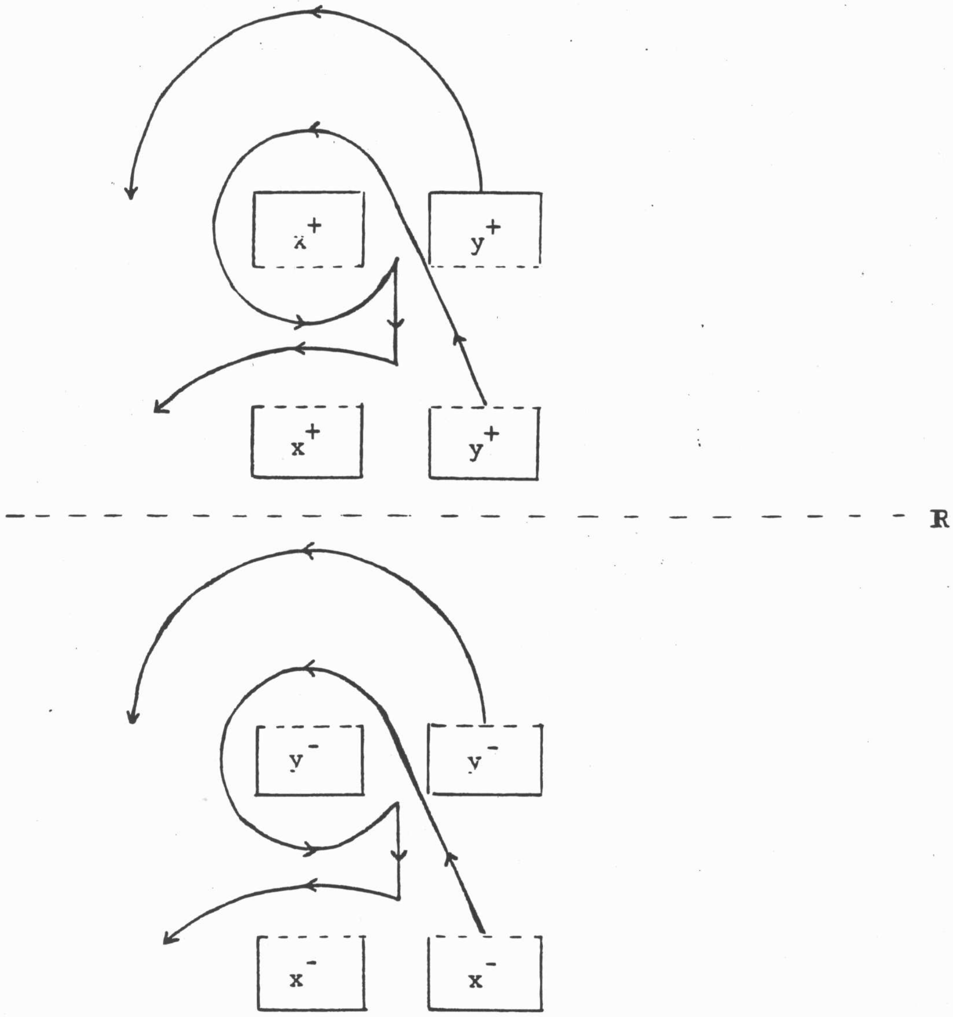
$$Q_1''(x+y) = Q_1''(x) + R_1''(x,y) + Q_1''(y)$$

(6.3.4) (Cartan formula) $Q_1'(xy) = x^2Q_1'(y) + xR_1'(x,y)y + Q_1'(x)y^2$

$$Q_1''(xy) = x^2Q_1''(y) + xR_1''(x,y)y + Q_1''(x)y^2$$

one considers the following figure, analogous to (4.4.3).

(6.3.5)



There are many more formulas of this kind, involving one or several of the operations Q_1' , Q_1'' , R_1' and R_1'' . To connect them to the operation Q_1 , we mention

(6.3.6) $$Q_1(x) = Q_1'(x) + R_{12}(x,x)$$

$$= Q_1''(x) + R_{13}(x,x) \qquad \text{in } H_*PS\mathbb{C}(g) ,$$

which are derived from $[\tilde{w}_1] = [\tilde{w}_2] + [\tilde{w}_{12}] = [\tilde{w}_3] + [\tilde{w}_{13}]$.

Chapter 7

The Geometry of Higher Genus Operations

The operations considered so far had as parameter spaces certain configuration spaces of the complex plane. We will now describe some operations parametrized by configuration spaces of surfaces. This enables us to use the homology of these spaces to induce homology operations; e.g. operations of degree 2.

In a second step we generalize this and the conformal class of the surface. Then the parameter space for operations is a fibrewise configuration space of the universal surface bundle over the moduli space itself.

