Chapter 3

Dipol Functions

In this chapter the function theoretic side of the Hilbert uniformization is developed. Based on the main existence theorem for dipol function, i.e. harmonic functions with a dipol singularity $\mathrm{Re}(\frac{1}{z})$ of prescribed strength and direction we study the gradient flow and its critical graph $\mathcal{K}_o$. The main work consists of extending or covering $\mathcal{K}_o$ by a branching graph, i.e. to add to $\mathcal{K}_o$ "virtual integral curves" to obtain a generic graph. Together with the dissected surface $F_o = F - \mathcal{K}_o$ this forms the branching complex from which the original surface can be reobtained. The mapping function w extends to the branching complex, and the image of its boundary will constitute a parallel slit domain, the objects of the next chapter.

3.1 The potential function.

Throughout this chapter let F be a fixed Riemann surface, compact, closed and connected, of arbitrary genus g ; furthermore, a point $P \in F$ and a direction $x \in \vec{T}(F)$ at P is given. The local parameters are denoted by $z_\alpha : Z_\alpha \longrightarrow \mathbb{C}$, $W_\alpha = z_\alpha(Z_\alpha)$ a region in $\mathbb{C}$.

Let $z = x + iy$ be a local parameter in a neighbourhood of the basepoint P . We call z directed if

$$(3.1.1) \qquad z(P) = 0 \ , \qquad \vec{D}z(x) = \frac{\vec{\partial}}{\partial x} = dx$$

holds. By composing with a translation and a rotation any parameter can be directed; two directed parameters are scalar multiples of each other. We need to consider projective classes of directed parameters near P : they are represented by pairs (z,a) , where z satisfies (3.1.1), and a is a positive real number; the equivalence is given by $(z,a) \sim (cz,ca)$ for $c > 0$; an equivalence class is denoted by $\alpha = (z : a)$. There is a bijective correspondence between projective parameter classes and positive real numbers, but there is no canonical identification.

The uniformization principle we will use has a strong heuristic background. Imagine an electrical dipole placed at P , pointing in the direction of x . The result will be a vector field or flow with a single (dipole) singularity and several stagnation points (zeroes). The positions of these stagnation points and the stream lines connecting them are completely determined by the complex structure of F ; vice versa, the positions of the stagnation points and the graph of the connecting stream lines - partly geometric and partly combinatoric data - determines the complex structure. To make this precise we study the potential function for such a flow. Let $\alpha = (z : a)$ and $b \in \mathbb{R}$

be given. A real function $u : F \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \infty$ is called a <u>dipol function</u> for α and b if u satisfies the following conditions:

(3.1.2) (i) u is finite and harmonic on F-P ; i.e.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ;$$

(ii) u has (as a function of z) the form

$$u(z) = \mathrm{Re}(\tfrac{a}{z}) + \phi(z) = \frac{ax}{x^2+y^2} + \phi(z)$$

for some real, smooth function ϕ defined in a neighbourhood of P such that $\phi(0) = b$.

The existence of dipol functions is a classical result in function theory. Earlier based only on heuristic arguments, the existence follows from Dirichlet's principle after Hilbert's re-establishing of the Dirichlet principle: u minimizes the (modified) Dirichlet integral away from P among all continuous, piecewise differentiable functions $h : F\text{-}P \longrightarrow \mathbb{R}$ with finite Dirichlet integral such that h-s is continuous near P , and continuously extendable to P , where $s(z) = \frac{x}{x^2+y^2} + \frac{x}{c^2}$ is a specially adapted function. We refer to [Hilbert 1909], [Courant 1950, p.51-55, 77], [Weyl 1913, §§14,15], [Springer 1957, p.211], [Siegel 1964, p.224-240] or [Farkas - Kra 1980, p.45-48] for a proof of the following statement.

(3.1.3) <u>Proposition.</u> <u>Let</u> F <u>be a Riemann surface,</u> $P \in F$, <u>and</u> x <u>a direction at</u> P . <u>Then for any</u> $\alpha = (z : a)$ <u>and</u> $b \in \mathbb{R}$ <u>there is a unique dipol function.</u> ■

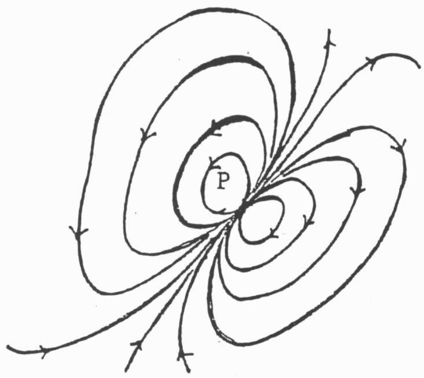
3.2. The critical graph of the gradient flow.

Viewed as a potential u induces a flow

$$(3.2.1) \qquad \Phi = \mathrm{grad}_u = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

in a local parameter $z = x + iy$. Near the dipol P the vector field Φ has stream lines as shown. The index at P is 2 .

(3.2.2)



$m = -2$

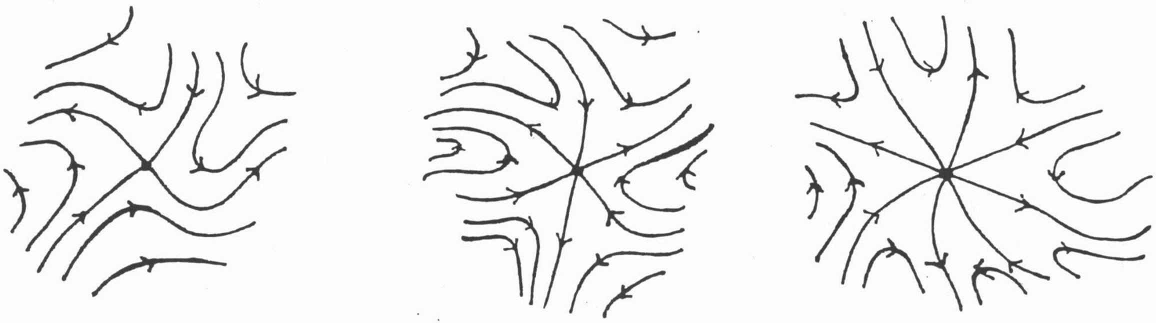
index $= +2$

A critical point S of u is a zero of Φ ,

$$(3.2.3) \qquad \mathrm{grad}_u(S) = 0 \ , \quad \frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial y}(0) = 0$$

for a local parameter $z = x + iy$ with $z(S) = 0$, S is called a stagnation point. Since u is harmonic, it is in some neighbourhood of S the real part of a holomorphic function f with $\frac{\partial f}{\partial z}(z_o) = 0$, hence $f(z) = a_1(z - z_o)^{m+1} + a_2(z - z_o)^{m+2} + \ldots$ for some local parameter z , $z_o = z(S)$ valid in this neighbourhood; the number $m = m(S) \geq 1$ is the multiplicity of S . The following figures show the stream lines near zeroes.

(3.2.4)



$m = 1$ index $= -1$ | $m = 2$ index $= -2$ | $m = 3$ index $= -3$

The indices of all critical points must add up to the Euler characteristic $\chi(F) = 2 - 2g$ of F ; since there is precisely one pole P of index -2 , there are $2g$ zeroes if counted with their multiplicities. We denote the set of stagnation points by $\mathcal{S}$.

To describe the combinatorial structure of the vector field Φ we use the graph of critical stream lines. Let S_o , S_1 , be two distinct points in $\mathcal{S} \cup \{P\}$, and take $t_o = u(S_o)$, $t_1 = u(S_1)$ as points in $\overline{\mathbb{R}}$; assume $t_o < t_1$. A curve $K : [t_o, t_1] \longrightarrow F$ is called an <u>integral curve from</u> S_o <u>to</u> S_1 , if it has the following properties:

(3.2.5) (i) $K(t_o) = S_o$, $K(t_1) = S_1$,

(ii) $u(K(t)) = t$,

(iii) K is continuous on $[t_o,t_1]$ and smooth on $]t_o,t_1[$, and

$$\frac{dK}{d\Theta}(t) = -\mathrm{grad}_u(K(t)) \neq 0 \quad \text{für} \quad t \in]t_o,t_1[\ .$$

Such a K is called an integral curve <u>leaving</u> S_o , or <u>entering</u> S_1 .

The graph $\mathfrak{K}$ has the points of $\mathcal{S}$ and P as vertices; a vertex $S \neq P$, i.e. a stagnation point, is called a <u>finite vertex</u>. There is exactly one edge K from S_o to S_1 , denoted by $K = (S_o,S_1)$ if and only if there is an integral curve K from S_o to S_1 . We denote the set of all edges entering S by $\vec{\mathfrak{K}}(S)$, and the set of all edges leaving S by $\overleftarrow{\mathfrak{K}}(S)$. We have

(3.2.6) $\quad \#\vec{\mathfrak{K}}(S) = \#\overleftarrow{\mathfrak{K}}(S) = m(S) + 1$.

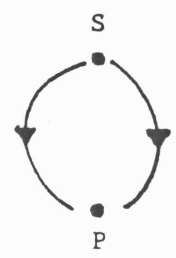
We call an edge <u>critical</u> if it leaves a finite vertex. All vertices together with the critical edges form a subgraph, called the <u>critical graph</u> $\mathfrak{K}_o$.

$\mathfrak{K}$ is a directed graph, imbedded into the surface F ; indeed, this fact will become particularity important.

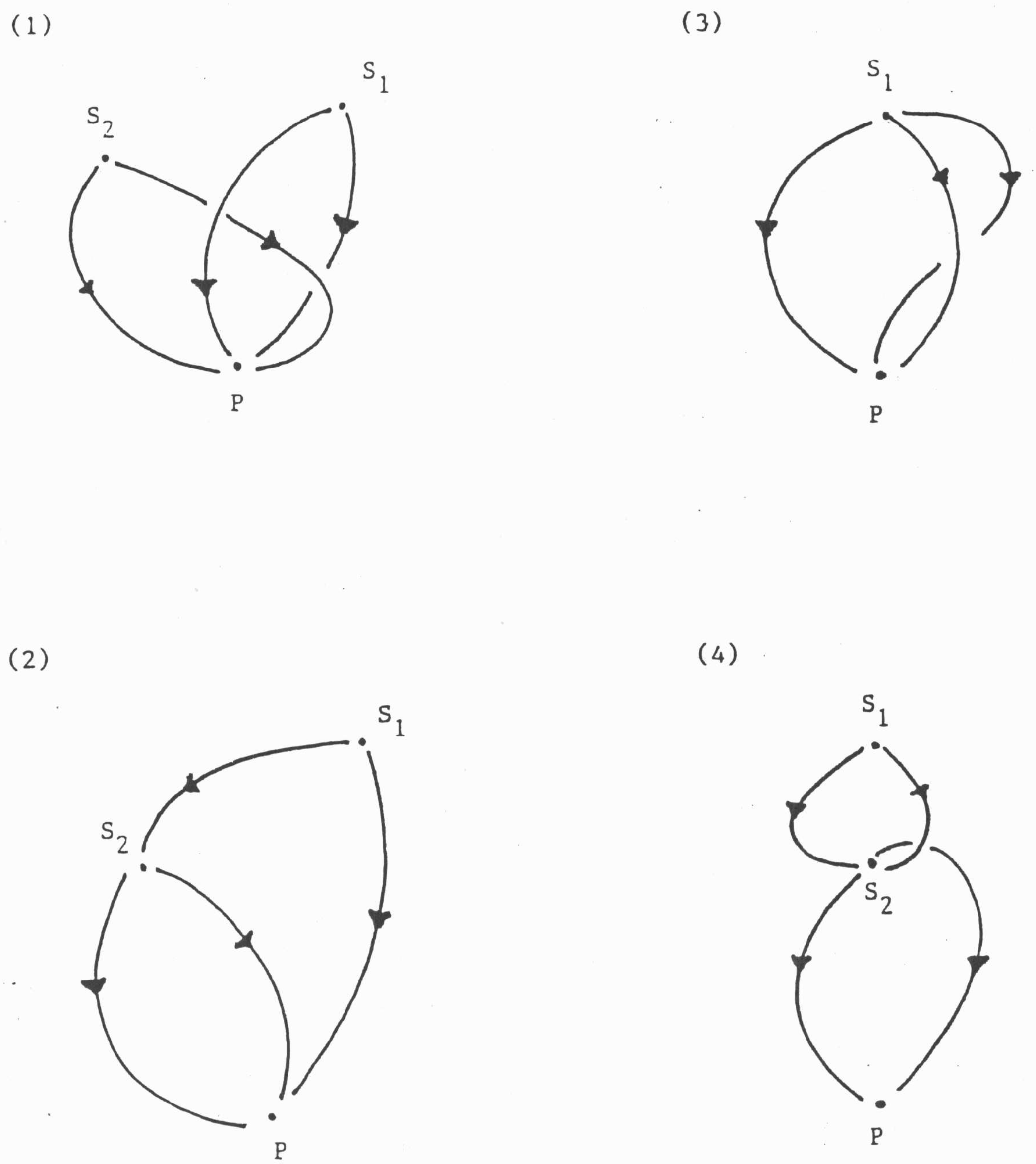
$\mathfrak{K}_o$ being a directed graph makes $\mathcal{S} \cup \{P\}$ a partially ordered set. If $K : S_o \longrightarrow S_1$, we call S_o <u>higher</u> than S_1 . There are, in general, several highest vertices in $\mathfrak{K}_o$; P is the unique lowest vertex P .