7.1 Dyer-Lashof operations from a single surface.

Let $\mathfrak{L}^o = [L^o] = [L_1^o, \dots, L_{4g_o}^o; \lambda^o]$ be a fixed parallel slit domain of genus g_o. Away from the dipole $P = \infty$ and the stagnation points $S_1^o, \dots, S_{4g_o}^o$ the associated surface $F(\mathfrak{L}^o)$ has canonical x,y-coordinate charts. The deleted surface

$$(7.1.1) \qquad F^{\smile}(\mathfrak{L}^o) = F(\mathfrak{L}^o) - \{\infty, \bar{S}_1^o, \dots, \bar{S}_{4g_o}^o\} ,$$

where $\bar{S}_i$ denotes the image of S_i^o in $F(\mathfrak{L})$, has $m = m(\mathfrak{L}^o)$ punctures; m can vary from 2 to $2g_o+1$. The configuration spaces

$$(7.1.2) \qquad \begin{array}{ccc} \tilde{C}^n(\mathfrak{L}^o) & = & \tilde{C}^n(F^{\smile}(\mathfrak{L}^o)) \\ \downarrow & & \\ C^n(\mathfrak{L}^o) & = & \tilde{C}^n(F^{\smile}(\mathfrak{L}^o)) / \Sigma_n \end{array}$$

depend on g_o and m. They are Eilenberg-MacLane-spaces by [Fadell-Neuwirth 1962; p.113], since the underlying surface is never closed. Their fundamental groups are sometimes called braid groups of the underlying manifold; presentations are given in [Scott 1970].

There is a new distance function $\varepsilon : C^n(\mathfrak{L}^o) \longrightarrow]0,\infty[$ induced by the maximal-coordinate norm where this time we take the punctures into account,

$$(7.1.3) \qquad \varepsilon(\{e_1, \dots, e_n\}) = \min\{\|z - z'\| \, z, z' \in \{e_1, \dots, e_n, \bar{S}_1^o, \dots, \bar{S}_{4g}^o\}, \ z \neq z'\} .$$

It is now clear what an ε-square $B_\varepsilon(e_i)$ around e_i means. Also the notion of an extended ε-square $\hat{B}_\varepsilon(e_i)$ and the order relation $<$ of (3.5.2) extends to the new situation.

To define the Dyer-Lashof maps in this situation

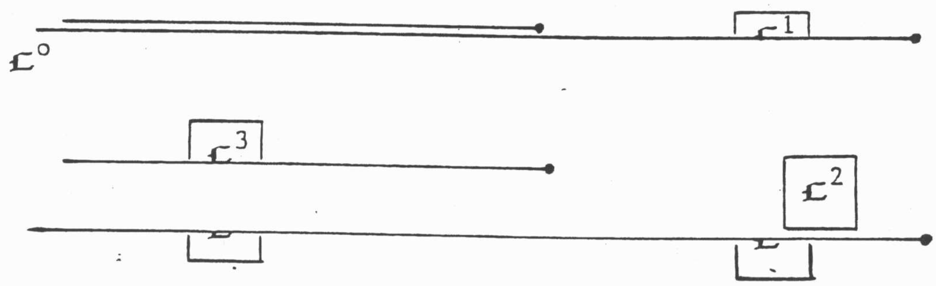
$$(7.1.4) \qquad \vartheta = {}_{\mathcal{L}^o}\vartheta^n_g : \tilde{C}^n(\mathcal{L}^o) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(g_o + ng)$$

we proceed exactly as in (3.5): only we assume that $\mathcal{L}^o$ has been implanted already (in its original position and size) as a first implantation. This defines maps

$$(7.1.5) \qquad \tilde{\vartheta} = \tilde{\vartheta}_{g_1,\dots,g_n} : \tilde{C}^n(\mathcal{L}^o) \times PS\mathbb{C}(g_1) \times \dots \times PS\mathbb{C}(g_n) \longrightarrow PS\mathbb{C}(g_o + g_1 + \dots + g_n$$

which give by Σ_n-invariance (7.1.4) when $g_1 = \dots g_n = g$.

(7.1.6)



If $g_o = 0$ and therefore $\mathcal{L}^o = [\emptyset]$, then $F^{\vee}([\emptyset]) = \mathbb{C}$, and we are in the old situation.

The associativity relations require to let the conformal structure of $\mathcal{L}^o$ vary as an additional parameter. Before doing so we briefly treat the symplectic operations. For the symplectic operations we used the notion of a vertical pair. This is still possible on any surface $F^{\vee}(\mathcal{L}^o)$ by using the harmonic function $h_{\mathcal{L}}$ (which is just the x-coordinate). Thus we can define $\tilde{E}^n(\mathcal{L}^o)$ for $\mathcal{L}^o \in PS\mathbb{C}_{\bullet}(g)$. The condition (5.1.2) (3'), defining the subspace $\tilde{E}^n_{\bullet}(\mathcal{L}^o)$, is now to be interpreted that no point $e_i^{\pm}$ lies on $\mathbb{R}$; that does not exclude a point moving from one half-plane to the other by crossing a bridging pair of slits. The distance function and the notion of an (extended) square also carries over. Proceeding as usual we find operation maps

$$(7.1.7) \qquad \vartheta = {}_{\mathcal{L}^o}\vartheta^n_g : \tilde{E}^n_{\bullet}(\mathcal{L}^o) \times_{\Sigma_n} PS\mathbb{C}_{\bullet}(g)^n \longrightarrow PS\mathbb{C}_{\bullet}(g_o+ng) \ .$$

7.2 The universal surface bundle over $PS\mathbb{C}(g)$.

The space $PS\mathbb{C}(g)$ supports - as the moduli space of directed Riemann surfaces - a universal surface bundle $FS\mathbb{C}(g_o)$. The easiest definition is $EDiff^+(F_g,x) \times_{Diff^+(F_g,x)} F_g \longrightarrow BDiff^+(F_g,x) \simeq B\vec{\Gamma}(g) \simeq PS\mathbb{C}(g)$, which is not useful for our purposes. As a set,

$$(7.2.1) \qquad FS\mathbb{C}(g) = \coprod_{\mathcal{L} \in PS\mathbb{C}(g)} F(\mathcal{L}) \ .$$