We say $\mathfrak{K}_o$ decomposes if there are connected, full subgraphs having all P and only P is common. Of particular interest is the case of a generic $\mathfrak{K}_o$; we call $\mathfrak{K}_o$ <u>generic</u>, if $m(S) = 1$ for all $S \in \mathcal{S}$. Then there are $2g$ vertices, all are highest vertices, and $\mathfrak{K}_o$ decomposes into $2g$ subgraphs as shown in the next figure.

(3.2.7)



(3.2.8) <u>Example</u>. The four possible critical graphs $\mathcal{K}_o$ for $g = 1$.



3.3 The conjugate harmonic function and the mapping function.

As a harmonic function u is, locally, the real part of a holomorphic function f. The conjugate harmonic function $v = \mathrm{Im}(f)$ is locally (up to an additive constant) determined by the Cauchy-Riemann equations $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$.
Before we can integrate these differential forms we must disect F along the critical curves of the flow $\Phi = \mathrm{grad}_u$. This yields an open sub-surface

$$(3.3.1) \qquad F_o = F - \mathcal{K}_o \quad .$$

Using the flow Φ one can construct a contraction of F_o ; in particular, F_o is simply-connected. Choose an arbitrary point $P' \in F_o$ and define

$$(3.3.2) \qquad v(\zeta) = \int_{P'}^{\zeta} [\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx]$$

for $\zeta \in F_o$; the integration is along any path in F_o from P' to ζ.
v is a harmonic function $F_o \longrightarrow \mathbb{R}$. We set

$$(3.3.3) \qquad w = u + iv \quad , \qquad w : F_o \longrightarrow \mathbb{C} \quad .$$

w is a holomorphic function, called the <u>complex potential</u>, or mapping function associated with u.

If we choose a different point $\tilde{P}' \in F_o$ in (3.3.2) to define another harmonic conjugate $\tilde{v}$, then they differ only by the constant

$$(3.3.4) \qquad b_v = v - \tilde{v} = \int_{P'}^{\tilde{P}'} [\frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx] \quad .$$

Thus w can be characterized as follows.

(3.3.5) Proposition. Let a direction x at $P \in F$ be given. For any projective parameter class $\alpha = (z : a)$ and any complex number $b = b_u + ib_v$ there is a unique harmonic function u and a unique holomorphic function w such that

(i) u is defined outside P ,

(ii) w is defined outside the critical graph of grad u ,

(iii) $u = \mathrm{Re}(w)$

(iv) $w(z) = \frac{a}{z} + g(z)$ for some g defined in a neighbourhood of P , with $g(0) = b$. ∎

w has the name mapping function because it is a uniformizer or (local) parameter for the dissected surface F_o with maximal image.

(3.3.6) Proposition. The image of $w = u + iv : F_o \longrightarrow \mathbb{C}$ is the whole complex plane minus a finite number of lines, parall[el] to the real axis, starting at some finite point, and being infinite to the left. ∎

For a proof, see [Hilbert 1909], [Courant 1950], for example. Of course, the missing lines "are" the critical integral curves.

3.4 The boundary of F_o .

We construct a surface $\overline{F}$ with boundary by adding ideal boundary points to $F_o = F - \mathcal{X}_o$. There are two kinds of boundary points. First, consider Cauchy sequences $\{P_n\}_{n \geq 1}$ in F_o which converge in $F - P$ to some point $P_o = \lim\limits_{\rightarrow} P_n$, and which have for any neighbourhood $N \subseteq F - P$ of P_o almost all P_n lie in the same component of $N - \mathcal{X}_o$. Two such sequences $\{P_n\}$ and $\{Q_n\}$ represent the same point if $\lim\limits_{\rightarrow} P_n = \lim\limits_{\rightarrow} Q_n = R$ and for any neighbourhood $N \subseteq F - P$ of R almost all P_n and Q_n lie in one and the same component of $N - \mathcal{X}_o$. We denote the space of equivalence classes by F_1 . The points in $F_1 - F_o$ are called finite boundary points.

Note that v has a unique extension to F_1 . Two equivalent sequences $\{P_n\}$, $\{Q_n\}$ can be deformed into each other within the same component of any $N - \mathcal{X}_o$: Therefore $\lim\limits_{\rightarrow} v(P_n) = \lim\limits_{\rightarrow} v(Q_n)$.

F_1 is not yet a bordered surface; there are finitely many points still missing. To add these in a second step we consider Cauchy sequences $\{P_n\}$ in F_1 ; two such sequences $\{P_n\}$ and $\{Q_n\}$ are regarded as representing the same point if $\lim\limits_{\rightarrow} w(P_n) = \lim\limits_{\rightarrow} w(Q_n)$ in $\mathbb{C}$. Here we use that $w = u+iv$ also has a unique extension to F_1 . We denote the space of these equivalence classes by $\overline{F}$. There is an inclusion

$$(3.4.1) \qquad F_o \longrightarrow F_1 \longrightarrow \overline{F} \ ;$$

the points in $\overline{F} - F_1$ are called infinite boundary points. There is a projection

$$\pi : \overline{F} \longrightarrow F \ , \quad \pi(\{P_n\}) = \lim\limits_{\rightarrow} P_n \ .$$

π is surjective, and $\pi(\bar{P}_0) = P_0$ if $\bar{P} \in F_0$. For $\pi^{-1}(P_0)$ there are the following possibilities:

(3.4.2) (i) If $P_0 \in F_0$, there is only one point over P_0 .

(ii) If $P_0 \in \mathcal{K}_0 - \mathcal{S}$, $P_0 \neq P$, there are two points $\bar{P}^+$ and $\bar{P}^-$ over P_0 ; they correspond to the right and left si of the integral curve K containing P_0 . Thus K has two well-distinguished lifts $\bar{K}^+$, $\bar{K}^-$ to $\bar{F}$.

(iii) If $P_0 \in \mathcal{S}$, then there are $m(P) + 1$ (finite) boundar points above P_0 .

(iv) There are the $4g - \sum_{s \in \mathcal{S}} (m(s)-1)$ (infinite) boundary points above P .

The potential function u extends to $\bar{F}$ by

(3.4.3) $u(\bar{P}) = u(\pi(\bar{P}))$.

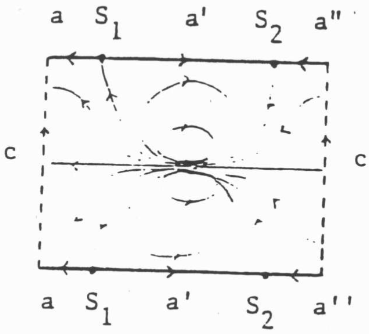
The boundary arcs of $\bar{F}$ are stream lines of the associated flow. We extend v to $\bar{F}$ by the formula

(3.4.4) $v(\bar{P}) = \lim\limits_{\rightarrow} v(P_n)$ of $\{P_n\}$ represents $\bar{P}$.

This is well-defined for the same reason as above. We summarize the results in the following

(3.4.5) <u>Proposition</u>. <u>The complex mapping function extends to a continu function</u> $w = u + iv : \bar{F} \longrightarrow \bar{\mathbb{C}}$. <u>The gradient flow of the real part</u> u $\sum_S (m(S)+1)$ <u>stagnation points and</u> $4g - \sum_S (m(S)-1)$ <u>poles, all in the boundary</u> $\bar{F}-F_0$. <u>The image of boundary arcs between stagnation points an poles are finite or semi-infinite horizontal lines in</u> $\bar{\mathbb{C}}$. ■

(3.4.6)



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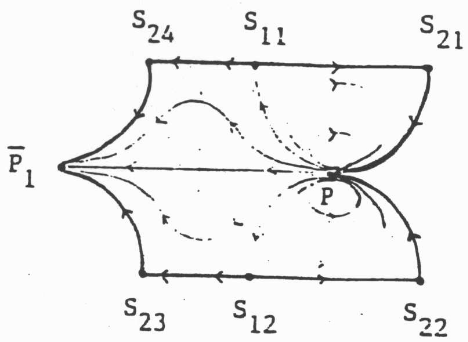


$\bar{F}$

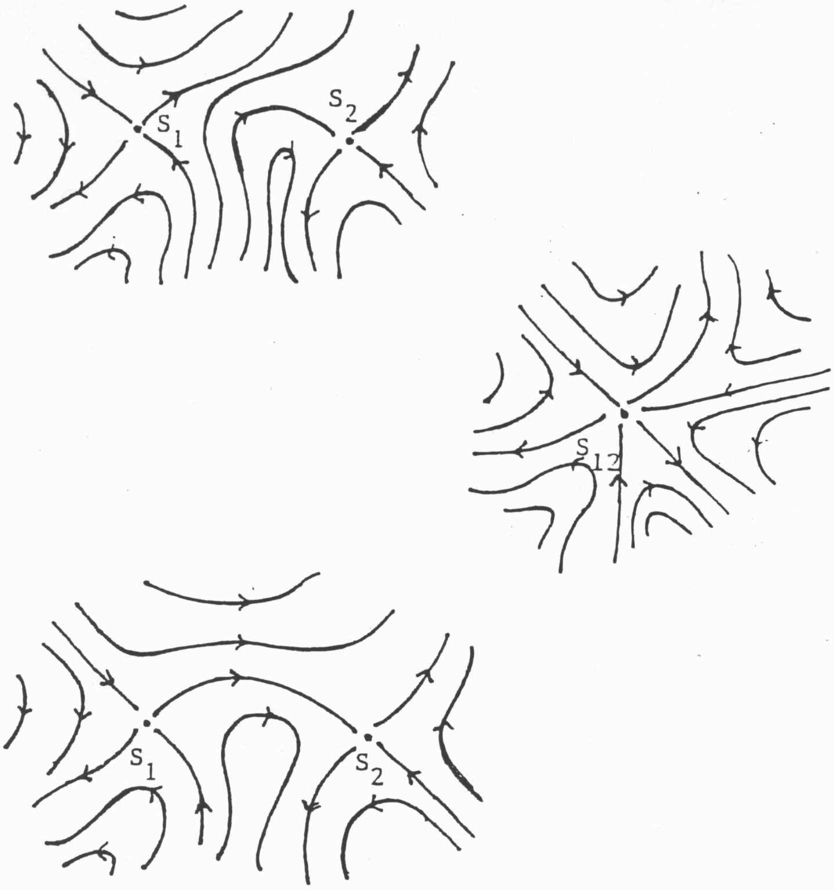
$g = 1$, $m(S_1) = m(S_2) = 1$

The identifications are indicated. The example corresponds to (3.2.8) (4).

3.5. The combinatoric of the branching graph.

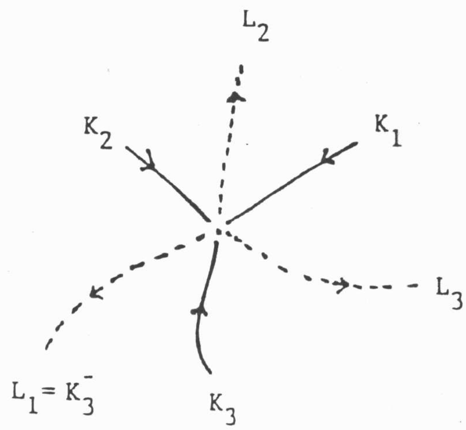
So far we have studied the singularities of the flow Φ_u of the harmonic function $u : F \longrightarrow \bar{R}$ by considering the graph of integral curves between stagnation points and the pole P . The easiest case is when the graph $\mathcal{K}_c$ of critical integral curves is generic. But if $\mathcal{K}_o$ has a stagnation point of multiplicity $m = m(S) \geq 2$, then there are only $m+1$ integral curves entering S , and the same number leaving S . Formally there should be $2m$ curves of each kind, and one should think of S as the sum or limit of m generic stagnation points each of multiplicity one.

(3.5.1)



On the other hand, the fact that $\mathcal{K}_o$ is a surface graph imposes strong conditions on $\mathcal{K}_o$ if merely considered as a graph. But another implication is a kind of orientability; namely the sets $\vec{\mathcal{K}}(S)$ and $\overleftarrow{\mathcal{K}}(S)$ have a unique cyclic ordering induced by the complex structure of F .

(3.5.2)



(We order the entering integral curves counter-clockwise, and the leaving ones clockwise.)

Furthermore, to each entering integral curve K there are two well-distinguished leaving integral curves K^- and K^+, considered as left and right branches into which K splits at S.

Thus $\mathcal{K}_o$ is more than a graph; and we will use the advantage of orientabilit to remedy the disadvantage of degeneracy. We will introduce virtual integral curves and complete $\mathcal{K}_o$ to a formally generic graph. The new graph $\mathcal{L}$ can be mapped to the complex plane by the mapping function $w = u+iv$, and this mapping describes the Riemann surface F.

To formulate the construction of $\mathcal{L}$ we need several notions about finite linear and cyclic sets.

Let $\underline{n}$ stand for the set $\{1,2,\ldots,n\}$. With the natural ordering $1 < 2 < \ldots < n$ it will be denoted by $[\underline{n}]$. A linear order on an arbitrary set A with n elements is then (determined by) a bijective function $\alpha : \underline{n} \longrightarrow A$. A cyclic ordering of A is represented by a bijective function $\alpha : \underline{n} \longrightarrow A$, but two such α_1, α_2 are called equivalent, if $\alpha_2^{-1} \circ \alpha_1$ is a cyclic permutation of $\underline{n}$. Let $\langle\underline{n}\rangle$ denote the set $\underline{n}$ with the (standard) cyclic ordering represented by the identity; we write $1 < 2 < \ldots < n < 1$.