To give the topology, let $\bar{\mathbb{C}}_o,\dots,\bar{\mathbb{C}}_{4g}$ be $4g+1$ disjoint copies of $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$; a point in $\bar{\mathbb{C}}_m$ will be denoted by (z,m) . Recall the space RegConf(g) from (I.4.5.1) the gluing rules (I.4.2.3) and the crossings (I.4.3.2) . In $\text{RegConf}(g) \times (\bar{\mathbb{C}}_o \cup \dots \cup \bar{\mathbb{C}}_{4g})$ we consider the subspace given by all $(L;z,m) = (L_1,\dots,L_{4g};\lambda;z,m)$ such that

$$(7.2.2) \qquad \text{Im}(S_m) \geq \text{Im}(z) \geq \text{Im}(S_{m+1}) \ .$$

In terms of the gluing process for $F(L)$, z lies in the m-th strip F^m of L; see (I.4.2.1). For $m = 0$ resp. $4g$ the left resp. right condition is to be disregarded. On this space we consider the equivalence relation $\approx$ generated simultaneously by crossings and gluings:

(7.2.3)

(1) if $(z,m) \sim (z',m')$ in L, then $(L;z,m) \approx (L;z',m')$;

(2) assume $L \approx L'$ by a crossing of L_{k-1} over the pair $L_k, L_{\lambda(k)}$ with $k < \lambda(k)$; set $h_k = i\mathrm{Im}(S_{\lambda(k)} - S_k)$;

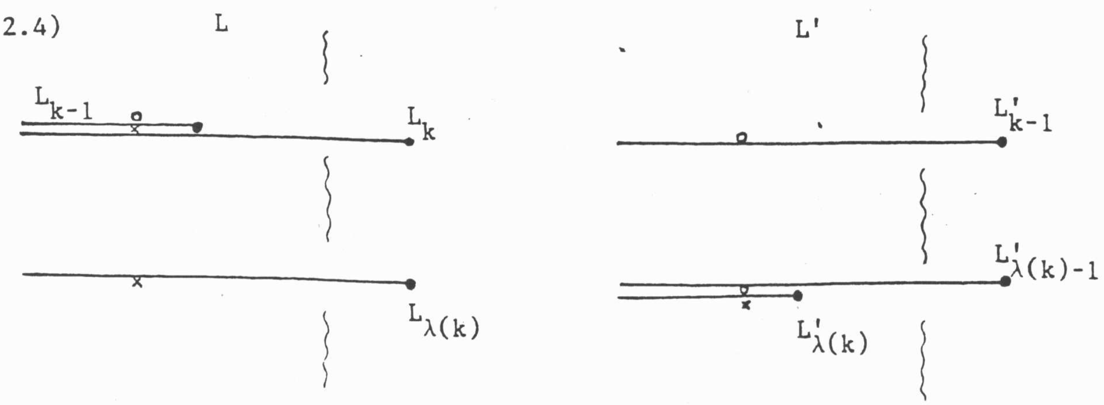
(2a) if $z \notin L_{j-1}, L_k, L_{\lambda(k)}$, then $(L;z,m) \approx (L';z,\rho(m))$ where ρ is the partial cyclic permutation $\rho = (\lambda(k) \ldots k\ k-1)$;

(2b) if $z \in L_{k-1} \subseteq L_k$ and $m = k-2$, then $(L;z,k-2) \approx (L';z,k-2) \approx (L';z+h_k;\lambda(k)-1)$;

(2c) if $z \in L_{k-1} \subseteq L_k$ and $m = k-1$, then $(L;z,k-1) \approx (L';z+h_k),\lambda(k))$

(2d) if $z \in L_{\lambda(k)}$ and $m = \lambda(k)$, then $(L;z,\lambda(k)) \approx (L';z,\lambda(k))$.

(7.2.4)



The map $\pi : FS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g)$, $\pi[L;z,m] = [L]$ is well-defined (and therefore continuous), because the crossings influence the gluings, but not vice versa.

(7.2.6) Proposition. π <u>is a locally trivial fibre bundle with fibres homeomorphic to a surface of genus</u> g .

Proof: We use the components of RegConf(g) (or rather their image in $PS\mathbb{C}(g)$) as parts over which π certainly is a product. Each such component is determined by a pairing function $\lambda \in \Sigma_{4g}$ which is connected in the sense of (I:4.4.6). Note that these components R_λ are not open (in $PS\mathbb{C}(g)$), the boundary consists of such configurations $L = (L_1,\dots,L_{4g};\lambda)$ with $L_{k-1} \subseteq L_k$, or $L_{k+1} \subseteq L_k$ for some k . In R_λ choose as a basepoint the configuration L' with $S'_k = -ki$. We contract R_λ to L' by the following homotopy: a configuration L is translated downwards below the horizontal $\mathrm{Im}(h) = -(4g+1)$; then the slit endpoints S_k are moved along a straight line to $S'_k = -ki$, one after the other, starting with S_1 . Note that L stays in the component R_λ all the time, and that L stays regular (since no forbidden subconfigurations are created). Let $F' = F(L')$ be the fibre over L' . The contraction just constructed lifts to a (tautological) deformation of $\pi^{-1}(R_\lambda)$: a point $(L;z,m)$ over L first follows the translation downwards, then remains constant while the first m slits move to their final position, and finally moves upwards with L_{m+1} , then remains constant again. From fibre to fibre, these are piecewise affine homeomorphisms. If we make the provision that z remains for all time a point on L_{m+1} if it was so at the beginning, then these homotopics extend over a neighbourhood of the boundary of R_λ . ∎