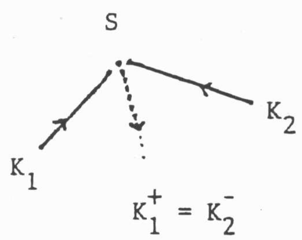
A different way to declare linear or cyclic orderings is with successor relations; in the case of a cyclic set A with n elements this is even a function $\mathrm{succ} : A \longrightarrow A$ with $\mathrm{succ}^k(a) \neq a$ for all $0 < k < n$, and $\mathrm{succ}^n(a) = a$.

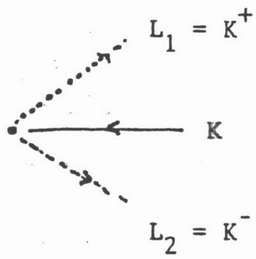
In contrast to linear sets there is of course no maximal or minimal element in a cyclic set.

<u>Example</u>.

The cyclic ordering on $\vec{\mathcal{K}}(S)$ and $\overleftarrow{\mathcal{K}}(S)$ is determined as follows. Let N be a parametric disc around S, containing no other stagnation point nor the pole P, and let $z : N \longrightarrow \mathbb{C}$ be a parameter. $N - \overleftarrow{\mathcal{K}}(S)$ has $m(S)+1$ components. For $K \in \vec{\mathcal{K}}(S)$ there are two integral curves L_1, L_2 leaving S which are in the boundary of the component intersecting K. Assume z is such that $z(S) = 0$, $z(K)$ is contained in the positive reals and $z(L_1)$ lies in the lower half-plane; then $K^- := L_1$ and $K^+ := L_2$. In $\vec{\mathcal{K}}(S)$ we declare: $\mathrm{succ}(K_1) = K_2$ if and only if $K_1^+ = K_2^-$. And in $\overleftarrow{\mathcal{K}}(S)$ we declare: $\mathrm{succ}(L_1) = L_2$ if and only if there is some $K \in \vec{\mathcal{K}}(S)$ with $L_1 = K^+$ and $L_2 = K^-$.

(3.5.3)





The choice of an element a in a cyclic set A with n elements determines a linear ordering by setting $a < \mathrm{succ}(a) < \ldots \mathrm{succ}^{n-1}(a)$. If we write $a_1 \leqq a_2 \leqq a_3$ for three elements of A, we mean $0 \leq k_2 \leq k_3 < n$ when $a_2 = \mathrm{succ}^{k_2}(a_1)$ and $a_3 = \mathrm{succ}^{k_3}(a_1)$. A function $f : A \longrightarrow B$ between two cyclic sets is called monotone, if $a_1 \leqq a_2 \leqq a_3$ always implies $f(a_1) \leqq f(a_2) \leqq f(a_3)$.

Let a , a' be two (distinct) elements of a cyclic set A with n elements such that $a' = \mathrm{succ}^{\ell}(a)$ for some $0 \leq \ell < n$. Then $[a,\dots,a']$ denotes the <u>interval</u> of all $\mathrm{succ}^{k}(a)$, $0 \leq k \leq \ell$. Note that it is a linear set.

Let A_i be a linear set for all $i \in I$. If I is itself linear with elements $i_1 < i_2 < \dots < i_r$, then we denote by

$$(3.5.4) \qquad \underset{i \in I}{[\ ]} A_i = [A_{i_1},\dots,A_{i_r}]$$

the <u>linear union</u> of all A_i , i.e. the disjoint union with the old ordering on each A_i , and $\max A_{i_k} < \min A_{i_{k+1}}$.

If I is only a cyclic set, then we denote by

$$(3.5.5) \qquad \underset{i \in I}{\langle\ \rangle} A_i = \langle A_{i_1},\dots,A_{i_r}\rangle$$

the <u>cyclic union</u> of all A_i , i.e. the disjoint union with the old successor relation extended by $\mathrm{succ}(\max A_i) = \min A_{\mathrm{succ}(i)}$.

Let B be a linear set $\beta : \underline{m} \longrightarrow B$. We introduce an abstract linear set $\check{\beta} : \underline{m+1} \longrightarrow \check{B}$ whose elements are to be interpreted as the gaps of B . The gap between two successive elements b_1, b_2 is denoted by $\overline{b_1 \surd b_2}$; the gap to the left of the minimum $\beta(1)$ of B , resp. to the right of the maximum $\beta(m)$ of B , is denoted by $\overline{\surd \beta(1)}$, resp. $\overline{\beta(m)\surd}$.

Let $\alpha : \underline{n} \longrightarrow A$ be another linear set; a <u>shuffle function</u> is a monotone function $\sigma : A \longrightarrow \check{B}$. It induces a decomposition of A into intervals $A_i := (\check{\beta}^{-1} \circ \sigma)^{-1}(i)$, $i \in \underline{m+1}$, such that $A = \underset{\check{B}}{[\ ]} A_i = [A_1,\dots,A_{m+1}]$.
The <u>shuffling</u> of A into B via σ is the linear union

$$A \underset{\sigma}{>} B := [A_1, \beta(1), A_2, \ldots, \beta(m), A_{m+1}] . \tag{3.5.6}$$

Now we begin with a single vertex S , which is a finite stagnation point, and set $m = m(S)$. Choose a cyclic set $\vec{\mathcal{C}}(S)$ with $2m$ elements and a surjective, monotone function

$$\varepsilon = \varepsilon_S : \vec{\mathcal{C}}(S) \longrightarrow \vec{\mathcal{K}}(S) . \tag{3.5.7}$$

This amounts to viewing each $K \in \vec{\mathcal{K}}(S)$ as a multiple edge such that there is a total number of $2m$ instead of $m+1$, but in addition the set $\varepsilon^{-1}(K)$ of replicas of K has a linear order. ε is called a <u>degeneracy function</u>. If $\varepsilon(C) = K$ we say C lies over K .

Corresponding to these entering edges we introduce a cyclic set $\overleftarrow{\mathcal{C}}(S)$ of $2m$ leaving integral curves, and another <u>degeneracy function</u>

$$\delta = \delta_S : \overleftarrow{\mathcal{C}}(S) \longrightarrow \overleftarrow{\mathcal{K}}(S) \tag{3.5.8}$$

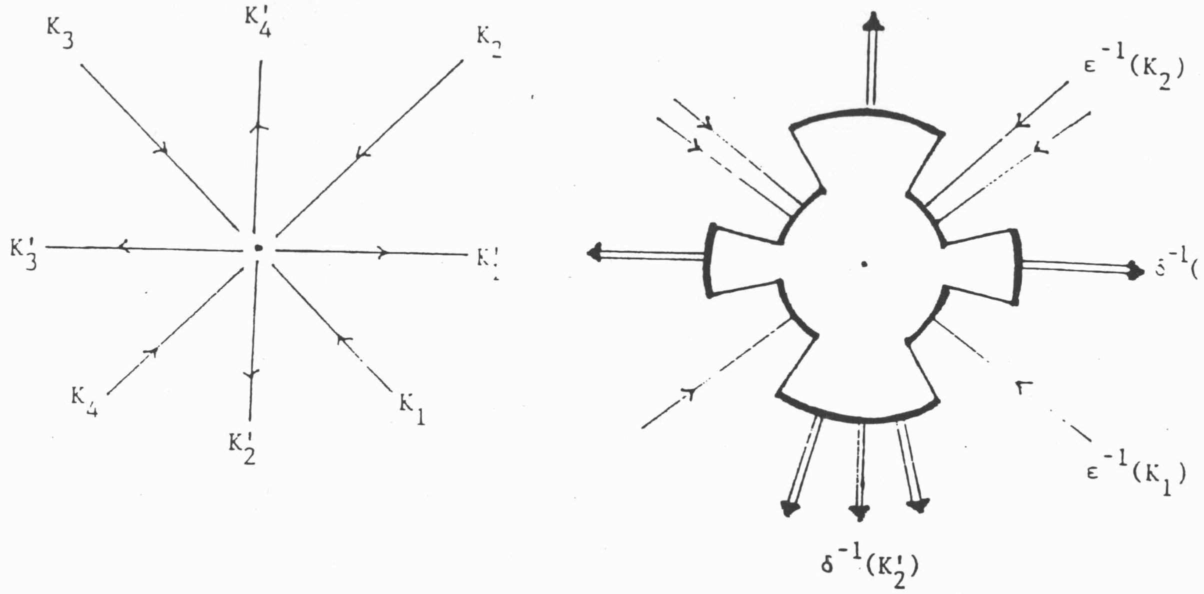
and two <u>branching functions</u>

$$\beta^+ = \beta_S^+ , \quad \beta^- = \beta_S^- : \vec{\mathcal{C}}(S) \longrightarrow \overleftarrow{\mathcal{C}}(S) \tag{3.5.9}$$

such that ε , δ , β^+ and β^- satisfy the following properties.

(3.5.10) (i) ε_S is monotone and surjective.

(ii) δ_S is monotone and surjective.

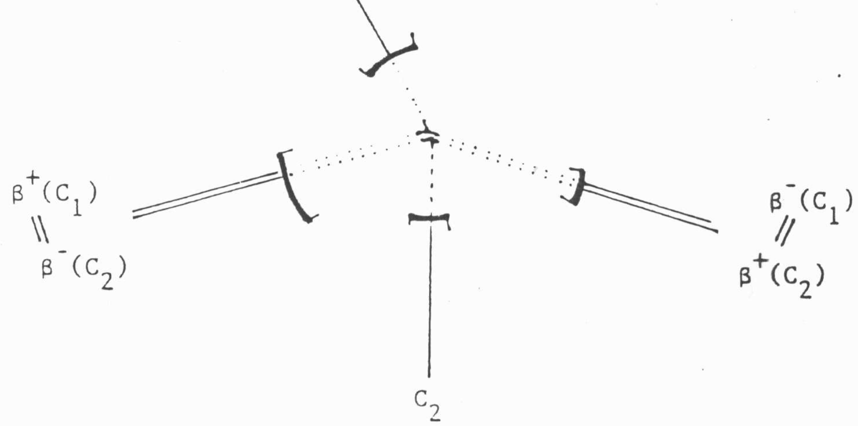


This figure shows a possible distribution of the cyclic sets $\vec{\mathcal{C}}(S)$ (inner circle) and $\overleftarrow{\mathcal{C}}(S)$ (outer circle) over $\vec{\mathcal{K}}(S)$ via ε , resp. over $\overleftarrow{\mathcal{K}}(S)$ via δ . Here $m(S) = 3$. For reasons suggested by the geometry we draw the leaving edges as double lines with a joint arrow.

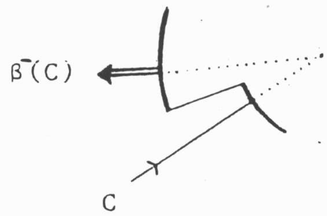
(iii) β^+ and β^- are bijective.

(iv) $\beta^+(C) \neq \beta^-(C)$, for all $C \in \vec{\mathcal{C}}(S)$.

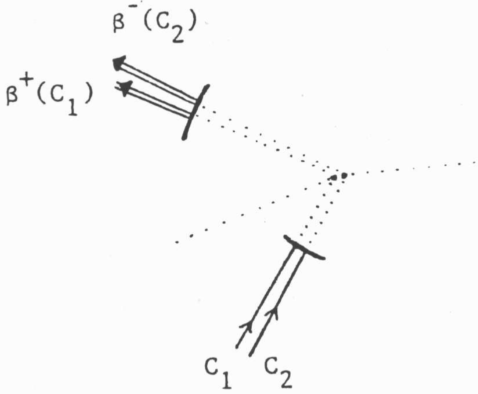
(v) $\beta^+(C_1) = \beta^-(C_2)$ if and only if $\beta^-(C_1) = \beta^+(C_2)$ for all $C_1, C_2 \in \vec{\mathcal{C}}(S)$.



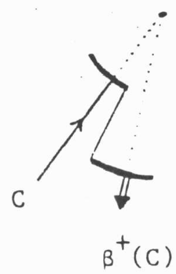
(vi) If C is minimal in $\varepsilon^{-1}(K)$ for some $K \in \vec{\mathfrak{X}}(S)$, then $\beta^{-}(C)$ is minimal in $\delta^{-1}(K^{-})$.



(vii) If $C_1 < C_2$ are successors in $\varepsilon^{-1}(K)$ for some $K \in \vec{\mathfrak{X}}(S)$, then $\beta^{+}(C_1) < \beta^{-}(C_2)$ are successors in $\delta^{-1}(K')$ for some $K' \in \overleftarrow{\mathfrak{X}}(S)$.



(viii) If C is maximal in $\varepsilon^{-1}(K)$ for some $K \in \vec{\mathfrak{X}}(S)$, then $\beta^{+}(C)$ is maximal in $\delta^{-1}(K^{+})$.



Thus we will have the following diagram.

(3.5.11)
$$
\begin{tikzcd}
\vec{\mathcal{K}}(S) & \vec{\mathcal{C}}(S) \arrow[l, "\varepsilon_S"'] \arrow[r, bend left, "\beta_S^-"] \arrow[r, bend right, "\beta_S^+"'] & \overleftarrow{\mathcal{C}}(S) \arrow[r, "\delta_S"] & \overleftarrow{\mathcal{K}}(S)
\end{tikzcd}
$$

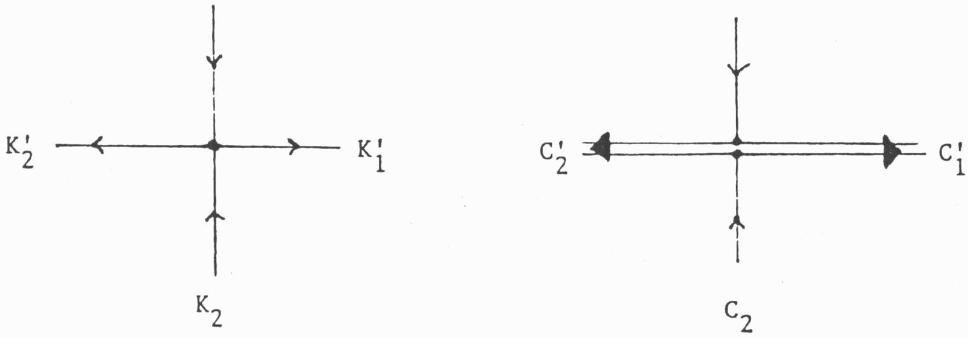
(3.5.12) <u>Lemma</u>. <u>There are degeneracy functions</u> ε_S , δ_S <u>and branching functions</u> β_S^+ , β_S^- <u>with the properties</u> (i) - (viii) <u>above</u>.

We will give an algorithm producing, for any choice of ε_S , the functions δ_S , β_S^+ and β_S^- . Since this involves several other choices, the result is not unique unless $m = 1$.

<u>Proof</u>. Let ε_S be any degeneracy function $\vec{\mathcal{C}}(S) \xrightarrow{\circ} \vec{\mathcal{K}}(S)$, satisfying (i) . The lemma is proved by induction on m .

If $m = 1$, there is no choice at all.

(3.5.13)



With the notation of (3.5.13) we have $\varepsilon(C_i) = K_i$ and $\delta(C_i') = K_i'$ $(i = 1,2)$, $\beta^+(C_1) = \beta^-(C_2) = C_2'$ and $\beta^-(C_1) = \beta^+(C_2) = C_1'$.

Assume $m \geqq 2$. There must be $K \in \vec{\mathcal{K}}(S)$ such that $\#\varepsilon^{-1}(K) = 1$; otherwise $\vec{\mathcal{E}}(S)$ would have more then $2m$ elements. Vice versa, there must be $K' \in \vec{\mathcal{K}}(S)$ such that $\#\varepsilon^{-1}(K') \geqq 2$; otherwise $\vec{\mathcal{E}}(S)$ would have less than $2m$ elements. Thus we can find some K and $K' = \mathrm{succ}(K)$ such that

(3.5.14) $\qquad \varepsilon^{-1}(K) = \{C_1\}$,

(3.5.15) $\qquad \varepsilon^{-1}(\mathrm{succ}(K)) = [C_2,\dots,C_r]$, $\quad 3 \leqq r < 2m$,

hold; then $C_2 = \mathrm{succ}(C_1)$ because of (i) and we write $\vec{\mathcal{E}}(S) = \langle C_1, C_2, \dots, C_{2m}\rangle$
Write $\overleftarrow{\mathcal{E}}(S) = \langle C_1', C_2', \dots, C_{2m}'\rangle$. To satisfy (vi) and (viii) with $C = C_1$ we set

(3.5.16) $\qquad \beta^+(C_1) := C_1'$ and $\delta(C_1') := K^+$,

$\qquad\qquad \beta^-(C_1) := C_2'$ and $\delta(C_2') := K^-$.

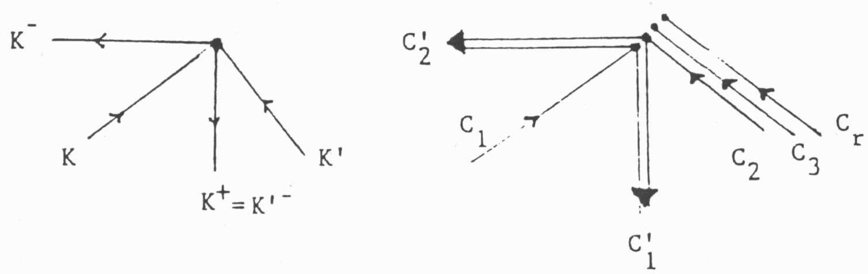
(3.5.17) To satisfy (vi) with $C = C_2$ we set

$$\beta^-(C_2) := C_1' .$$

And (v) forces

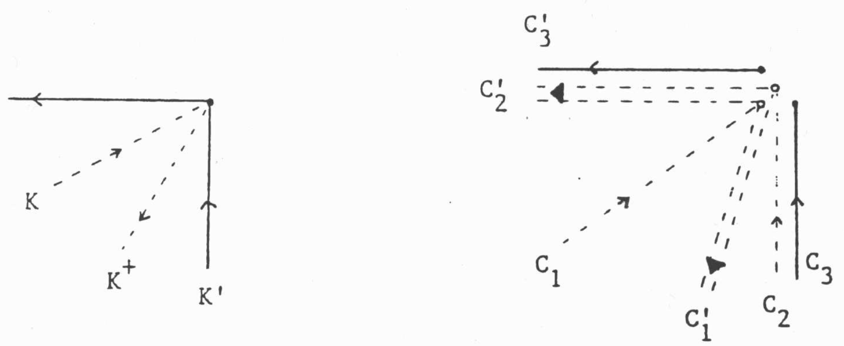
(3.5.18) $\qquad \beta^+(C_2) := C_2'$.

(3.5.19)



Consider the remaining entering edges $\vec{\mathcal{C}}(S) \setminus \{C_1, C_2\} = \langle C_3, \dots, C_{2m} \rangle$ as a cyclic set over the cyclic set $\vec{\mathcal{K}}(S) \setminus \{K\}$, and similarily $\overleftarrow{\mathcal{C}}(S) \setminus \{C'_1, C'_2\} = \langle C'_1, \dots, C'_{2m} \rangle$ over $\overleftarrow{\mathcal{K}}(S) \setminus \{K^+\}$ as cyclic sets.

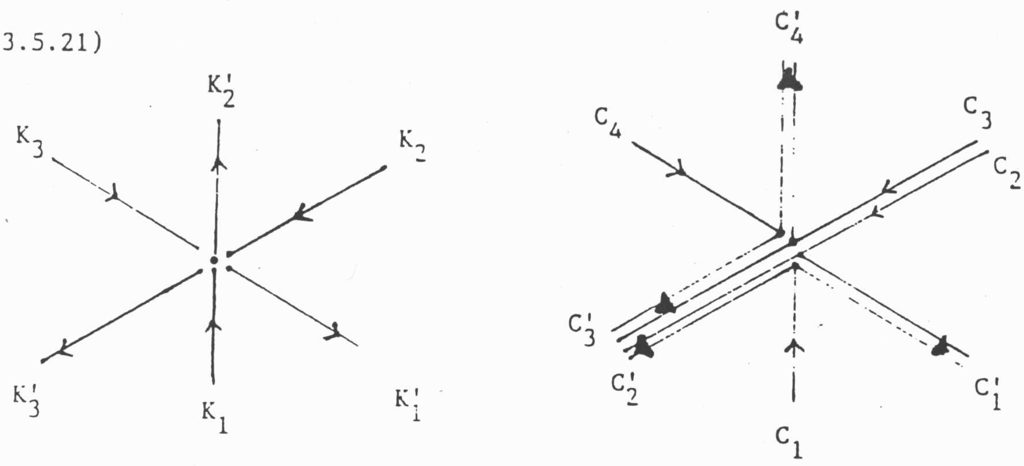
(3.5.20)



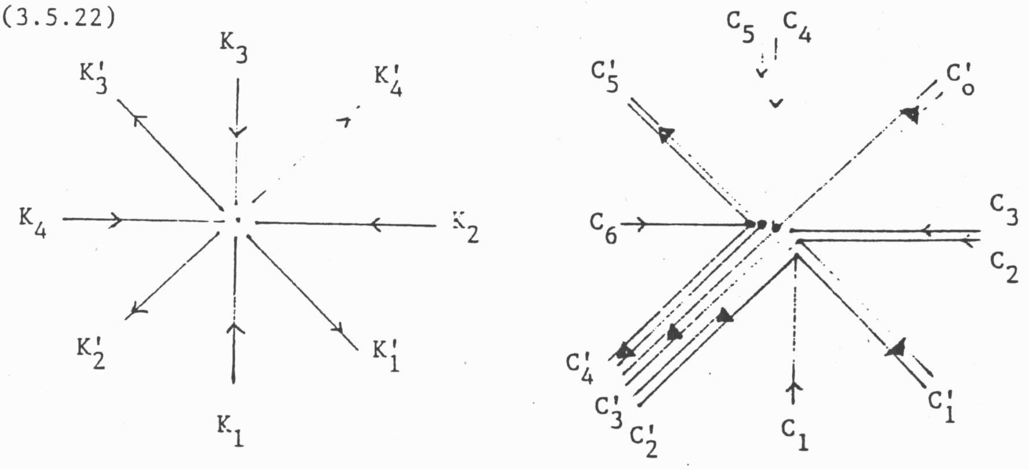
The restriction $\varepsilon' : \vec{\mathcal{C}}(S) - \{C_1, C_2\} \longrightarrow \vec{\mathcal{K}}(S) \setminus \{K^+\}$ is still a degeneracy function, satisfying (i) . By induction there exist extensions δ , β^+ , β^- of our definitions (3.5.16 - 18) to all of $\vec{\mathcal{C}}(S)$, resp. $\overleftarrow{\mathcal{C}}(S)$, with the properties (i) - (viii) in the restricted situation. Obviously these extensions are surjective and monotone in the case of δ , and bijective in the case of β^- , β^+ ; thus (ii) and (iii) hold. (iv) is true by inspection; (v), (vi) and (viii) hold by construction. Note that (vii) corresponds to (vi) in the restricted situation. ∎

The following figures show examples for this algorithm.

(3.5.21)



(3.5.22)

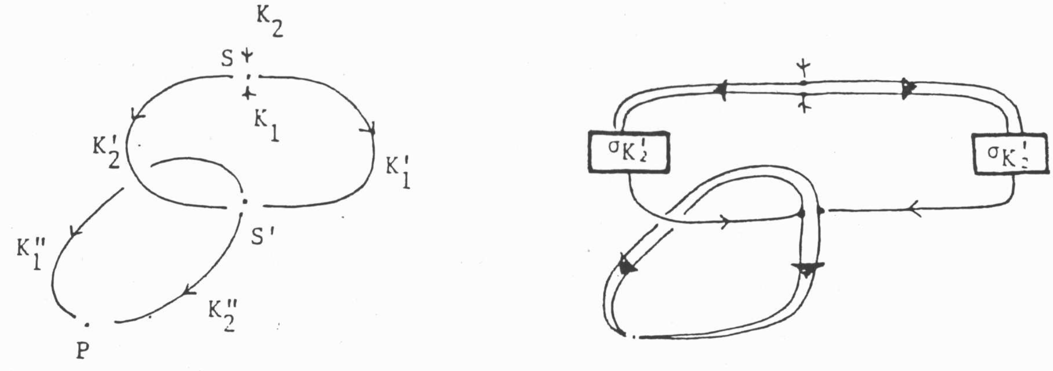


Next we combine the data so far constructed for each vertex S . Let $K = (S_1, S_2)$ be an edge in $\mathcal{K}_0$. The set $\delta_{S_1}^{-1}(K)$ is a linear subset of $\overleftarrow{\mathcal{C}}(S_1)$, and $\varepsilon_{S_2}^{-1}(K)$ is a linear subset of $\vec{\mathcal{C}}(S_2)$. We choose a shuffle function

(3.5.23) $$\sigma = \sigma_K : \delta_{S_1}^{-1}(K) \longrightarrow \varepsilon_{S_2}^{-1}(K)^{\smile} .$$

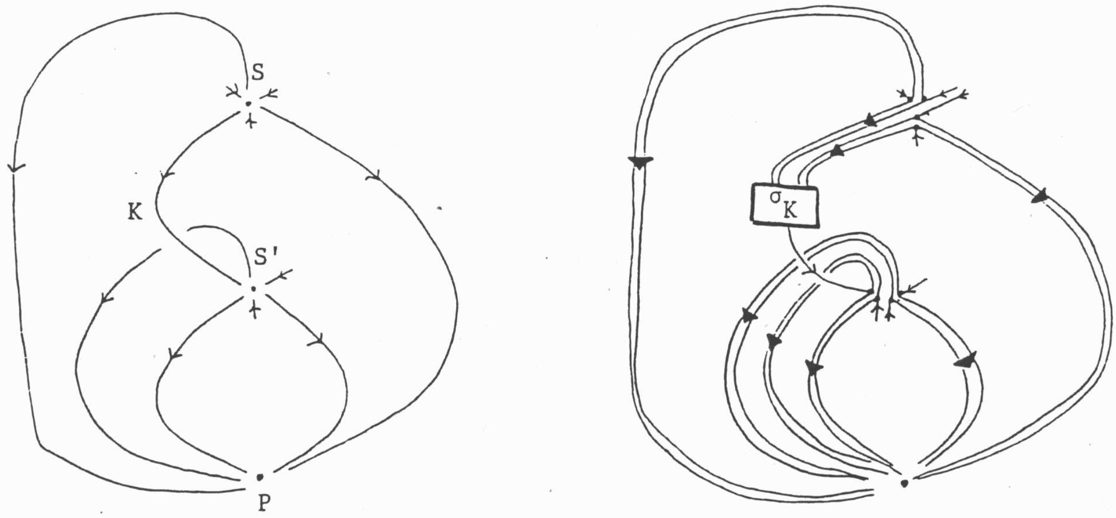
The following figures show examples.