7.3 Operations from families of surfaces.

In each fibre $F(\mathcal{L}) \subseteq FS\mathbb{C}(g)$ the set $\mathcal{S}(\mathcal{L})$ of all stagnation points is the image of the set of slit endpoints S_i $(i = i,\ldots,4g)$ under the gluing identifications. If counted with multiplicities, $\mathcal{S}(\mathcal{L})$ has $2g$ elements. Furthermore, $\mathcal{S}(\mathcal{L})$ varies continuously with $\mathcal{L}$, i.e. there is a section $\mathcal{S} : PS\mathbb{C}(g) \longrightarrow SP_{2g}(\pi)$ into the 2g-fold fibrewise symmetric product of $FS\mathbb{C}(g)$. More precisely, define $\widetilde{SP}_{2g}(\pi) = \{(W_1,\ldots,W_{2g}) \in FS\mathbb{C}(g)^{2g} \mid \pi(W_1) = \ldots = \pi(w_{2g})\}$, and set $SP_{2g}(\pi) = \widetilde{SP}(\pi)/\Sigma_{2g}$. The cardinality of $\mathcal{S}(\mathcal{L})$ varies between 1 and 2g ; thus the image of $\mathcal{S}$ - taken as a union of the sets $\mathcal{S}(\mathcal{L})$ - is a finite, branched covering over $PS\mathbb{C}(g)$.

Similarly, let $\overline{\mathcal{S}}(\mathcal{L}) = \mathcal{S}(\mathcal{L}) \cup \infty_{\mathcal{L}}$ define an element in $SP_{2g+1}(F(\mathcal{L})) \subseteq SP_{2g+1}(\pi)$. We are interested in the complement $F^{\vee}(\mathcal{L}) - \overline{\mathcal{S}}(\mathcal{L})$. Denote by

(7.3.1) $$FS\mathbb{C}^{\vee}(g) = FS\mathbb{C}(g) - \bigcup_{\mathcal{L}} \mathcal{S}(\mathcal{L}) = \bigcup_{\mathcal{L}} F^{\vee}(\mathcal{L})$$

the complement of all stagnation points , and the restriction of π to $FS\mathbb{C}^{\vee}(g)$ is denoted by $\pi^{\vee}$. Although it is not a fibration, we can form the fibrewise configuration space

(7.3.2) $$\tilde{C}^n(g) = \{(W_1,\ldots,W_n) \in FS\mathbb{C}^{\vee}(g)^n \mid W_i \neq W_j \text{ for } i \neq j , \; \pi^{\vee}(W_i) = \ldots = \pi^{\vee}(W_n)\}$$

The $W^i = [L^i,z^i,m^i]$ in such a configuration must have the same $[L_i] = [L]$; so we can write $[L;(z^1,m^1),\ldots,(z^n,m^n)]$. We set

(7.3.3) $$C^n(g) = \tilde{C}^n(g) / \Sigma_n .$$

Both spaces are spaces over $PS\mathbb{C}(g)$,

$$(7.3.4) \qquad \begin{array}{ccc} \tilde{C}^n(g) & \longrightarrow & C^n(g) \\ \downarrow & & \downarrow \\ PS\mathbb{C}(g) & = & PS\mathbb{C}(g) \end{array}$$

with configuration spaces $\tilde{C}^n(F^{\smile}(\mathcal{L}))$ of deleted surfaces as fibres. For each n there are sections

$$(7.3.5) \qquad \tilde{C}^n : PS\mathbb{C}(g) \longrightarrow \tilde{C}^n(g),$$

with $z^k = (a_+([L]) + i(b_+([L]+1)$.

The distance function $\varepsilon : C^n(g) \longrightarrow]0,\infty[$ is determined by the distance functions in the fibres $C^n(F^{\smile}(\mathcal{L}))$.

Our aim are maps

$$(7.3.6) \qquad \tilde{\vartheta} : \tilde{C}^n g_o) \times PS\mathbb{C}(g_1) \times \ldots \times PS\mathbb{C}(g_n) \longrightarrow PS\mathbb{C}(g_o+g_1+\ldots+g_n) \ ,$$

and $$\vartheta : \tilde{C}^n(g_o) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n \longrightarrow PS\mathbb{C}(g_n+ng) \quad ,$$

for any $n \geqq 1$ and $g_i, g \geqq 0$. Again we will not give all details. The left sides are spaces over $PS\mathbb{C}(g_o)$; the fibre over $\mathcal{L}^o$ is $\tilde{C}^n(\mathcal{L}^o) \times PS\mathbb{C}(g_n)$. On this space the map $\tilde{\vartheta}$ was given in (7.1.4). So $\tilde{\vartheta}$ is defined on the entire space. The continuity follows if $\tilde{\vartheta}$ is well-defined with respect to the equivalence classes $\mathcal{L}^o$; we omit this here.

To formulate the associativity let

(7.3.7) $\tilde{t} : \tilde{C}^r(h_o) \times \tilde{C}^{n_1}(h_1) \times \ldots \times \tilde{C}^{n_r}(h_r) \longrightarrow \tilde{C}^n(h)$

with $n = n_1 + \ldots + n_r$ and $h = h_o + h_1 + \ldots h_r$, implant r configurations of respective lengths n_i , lying in some parallel slit domains $\mathcal{L}^i$ of genus h_i , into squares around r points, lying in a parallel slit domain of genus h_o .