(3.5.24)



$g = 1$, $m(S) = m(S') = 1$

(3.5.25)



$g = 2$, $m(S) = m(S') = 2$

Out of the functions ε_S , δ_S , β_S^+ and β_S^- for each finite vertex S of $\mathcal{K}_o$, and of the functions σ_K for each edge K between finite vertices of $\mathcal{K}_o$ the <u>branching graph</u> $\mathcal{B}$ is constructed.

The vertices of $\mathcal{B}$ are the elements of $\coprod \vec{\mathcal{C}}(S)$, S a finite vertex, and an additional vertex $\bar{P}$. We write

(3.5.26) $\mathcal{L}(S) := \overleftarrow{\mathcal{C}}(S)$,

$$\mathcal{L}(P\ ) := \{\bar{P}\ \}$$

$$\mathrm{vert}(\mathcal{L}) := \coprod_{S} \mathcal{L}(S) \amalg \{\bar{P}_o\}$$

for the set of <u>vertices over S</u> , resp. of all vertices. $\bar{P}$ is the only vertex over P .

For each vertex $C \neq \bar{P}$ there will be exactly two edges $B^+(C)$ and $B^-(C)$ from C to $\bar{P}$, called the right and left branch of C ; there will be no other edges.

(3.5.27) $$\mathrm{edg}(\mathcal{L}) := \coprod_{C} \{B^+(C),B^-(C)\}$$

This describes $\mathcal{L}$ as a graph completely.

But we need an edge B of $\mathcal{L}$ to have an edge path $(K_1,K_2,\dots,K_n)$ in $\mathcal{K}_o$ from S to P associated to it, when B is $B^+(C)$ or $B^-(C)$, and C lies over S . Therefore we will define sets $\mathcal{L}(K)$ for each edge K in $\mathcal{K}_o$ whose elements are the edges B of $\mathcal{L}$; and their will be, for any B , exactly one edge path $(K_1,\dots,K_n)$ such that $B \in \mathcal{L}(K_i)$ for $i = 1,\dots,n$. Corresponding to the cyclic orderings on the $\mathcal{L}(S)$ there will be linear ordering on each $\mathcal{L}(K)$.

Let S be any vertex in $\mathcal{K}_o$. We duplicate $\overleftarrow{\mathcal{C}}(S)$ to obtain

(3.5.28) $$\overleftarrow{\mathcal{C}}^{\pm}(S) = \langle\ \rangle_{C \in \overleftarrow{\mathcal{C}}(S)} [C^-,C^+]$$

where $[C^-,C^+]$ is the linear set consisting of two copies of C with the ordering $C^- < C^+$. As a new degeneracy function one has

(3.5.29) $$d_S = d : \overleftarrow{\mathcal{C}}^{\pm}(S) \longrightarrow \overleftarrow{\mathcal{K}}(S) , \quad d(C^-) = d(C^+) := \delta(C) .$$

Because of this doubling we drew the leaving edges always as double lines. Now let S be a highest vertex in $\mathcal{K}_o$. For an edge $K = (S,S')$ leaving S we define

(3.5.30) $$\mathcal{L}(K) := d^{-1}(K) = \bigsqcup_{d(C)=K} [C^-,C^+]$$

as a subset of $\overleftarrow{\mathcal{C}}^{\pm}(S)$.

Here we identify the edge $B^+(C)$ for $C \in (S) \subseteq \mathrm{ver}(\mathcal{L})$ with $\beta^+(C)^+ \in \overleftarrow{\mathcal{C}}^{\pm}(S)$ and $B^-(C)^- \in \overleftarrow{\mathcal{C}}^{\pm}(S)$. Note that $\mathcal{L}(K)$ is a linear set. The shuffle function

$$\overleftarrow{\mathcal{C}}(S) \supseteq \delta^{-1}(K) \xrightarrow{\sigma_K} \varepsilon_{S'}^{-1}(K)^{\check{}}$$

extends to a new shuffle function

(3.5.31) $$s_K = s : \mathcal{L}(K) = d^{-1}(K) \longrightarrow \varepsilon_{S'}^{-1}(K)^{\check{}} , \quad s(C^-) = s(C^+) = \sigma(C) .$$

Assume that S' is a finite vertex in $\mathcal{K}_o$, and that for all edges $K = (S,S')$ in $\mathcal{K}_o$ we have defined linear sets $\mathcal{L}(K)$ and shuffle functions

$$s_K : \mathcal{L}(K) \longrightarrow \varepsilon_{S'}^{-1}(K)^{\check{}} .$$

Let $K' = (S',S'')$ be an edge leaving S' . Consider a gap γ of the linear set $d_{S'}^{-1}(K') = [C_1^-,C_1^+,C_2^-,C_2^+,\ldots,C_r^-,C_r^+]$ of the special form

(3.5.32)
$$\gamma_o = \overline{}\surd\overline{C_1^-} ,$$
$$\text{or} \quad \gamma_i = \overline{C_i^+}\surd\overline{C_{i+1}^-} , \quad (i = 1,\ldots,r-1)$$
$$\text{or} \quad \gamma_r = \overline{C_r^+}\surd\overline{} .$$

They are linearly ordered (by their index). Gaps of this form correspond to gaps of $\vec{\mathcal{C}}(S')$ via

$$\gamma_o' = \overline{}\sqrt{(\beta^-)^{-1}C_1} \;,$$

resp. $\gamma_i' = \overline{(\beta^+)^{-1}C_i \sqrt{(\beta^-)^{-1}C_{i+1}}} \;, \qquad (i = 1,\dots,r-1)\;,$

resp. $\gamma_r' = \overline{(\beta^+)^{-1}C_r \sqrt{}} \;.$

Note that these γ_j' are indeed gaps of different $\varepsilon_{S'}^{-1}(K')$ because β^- , β^+ have the properties (2.7.10)(vi) - (viii) . Thus there is an interval

(3.5.33) $\qquad \mathcal{L}_\gamma := s_{K'}^{-1}(\gamma') \subseteq \quad (K')$

assigned to each $\gamma = \gamma_o,\dots,\gamma_r$. And we define

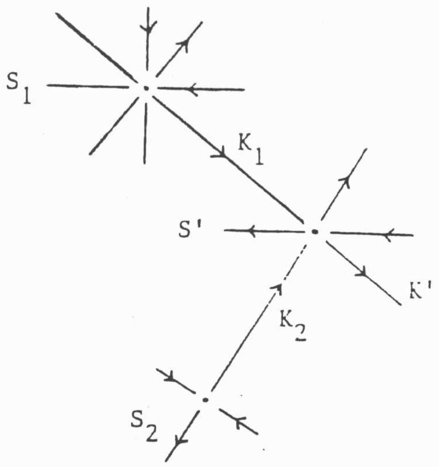
(3.5.34) $\qquad \mathcal{L}(K') := [\;]_\gamma \,\mathcal{L}_\gamma = [\mathcal{L}_{\gamma_o},\dots,\mathcal{L}_{\gamma_r}]$

where the union is over all gaps in $d^{-1}(K')$ of the forms (3.5.31) .

Since S" is allowed to be P in this last step, this finishes the construction of $\mathcal{L}$.

The following figure shows the shuffling at a vertex S' over an edge K' = (S',S") ; here S_1 and S_2 are highest vertices in $\mathcal{K}_o$.

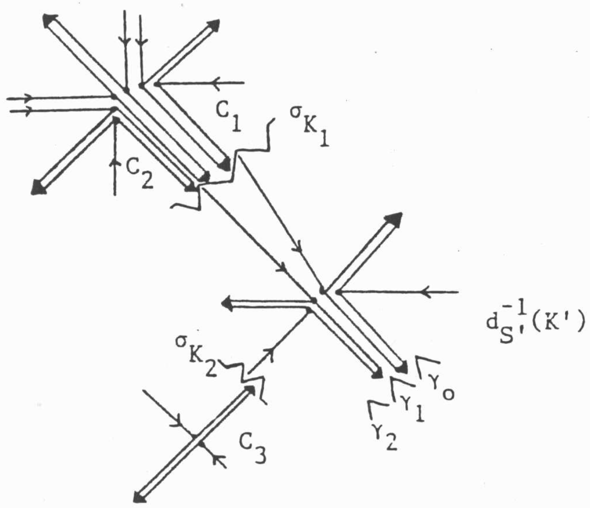
(3.5.35)



$m(S_1) = 3$

$m(S') = 2$

$m(S_2) = 1$



$\gamma_1' = [C_1^-, C_1^+, C_2^-, C_2^+]$

$\gamma_o' = \emptyset$

$\gamma_2' = [C_3^-, C_3^+]$

The figure shows the doubles C^- , C^+ still with one arrow , since for the shuffling the gaps $\overline{C^-}\sqrt{C^+}$ are disregarded.

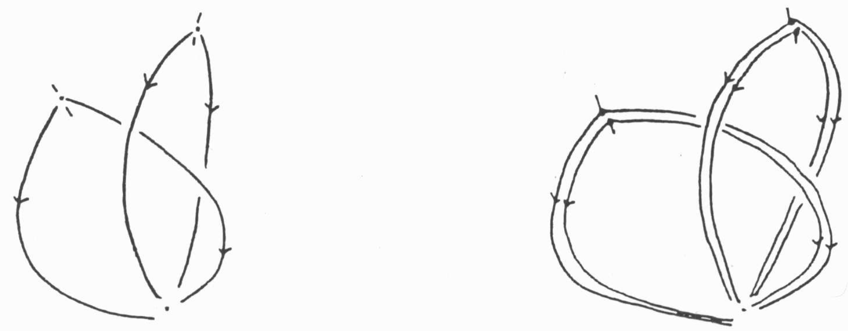
Summing up we have:

- a cyclic set $\mathcal{L}(S)$ over each vertex S in $\mathcal{K}_o$, and $\bar{P}_o$ alone over P_o ;
- a degeneracy function $\varepsilon_S : \mathcal{L}(S) \longrightarrow \vec{\mathcal{K}}(S)$;
- a linear set $\mathcal{L}(K)$ over each edge K in $\mathcal{K}_o$;
- a shuffle function $s_K : \mathcal{L}(K) \longrightarrow \varepsilon^{-1}(K)^{\sim}$.
- The set of vertices is $\mathrm{vert}(\mathcal{L})$ consists of $\bar{P}_o$ and all elements in some $\mathcal{L}(S)$; thus the vertices are in addition also distributed over the entering edges of $\mathcal{K}$ via $\varepsilon : \mathrm{vert}(\mathcal{L}) \longrightarrow \mathrm{edg}(\mathcal{K})$.
- The edges of $\mathcal{L}$ consist of a vertex $C \in \mathcal{L}(S)$ and an edge path $(K_1,\dots,K_n)$ from S to P_o in $\mathcal{K}_o$; abstractly $B=(C;K_1,\dots,K_n)$ is an element of $\mathcal{L}(K)$ precisely if $K = K_i$ for some $i = 1,\dots,n$.

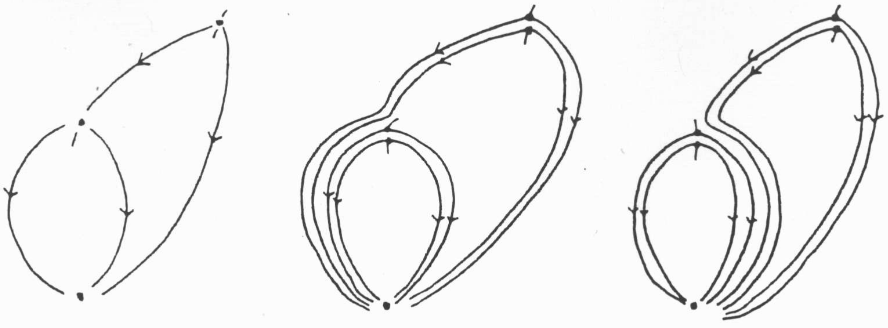
As final examples the next figure shows all possibilities of branching graphs for $g = 1$. The cases (1) - (4) correspond to the table (3.2.8) ; (1) generic case, (2) 2 choices for σ_K , (3) 3 choices for ε_S , (4) 2 choices for each of σ_{K_1} , σ_{K_2} .

(3.5.36)

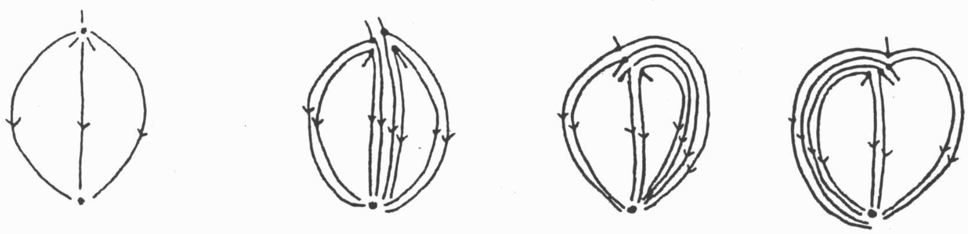
(1)



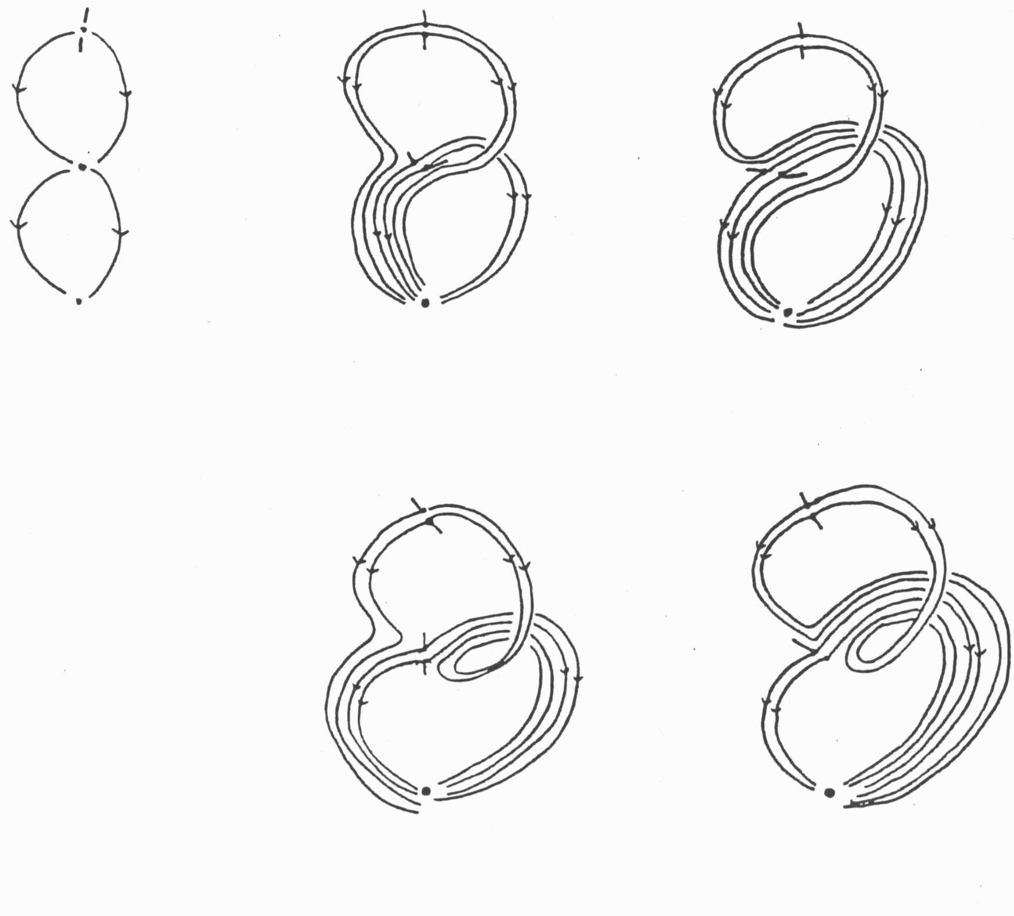
(2)



(3)



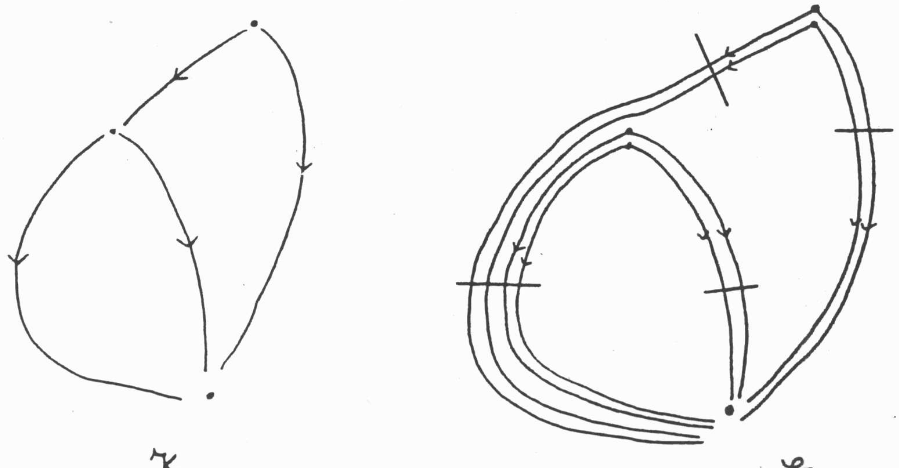
(4)



3.6 The branching complex.

The graph $\mathcal{L}$ associated to the degeneracy functions ε_S , δ_S , the branching functions β_S^+ , β_S^- , and the shuffle functions σ_K will now be regarded as a one-dimensional complex. The vertices $C \in \mathcal{L}(S)$, $S \in \mathcal{S}$, and $\bar{P}_o$ are the 0-cells. For each edge $B = (C;K_1,\ldots,K_n)$ with $K_i = (S_i,S_{i+1})$, $S_1 = S$ and $S_{n+1} = P$ we take a copy of $[-\infty,u(S_1)]$ and identify $-\infty$ with $\bar{P}$, and $u(S_1)$ with C . The curves $K_i : [u(S_{i+1}),u(S_i)] \longrightarrow F$ give an injective map $\pi : B \cong [-\infty,u(S_1)] \to F$, $\pi(t) = K_i(t)$ if $u(S_{i+1}) \leqq t \leqq u(S_i)$. Topologically, $\mathcal{L}$ is a bouquet of circles formed by $B_C = B^+(C) \cup B^-(C)$; since there are $2m(S)$ vertices C over S in $\mathcal{L}$, we have $\sum_S 2m(S) = 4g$ such circles; all have the point $\bar{P}$ and only $\bar{P}$ in common.

(3.6.1)



$\mathcal{K}_o$ $\qquad\qquad$ $\mathcal{L}$

Recall the bordered surface $\pi : \bar{F} \longrightarrow F$ over F , constructed as the closure of the open surface $F_o = F - \mathcal{K}_o$. The two harmonic functions u, v are extended to $\bar{F}$, and $u(\pi(z)) = z(z)$ for all $z \in \bar{F}$, and $v(\pi(z)) = v(z)$ for all z in the interior of $\bar{F} = \pi^{-1}(F_o)$. In the boundary of $\bar{F}$ there are $m(P\ )+1$ infinite boundary points, all lying over the pole P . We identify all infinite boundary points to a single point $\bar{P}$, and identify $\bar{P} \in \bar{F}$ with $\bar{P} \in \mathcal{L}$; this attaches the bouquet $\mathcal{L}$ to $\bar{F}$. The two-dimensional complex $\bar{F} \cup \mathcal{L}$ is called the <u>branching complex</u>.

Extending the functions u and v from $\bar{F}$ to $\bar{F} \cup \mathcal{L}$ is easily done for u by

(3.6.2) $\qquad u(t) := t \quad \text{for} \quad t \in B^+(C) \cong B^-(C) \cong [-\infty, u(S)] \quad \text{if} \quad C \in \mathcal{L}(S).$

Note that $u(\pi(t)) = u(t)$ because of (3.2.5)(iii).

To define v on $\mathcal{L}$ remember that $B^+(C)$ and $B^-(C)$ are to be thought of as virtuell integral curves of the flow $\bar{\Phi}_u$. Hence v will be constant on these 1-cells, and moreover with the same constant value $v(C)$ on $B^+(C)$ and $B^-(C)$. Defining these values is an inductive procedure. Let M be the set of all (non-critical) edges $K = (P\ ,S)$ in $\mathcal{K} - \mathcal{K}_o$; and let $\vec{\mathcal{C}}_M$ be the set of all vertices C of $\mathcal{L}$ such that $\varepsilon(C) \in M$. If $C \in \vec{\mathcal{C}}_M$ then v is defined on $K = \varepsilon(C)$, or in other words there is a unique lift $\bar{K}$ to $\bar{F}$; we set

(3.6.3) $\qquad v(C) = v(\varepsilon_S(C)) = v(K) = v(\bar{K}) \ , \qquad C \in \vec{\mathcal{C}}_M \ .$

It is implied that v has the constant value $v(C)$ along the 1-cells $B^+(C)$ and $B^-(C)$.

We need the following fact for the inductive step.

(3.6.4) Let $K' = (S,S')$ be an edge leaving S , and let C_1, C_2 be two vertices of $\mathcal{L}$ over S such that $B^+(C_1)$ and $B^-(C_2)$ are successive in $\varepsilon^{-1}(K'')$ for some (non-critical) $K'' = (S'',S)$. This follows from (3.5.10)(vii) .

There are three immediate consequences of (3.5.3, 4) which we formulate as hypotheses for the induction to come. First recall that there are two lifts $\overline{K}^+$ and $\overline{K}^-$ of K to $\overline{F}$, since K is critical.

(3.6.5) If B_o is minimal in $\mathcal{L}(K')$, then $v(B_o) = v(\overline{K}^-)$.

If $B_1 = B^+(C_1)$ and $B_2 = B^-(C_2)$ are successive in $\mathcal{L}(K')$ then $v(B_1) = v(B_2)$.

If B_m is maximal in $\mathcal{L}(K')$, then $v(B_m) = v(\overline{K}^+)$.

In the inductive step we consider $C \in \mathrm{vert}(\mathcal{L})$ with $\varepsilon(C) = K$ critical; and we assume that $v(C')$ is defined for all $C' \in \mathrm{vert}(\mathcal{L})$ with $B^+(C')$ or $B^-(C')$ in $\mathcal{L}(K)$. Reading the shuffle function $s_K : \mathcal{L}(K) \longrightarrow \varepsilon^{-1}(K)^\sim$ backwards $\varepsilon^{-1}(K)$ is distributed over gaps of $\mathcal{L}(K)$, but only over those gaps of the special form $\overline{B^+(C_1)\bigvee B^-(C_2)}$, or the minimal and the maximal (end) gap. (3.6.5) is used as part of the inductive hypothesis to extend v from $\mathcal{L}(K)$ to $\mathcal{L}'(K) = (K) \triangleright_{s_K} \varepsilon^{-1}(K)$ in a gapwise constant manner.

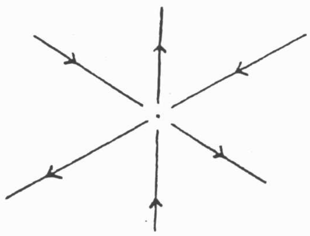
(3.6.6) If $C \in \varepsilon^{-1}(K)$ is in $\mathcal{L}'(K)$ smaller then the minimal element B_o of $\mathcal{L}(K)$, then $v(C) := v(B) = v(\overline{K}^-)$.

If $C \in \varepsilon^{-1}(K)$ lies in $\mathcal{L}'(K)$ between $B_1, B_2 \in \mathcal{L}(K)$, then $v(C) := v(B_1) = v(B_2)$.

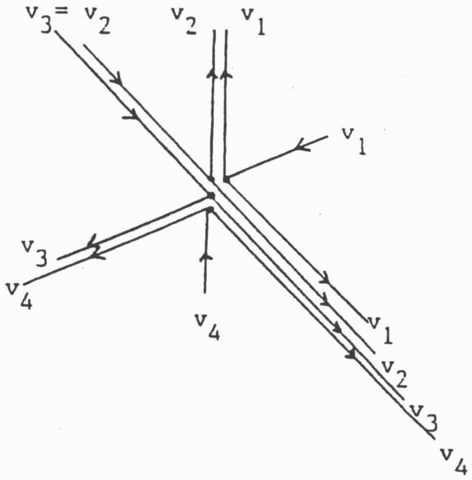
If $C \in \varepsilon^{-1}(K)$ is in $\mathcal{L}'(K)$ larger then the maximal element B_m of $\mathcal{L}(K)$, then $v(C) := v(B_m) = v(\overline{K}^+)$.

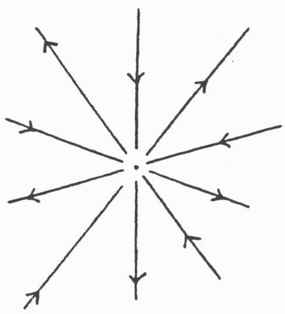
Of course, v has the constant value $v(C)$ on $B^+(C)$ and $B^-(C)$. Note that (3.6.5) is still valid. This completes the definition of v on the 1-cells.