(7.3.8) (<u>associativity</u>) <u>For</u> $r,n \geqq 1$, h_o, g_o, $g \geqq 0$ <u>the following diagram is commutative.</u>

$$\begin{array}{ccc}
\tilde{C}^r(h_o) \underset{\Sigma_r}{\times} (\tilde{C}^n(g_o) \underset{\Sigma_n}{\times} PS\mathbb{C}(g)^n)^r & \xrightarrow[\Sigma_r]{id \times (\vartheta)^r} & \tilde{C}^r(h_o) \underset{\Sigma_r}{\times} PS\mathbb{C}(g_o+ng) \\
\Big\downarrow {\scriptstyle t \underset{\Sigma_r}{\times} (id)^r} & & \Big\downarrow {\scriptstyle \vartheta} \\
\tilde{C}(h_o+rg_o) \underset{\Sigma_{rn}}{\times} PS\mathbb{C}(g)^{rn} & \xrightarrow[\vartheta]{} & PS\mathbb{C}(h_o+rg_o+rng)
\end{array} \quad \blacksquare$$

All Dyer-Lashof operations constructed are special cases of these last ones.

For the symplectic operations we proceed similarly. One defines the parameter spaces $\tilde{E}^n_{\cdot}(g)$ as subspaces of $\tilde{C}^{2n}(g)$, $\tilde{E}^n_{\cdot}(g) = \cup \tilde{E}^n_{\cdot}(\mathcal{L})$, $\mathcal{L} \in PS\mathbb{C}_{\cdot}(g)$. then there are operations

(7.3.9) $\tilde{E}^n_{\cdot}(g_o) \underset{\Sigma_n}{\times} PS\mathbb{C}_{\cdot}(g)^n) \longrightarrow PS\mathbb{C}_{\cdot}(g_o+ng)$.

Chapter 8

The Homology of Higher Genus Operations

8.1 Some examples of higher operations.

The parameter spaces $C^n(g)$ for the higher operations have a rich homological structure. For example, it allows to define operations of degree 2 and higher. We confine ourselves to some examples, since we have so far no systematic treatment for these homology operations.

8.1 Some examples of higher operations.

About the homology of the higher genus operations we do have very little systematic knowledge so far. But we can give some justification and examples.

We remark first that the homology $H_*C^n(\mathcal{C}^o)$ of configuration spaces of (deleted) surfaces $F^{\vee}(\mathcal{C}^o)$ is known by [Bödigheimer-Cohen-Taylor 1989], [Bödigheimer-Cohen 1988] and [Bödigheimer-Cohen-Milgram 1989]. The coefficients are in a finite field or in $\mathbb{Q}$. For operations in mod-2 homology one uses $C^2(\mathcal{C}^o)$ as a parameter space. If the genus g_o of $\mathcal{C}^o$ is positive, then there are several two-dimensional classes, in contrast to $C^2(\mathbb{C})$ and $E^2_{\bullet}(\mathbb{C})$. In fact the rank of $H_2C^2(\mathcal{C}^o)$ grows quadratically in g_o . These classes induce operations of degree two.

Note furthermore, that the parameter spaces $C^2(g_o)$ have non-trivial homology; since there are sections to the map $C^2(g_o) \longrightarrow PS\mathbb{C}(g_o)$, the homology of $PS\mathbb{C}(g_o)$ is a direct summand of the homology of $C_2(g_o)$.

We finish with some examples.

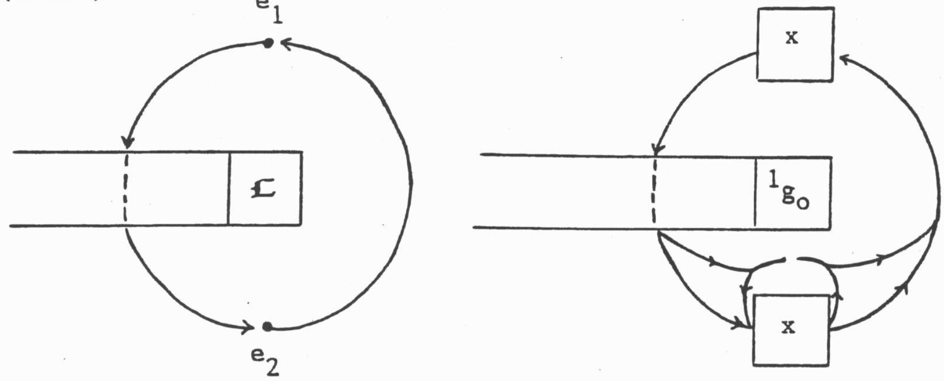
(1) Let $\mathcal{C} \in PS\mathbb{C}(g_o)$ be fixed and consider the curve w in $C^1(\mathcal{C})$ given in figure (8.1.1). If ϑ is the operation $C^1(\mathcal{C}) \times PS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g_o+g)$, the induced homology operation $\theta_1 : H_qPS\mathbb{C}(g) \longrightarrow H_{q+1}PS\mathbb{C}(g_o+g)$, $\theta_1(x) = \vartheta_*[w \otimes x]$, is depicted in the same figure. We clearly have $\theta_1(x) = R_1(1_{g_o}, x)$ for $1_{g_o} \in H_oPS\mathbb{C}(g_o)$.