(3.6.7)

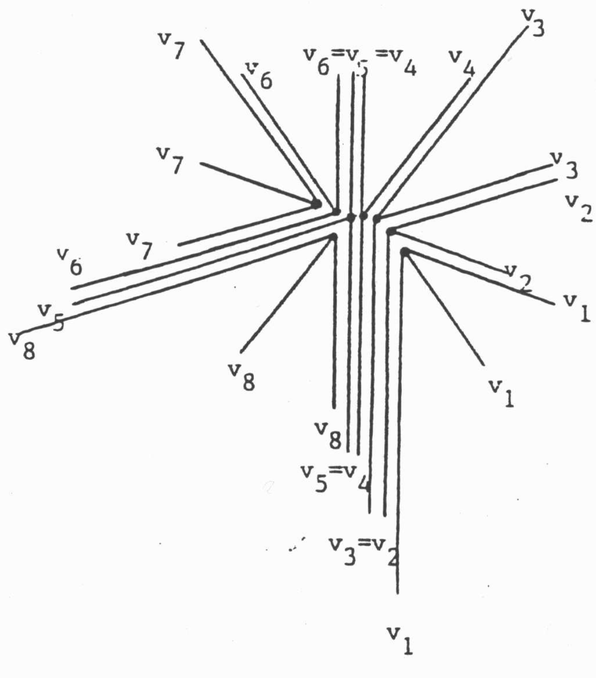


m = 2



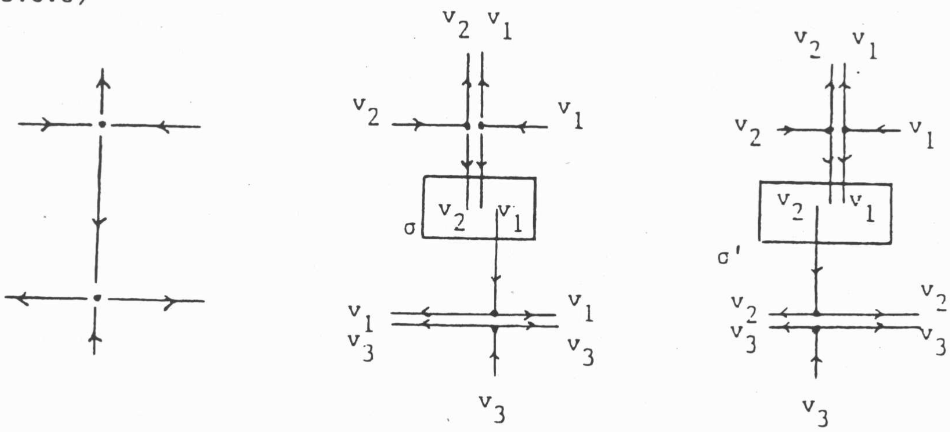


m = 4

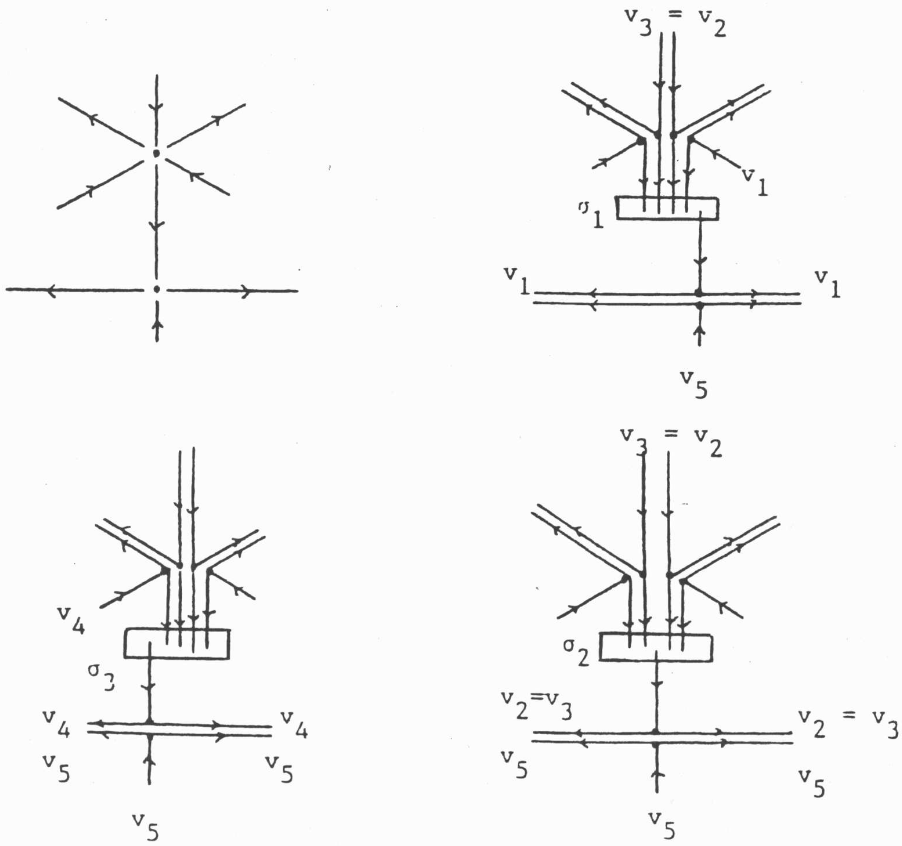


The two figures show the value distribution over a highest stagnation point.

(3.6.8)



Here we have two value distributions according to the two choices of the shuffle function. In the next figure there are three choices.



Our next aim is a linear ordering of the vertices of $\mathscr{L}$. The values $v(C)$ establish a partial ordering on $\mathrm{vert}(\mathscr{L})$; this will be sharpened to a total, i.e. linear ordering.

Consider again the set M of (3.8.3). The constant values of v on each $K \in M$ must be distinct, since the mapping function $w = u + iv$ is injective on $F_o = F - \mathcal{K}_o$. Thus we can regard M with the linear ordering induced by v . Furthermore, $\varepsilon^{-1}(K)$ is a linear set for each $K \in M$. This turns $\vec{\mathscr{C}}_M$ into a linear set by

(3.6.9) $$\vec{\mathscr{C}}_M = \bigsqcup_{K \in M} \varepsilon^{-1}(K) \quad .$$

Arguing as always by downward induction in $\mathcal{K}_o$ we see the following fact.

(3.6.10) If $C_1,C_2 \in \mathrm{vert}(\mathscr{L})$ with $v(C_1) = v(C_2)$, then there is some edge K in $\mathcal{K}_o$ such that

either (i) $B^+(C_1),B^-(C_2) \in \mathscr{L}(K)$,

or (ii) $B^+(C_2),B^-(C_1) \in \mathscr{L}(K)$ holds.

This is an immediate consequence of (3.6.6) where no new value is introduced, only existing values are given to successors and predecessors. Note also that the alternative in (3.6.10) is independent of K , since the shuffle functions are monotone.

Now we are ready to define the ordering. Given $C_1,C_2 \in \mathrm{vert}(\mathscr{L})$ we declare:

(3.6.11) $C_1 < C_2$ if $v(C_1) < v(C_2)$ in $\mathbb{R}$

or if $v(C_1) = v(C_2)$ in $\mathbb{R}$ and $B^+(C_1) < B^-(C_2)$ in $\mathscr{L}(K)$ for some edge K satisfying (3.6.10) .

Because (3.6.10) is a true alternative, $<$ is a well-defined linear ordering.

There is an obvious pairing of the vertices C in $\mathcal{L}$, i.e. a decomposition into disjoint subsets of exactly two elements. Already for each $\mathcal{L}(S)$ we have a pairing, induced by the function

$$(3.6.12) \qquad \lambda_S = (\beta_S^+)^{-1} \circ \beta_S^- : \overrightarrow{\mathcal{C}}(S) \longrightarrow \overleftarrow{\mathcal{C}}(S) \longrightarrow \overrightarrow{\mathcal{C}}(S) \quad .$$

λ_S is a bijection of $\mathcal{L}(S) = \overrightarrow{\mathcal{C}}(S)$. Because of (3.5.10)(iii, iv, v) it has the following properties:

$$(3.6.13) \qquad \lambda_S(C) \neq C \;, \quad \lambda_S^2(C) = C \qquad \text{for all } C \in \mathcal{L}(S) \;,$$

$$(3.6.14) \qquad (\beta_S^+)^{-1} \circ \beta_S^- = (\beta_S^-)^{-1} \circ \beta_S^+ \;.$$

On the set $\mathrm{vert}(\mathcal{L}) = \coprod \mathcal{L}(S)$ the λ_S induce a fixpoint-free involution λ, i.e. a pairing.

Altogether we have for the branching complex

$$\begin{tikzcd}
\bar{F} \cup \mathcal{L} \arrow[rr, "w = u + iv"] \arrow[d, "\pi"'] & & \bar{\mathbb{C}} \\
F \arrow[urr, "u"'] & &
\end{tikzcd}$$

where $w = u + iv$ is the extended complex potential or mapping function.

For C a vertex of $\mathcal{L}$ over a stagnation point S we denote by

$$(3.6.15) \qquad L_C = w(B_C) = w(B^+(C)) = w(B^-(C))$$

the image of the cycle B_C under w . L_C is a semi-infinite slit in the complex plane, parallel to the x-axis, with right endpoint

$$(3.6.16) \qquad S_C = (u(C),v(C)) = (u(S),v(C)) \; ,$$

$$L_C = \{z = x + iy \mid x \leqq u(S),\ y = v(C)\} \; .$$

These slits are not necessarily distinct nor disjoint.

Using the ordering of $\mathrm{vert}(\mathscr{L})$ we can write

$$(3.6.17) \qquad L = (L_{C_1}, L_{C_2}, \ldots, L_{C_{4g}})$$

if $C_1 < C_2 < \ldots < C_{4g}$ in $\mathrm{vert}(\mathscr{L})$. Note that the pairing λ becomes a permutation of L .

Chapter 4

Parallel Slit Domains

A parallel slit domain is the image of the boundary of the branching complex together with the combinatorial data of the branching graph. Thus associated to a dipol function is a configuration of slits in the complex plane together with boundary coordinations. The space $PS\mathbb{C}(g)$ of these parallel slit domains will be a concrete model for the moduli space $\vec{\mathfrak{M}}(g)$.

After introducing certain configurations of slits as representatives we describe the regularity conditions and the equivalence relations. With the help of some basic moves of configurations we prove that $PS\mathbb{C}(g)$ is connected. Various othe properties and notions are introduced.

4.1 Configuration of slit pairs.

A slit is a semi-infinite line, i.e. a subset of $\mathbb{C}$ of the form

$$(4.1.1) \qquad L_o = \{z = x + iy \in \mathbb{C} \mid x \leq a,\ y = b\}$$

for some point $S_o = a + ib$ of $\mathbb{C}$. Thus a slit determines a point $S_o = S(L_o)$ as its right endpoint, and vice versa, a point S_o determines a slit $L_o = L(S)$ as above. A translation L_o' of L_o by some $t \in \mathbb{C}$,

$$L_o' = L_o + t = \{z+t \mid z \in L_o\}\ , \quad S_o' = S_o + t$$

we call parallel, if t is imaginary. A configuration of slit pairs of genus g consists of an ordered sequence $L_1, \ldots, L_{4g}$ of $4g$ slits together with a pairing function $\lambda \in \Sigma_{4g}$ such that the following conditions are satisfied:

(4.1.2) (i) the sequence has decreasing imaginary parts, i.e.

$$\mathrm{Im}(S(L_i)) \geq \mathrm{Im}(S(L_{i+1})) \quad \text{for } i = 1, \ldots, 4g-1\ ;$$

(ii) the orbits of λ partition the indices $\{1, \ldots, 4g\}$ into $2g$ pairs of indices, i.e.

$$\lambda(i) \neq i,\ \lambda^2(i) = i \text{ for } i = 1, \ldots, 4g.$$

(iii) the slits of a pair are parallel, i.e.

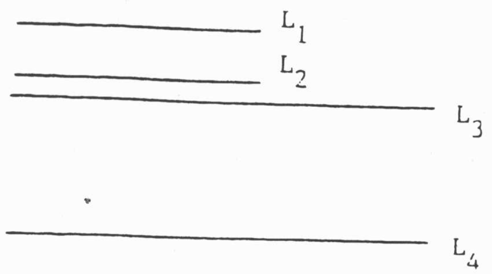
$$\mathrm{Re}(S(L_i)) = \mathrm{Re}(S(L_{\lambda(i)})) \quad \text{for} \quad i = 1, \ldots, 4g\ .$$

Note that the slits of such a configuration are not necessarily distinct; some of them can be equal, or contained in each other.

We denote a configuration as $L = (L_1, \ldots, L_{4g}; \lambda)$, and abbreviate $S_i = S(L_i)$.

<u>Examples</u>, and how to draw them.

(1)



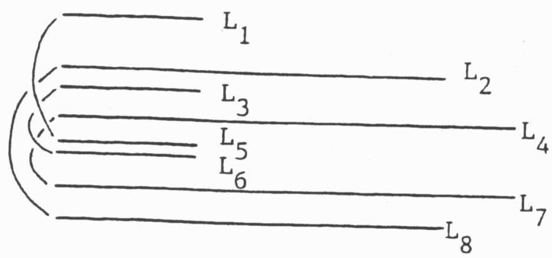
$g = 1$

$L = (L_1, L_2, L_3, L_4; \lambda)$

$\lambda = (12)\ (34)$

Of course, the slits are infinite to the left. The pairing function will usually be written as a permutation in cycle notation.