(8.1.1)

w $\qquad\qquad$ $\theta_1(x)$

(2) For the same $\mathcal{L}$, take w' to be the curve in $\tilde{C}^2(\mathcal{L})$ shown in (8.1.2). Then $\vartheta : \tilde{C}^2(\mathcal{L}) \underset{\Sigma_2}{\times} PS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g_o \times 2g)$ induces $\Theta_1 : H_q PS\mathbb{C}(g) \longrightarrow H_{2q+1} PS\mathbb{C}(g_o+g)$, $\Theta_2(x) = \vartheta_*[w' \otimes x \otimes x]$. This time we find $\Theta_2(x) = Q_1(x) + R_1(1_{g_o}, x)$.

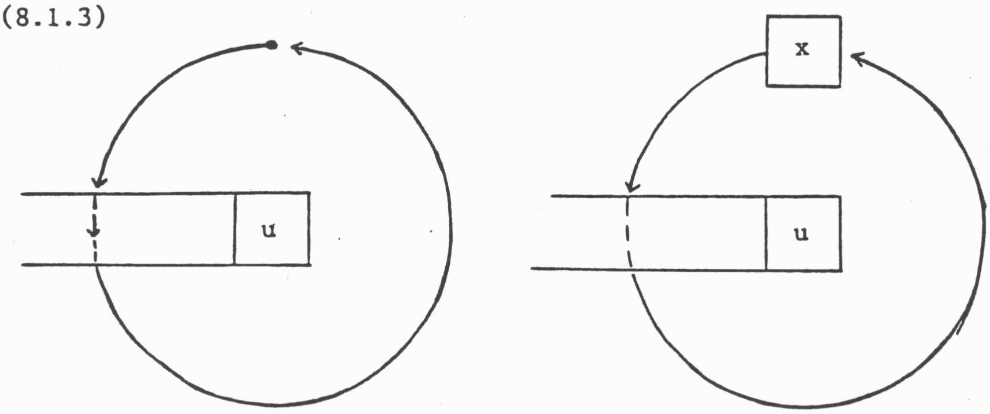
(8.1.2)



w' $\qquad$ $\theta_2(x)$

(3) Assume $u \in H_p PS\mathbb{C}(g_o)$ is some homology class. (8.1.3) gives a (p+1)-dimensional class u' in $C^1(g_o)$. The operation $\Theta_3(x) = \vartheta_*[u' \otimes x]$ induced by $\vartheta : C^1(g_o) \times PS\mathbb{C}(g_o) \longrightarrow PS\mathbb{C}(g_o+g)$ has degree p+1 , $\Theta_3(x) \in H_{q+p+1} PS\mathbb{C}(g_o+g)$ for $x \in H_q(PS\mathbb{C}(g)$. Θ_3 is not new, since $\Theta_3(x) = R_1(u,x)$.

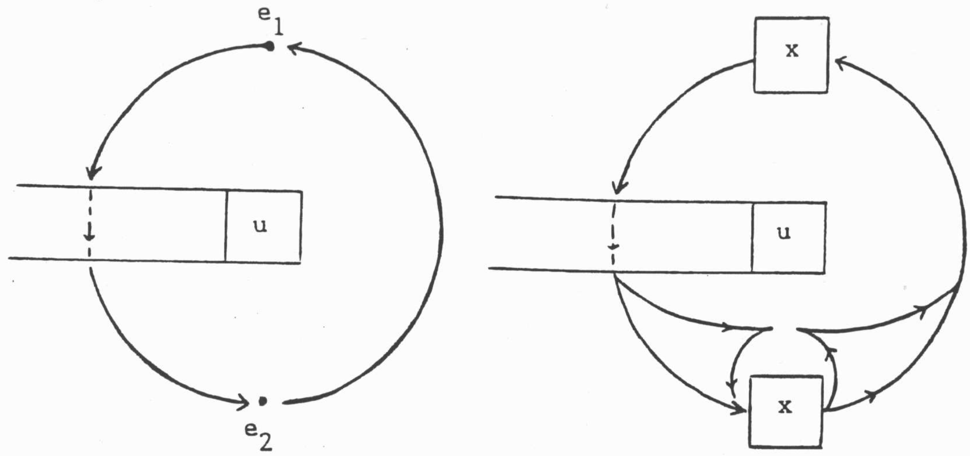
(8.1.3)



u' $\qquad$ $\theta_3(x)$

(4) The next figure then shows an operation $\Theta_4(x) = x\Theta_3(x) + uQ_1(x)$.

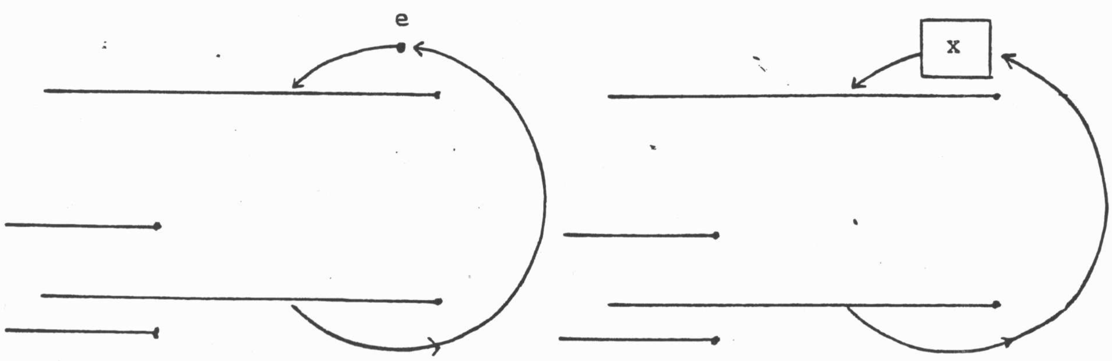
(8.1.4)



$\theta_4(x)$

(5) Let $\mathcal{L} \in PS\mathbb{C}(g_o)$ be fixed, and let v be a closed, non-trivial curve in $F(\mathcal{L})$, as e.g. in (8.1.5). The induced operation $\Theta_5 : H_q PS\mathbb{C}(g) \longrightarrow H_{q+1} PS\mathbb{C}(g_o + g)$ is the first new one.

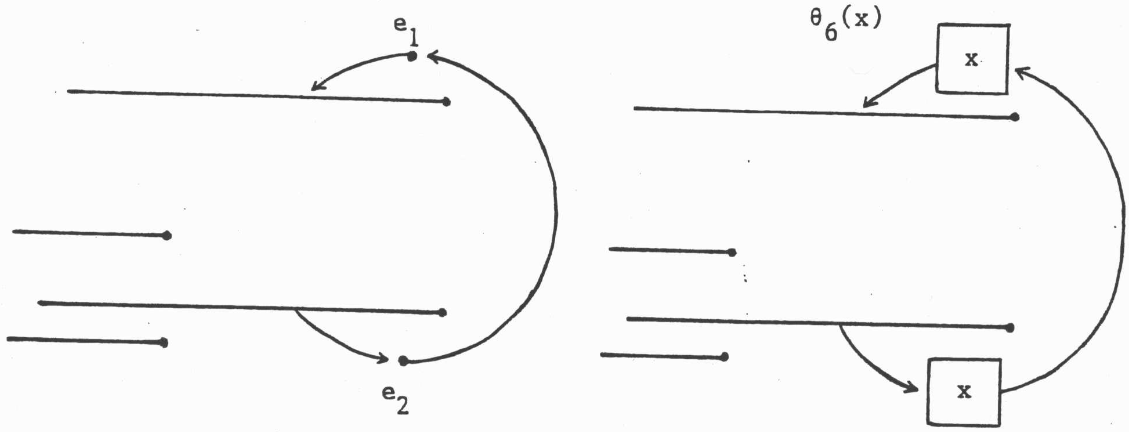
(8.1.5)



v $\qquad$ $\theta_5(x)$

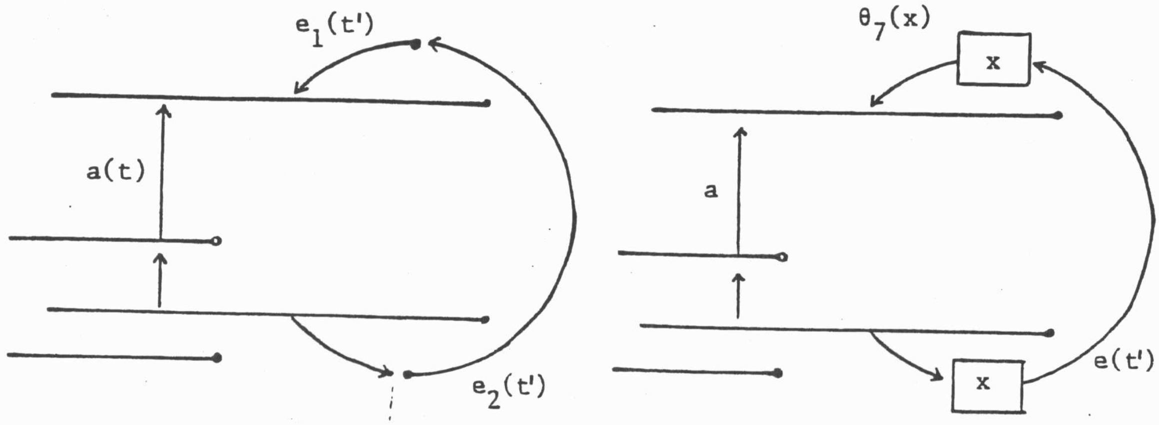
(6) With everything as above, we can also have a $\Theta_6 : H_q PS\mathbb{C}(g) \longrightarrow H_{2q+1} PS\mathbb{C}(2g)$.

(8.1.6)



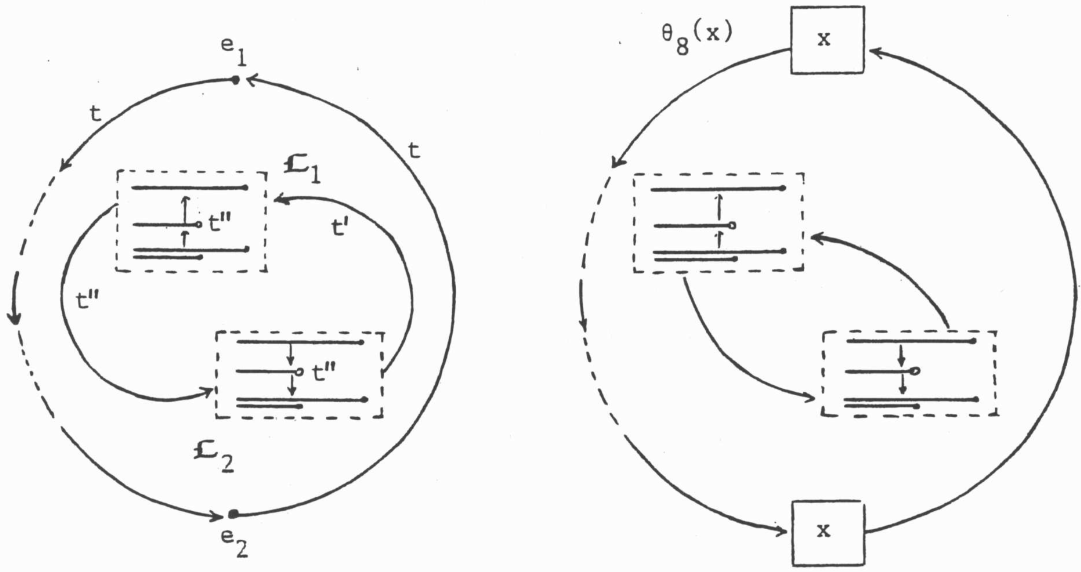
(7) An operation of degree 2 can be constructed as follows. Let a be the curve in $PS\mathbb{C}(1)$ given in (8.1.7). It represents a homology class $[a] \in H_1 PS\mathbb{C}(1)$. Together with the points e_1, e_2 moving in the varying parallel slit domain, we have a torus embedded in $C^2(1)$. The operation $\vartheta : \tilde{C}^2(1) \underset{\Sigma_2}{\times} PS\mathbb{C}(g)^2 \longrightarrow PS\mathbb{C}(2g+1)$ induces $\Theta_7(x) = \vartheta_*[a' \otimes x \otimes x]$, $\Theta_7 : H_q(PS\mathbb{C}(g) \longrightarrow H_{2q+2} PS\mathbb{C}(2g+1)$, where a' is the chain in $\tilde{C}^2(1)$ given geometrically by $(a(t),(e_1(t'),e_2(t')))$.

(8.1.7)



(8) The last figure shows a degree 3 operation $\theta_8 : H_q PS\mathbb{C}(g) \longrightarrow H_{2q+3} PS\mathbb{C}(2g+2)$, parametrized by an embedded 3-manifold in $C^2(2)$, which projects to a Klein-bottle embedded in $PS\mathbb{C}(2)$.

(8.1.8)



References

E. Artin [1925]
Theorie der Zöpfe.
Hamburger Abh. 4 (1925), 47-72 .

J.S. Birman [1974]
Braids, Links and Mapping Class Groups.
(Princeton University Press, Ann. Math. Studies vol.82, 1974)

J. Boardman, R. Vogt [1968]
Homotopy-everything H-spaces.
Bull. Amer. Math. Soc. 74 (1968), 1117-1122 .

C.-F. Bödigheimer, F. Cohen [1988]
Rational cohomology of configuration spaces of surfaces.
in: *Algebraic Topology and Transformation Groups.*
Proc. Conf. Göttingen 1987. (Springer Verlag, Lecture Notes in Math., vol. 1361, 1988, 7-13).

C.-F. Bödigheimer, F. Cohen, L. Taylor [1989]
On the homology of configuration spaces.
Topology 28 (1989), 111-123 .

C.-F. Bödigheimer, F. Cohen, J. Milgram [1989]
On truncated symmetric products.
preprint, Math. Gott. Heft 10 (1989).

C.-F. Bödigheimer [1990]
On the topology of moduli spaces of Riemann surfaces.
Part I: Hilbert uniformization. (2 Teile)
preprint, Math. Gott., Hefte 7+8 (1990).

W. Browder [1960]
Homology operations and loop spaces.
Ill. J. Math. 4 (1960), 347-357 .

F. Cohen [1976]
The homology of C_{n+1}-spaces, $n \geqq 0$.
in: F. Cohen, Th. Lada, P. May
The Homology of Iterated Loop Spaces.
(Springer Verlag, Lecture Notes in Math., vol. 533, (1976), 207-351).

F. Cohen [1987]
Homology of mapping class groups for surfaces of low genus.
Contemp. Math. 58 (1987), 21-30 .

E. Dyer, R.K. Lashof [1962]
Homology of iterated loop spaces.
Amer. J. Math. 84 (1962), 35-88 .

E. Fadell, L. Neuwirth [1962]
Configuration spaces.
Math. Scand. 10 (1962), 111-118 .

J. Harer [1985]
Stability of the homology of the mapping class groups of orientable surfaces.
Ann. Math. 121 (1985), 215-249 .

J. Harer [1986]
The virtual cohomological dimension of the mapping class group of an orientable surface.
Invent. math. 84 (1986), 157-176 .

N.V. Ivanov [1987]
Complexes of curves and the Teichmüller modular group.
Russian Math. Surv. 42 (1987), 55-107 .

J. Maginnis [1987]
Thesis, Stanford University (1987).

J.P. May [1970]
A general algebraic approach to Steenrod operations.
in: *The Steenrod Algebra and Its Applications.*
(Springer Verlag, Lecture Notes in Math., vol. 168 (1970), 153-231).

J.P. May [1972]
The Geometry of Iterated Loop Spaces.
(Springer Verlag, Lecture Notes in Math., vol. 271, 1972)

R.J. Milgram [1966]
Iterated loop spaces.
Ann. Math. 84 (1966), 386-403 .

E. Miller [1986]
The homology of the mapping class group.
J. Diff. Geom. 24 (1986), 1-14 .

G.P. Scott [1970]
Braid groups and the group of homeomorphisms of a surface.
Proc. Cambr. Phil. Soc. 68 (1970), 605-617 .