(2)



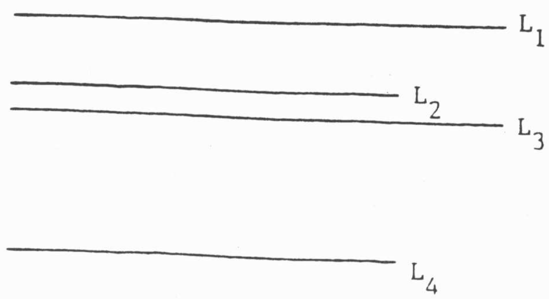
$g = 2$

$L = (L_1, \ldots, L_8; \lambda)$

$\lambda = (15)\ (28)\ (47)\ (36)$

In this example the position of the slits does no longer (as in Ex. (1), (3)) already determine the pairing by (4.1.2)(iii) here $\lambda' = (16)(28)(47)(35)$ is another possible pairing. It is sometimes convenient to indicate the pairing as above.

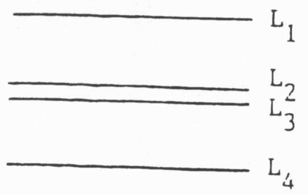
(3)



$L = (L_1, L_2, L_3, L_4; \lambda)$

$\lambda = (13)\ (24)$

(4)

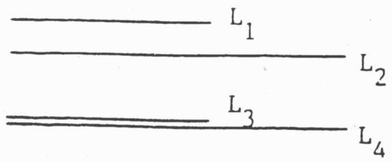


$\lambda = (13)\ (24)$

$\lambda' = (12)\ (34)$

There are two possible pairings.

(5)



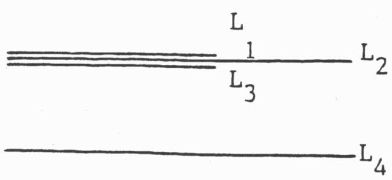
$L = (L_1, L_2, L_3, L_4)$ and

$L_3 \subseteq L_4$

$\lambda = (13)\ (24)$

To indicate that $L_i \subseteq L_{i+1}$ we draw the lines close to each other.

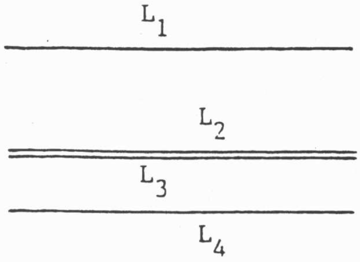
(6)



$\lambda = (13)\ (24)$

$L_1 = L_3 \subseteq L_2$

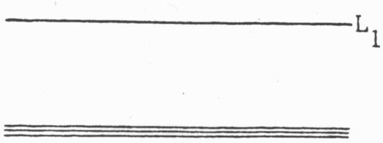
(7)



$\lambda = (13)\ (24)$

$L_2 = L_3$

(8)



$L_2 = L_3 = L_4$

$\lambda = (13)\ (24)$

4.2 The space F(L).

It is straight forward why a configuration $L = (L_1, \ldots, L_{4g}; \lambda)$ of slit pairs might represent a Riemann surface: one cuts $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ open along the slits L_i and then reglues upper to lower edges of the boundary according to the pairing function λ. But not every L represents a surface; in general, this gluing leads only to a two dimensional complex; furthermore, different configurations may represent the same surface. To single out the admissible configurations we next describe the space F(L) associated with L.

Let $L = (L_1,\ldots,L_{4g};\lambda)$ be given. The 4g+1 sets

$$(4.2.1) \quad \begin{aligned} F^o &= \{z \in \bar{\mathbb{C}} \mid Im(z) \geqq Im(S_1)\} \\ F^k &= \{z \in \bar{\mathbb{C}} \mid Im(S_k) \geqq Im(z) \geqq Im(S_{k+1})\} \text{ for } k = 1,\ldots,4g-1 \\ F^{4g} &= \{z \in \bar{\mathbb{C}} \mid Im(S_{4g}) \geqq Im(z)\} \end{aligned}$$

are closed subspaces of $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$, and all contain ∞. F^o and F^{4g} are homeomorphic to the extended upper and lower half-plane, respectively; and F^k for $k = 1,\ldots,4g-1$ is a horizontal strip containing ∞ , which may degenerate into a line if $L_k \cap L_{k+1} \neq \emptyset$. Since F^k are not disjoint, we denote a point in F^k by (z,k) with $z \in F^k$, $z \neq \infty$. Apart from ∞ the F^k are now disjoint. On the disjoint union

$$(4.2.2) \qquad F' = F^o \amalg \ldots \amalg F^{4g}$$

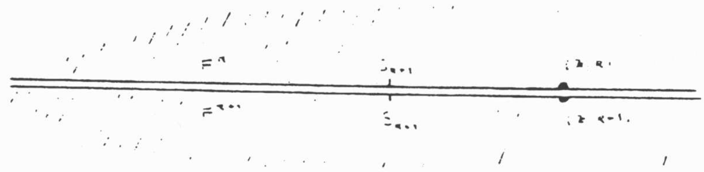
points are identified by the following

(4.2.3) <u>gluing-rules:</u>

(1) $F^k \ni (z,k) \sim (z,k+1) \in F^{k+1}$

if $Im(z) = Im(S_k)$ and $Fe(z) \geq Re(S_{k+1})$.

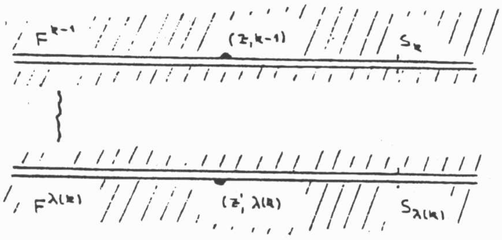
$(k = 0,\dots,4g-1)$



(2) $F^{k-1} \ni (z,k-1) \sim (z',\lambda(k)) \in F^{\lambda(k)}$

if $z \in L_k$, $z' \in L_{\lambda(k)}$ and $Re(z) = Re(z')$.

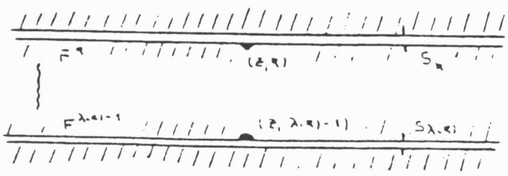
$(k = 1,\dots,4g)$



(3) $F^k \ni (z,k) \sim (z',\lambda(k)-1) \in F^{\lambda(k)-1}$

if $z \in L_k$, $z' \in L_{\lambda(k)}$ and $Re(z) = Re(z)$

$(k = 1,\dots,4g)$



Then we define

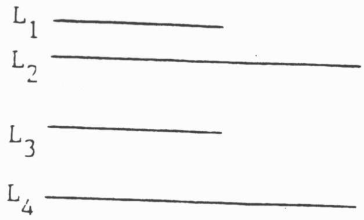
(4.2.4) $\quad F(L) := F'/\sim$

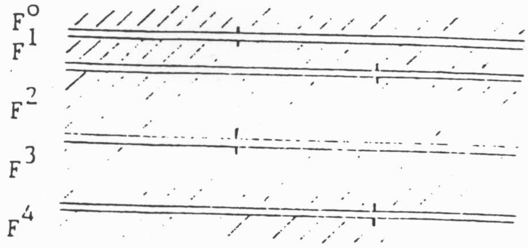
as the (2-dimensional) complex associated with the configuration L. L is called <u>non-degenerate</u> or <u>regular</u> if F(L) is a closed surface of genus g. Note that F(L) can fail to be a surface for two reasons: by not being a 2-manifold, and - more surprisingly - by not having genus g. (The following figures show the gluing process in the case of degenerate and non-degenerate configurations.) Our definition of regularity is inappropriate; but we will replace it in 4.4 by a direct criterion for the configuration itself.

The following figures (4.2.5) show some examples of the gluing-process.

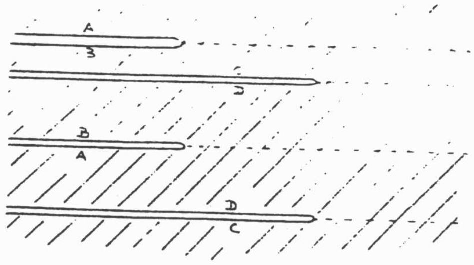
(4.2.5)

(1)



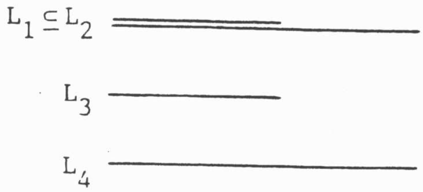


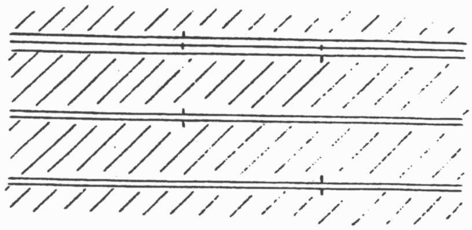
F'



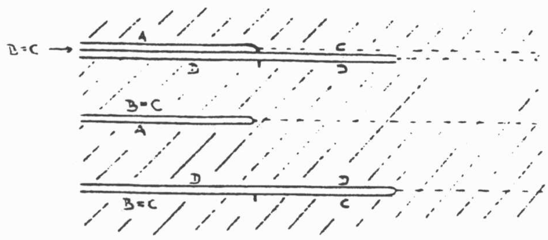
F(L)

(2)





F'

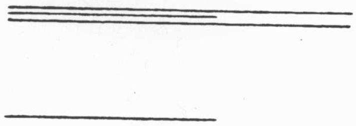


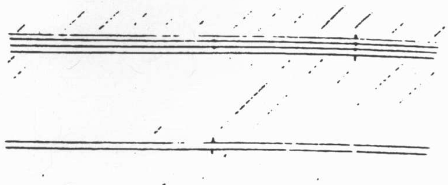
F(L)

(3)

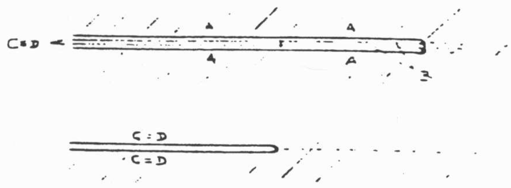
$L_2 \supseteq L_1 = L_3$

L_4

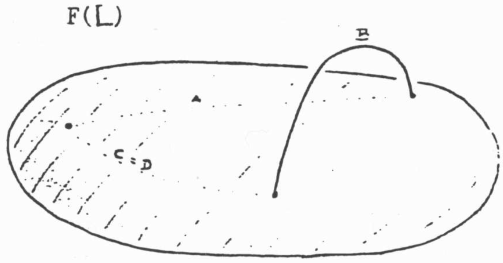




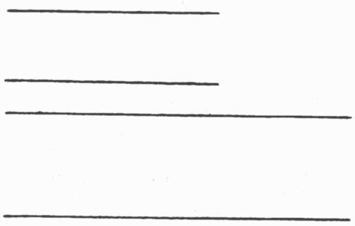
F'

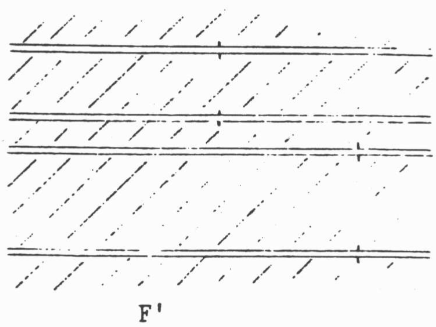


F(L)

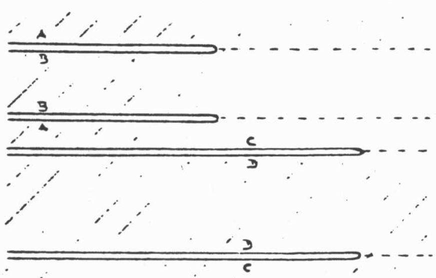


(4)

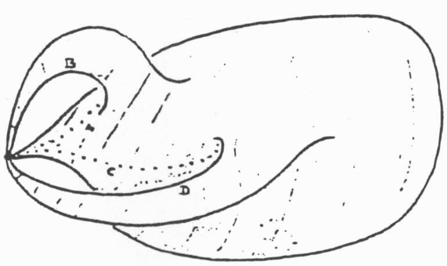




F'



F(L)



4.3 Equivalence of configurations.

That different configurations L and L' may give the same surface $F(L) = F(L')$ one can see from the following examples.

(4.3.1)

$\lambda = (13)(24)$ $\qquad$ $\lambda' = (13)\ (24)$

We see here the basic phenomenon: if a slit L_i touches a longer slit L_j from above/below, it jumps by a vertical shift up or down to the other side of the pair $(j,\lambda(j))$ to touch now $L_{\lambda(j)}$ from below/above.

In general, let a configuration $L = (L_1,\ldots,L_{4g};\lambda)$ be given. There are two kinds of crossings (or jumps). We need some notation to describe them. For $k = 1,\ldots,4g$ set $h_k = \operatorname{Im}(S_{\lambda(k)}) - \operatorname{Im}(S_k)$. For $1 \leqq m < n \leqq 4g$ let $\rho_{m,n}$ denote the cyclic permutation of the subset $\{m,\ldots,n\}$ in the set of all indices, i.e. $\rho_{m,n} = (m\ m{+}1\ \ldots\ n)$ as an element of Σ_{4g} in circle notation.

(4.3.2)

(i) If $L_{k-1} \subseteq L_k$ and $k < \lambda(k)$, then $L \approx L' = (L'_1, \ldots, L'_{4g}; \lambda')$ where

$$L'_i = L_i \qquad \text{for } i = 1, \ldots, k-2 \quad ,$$

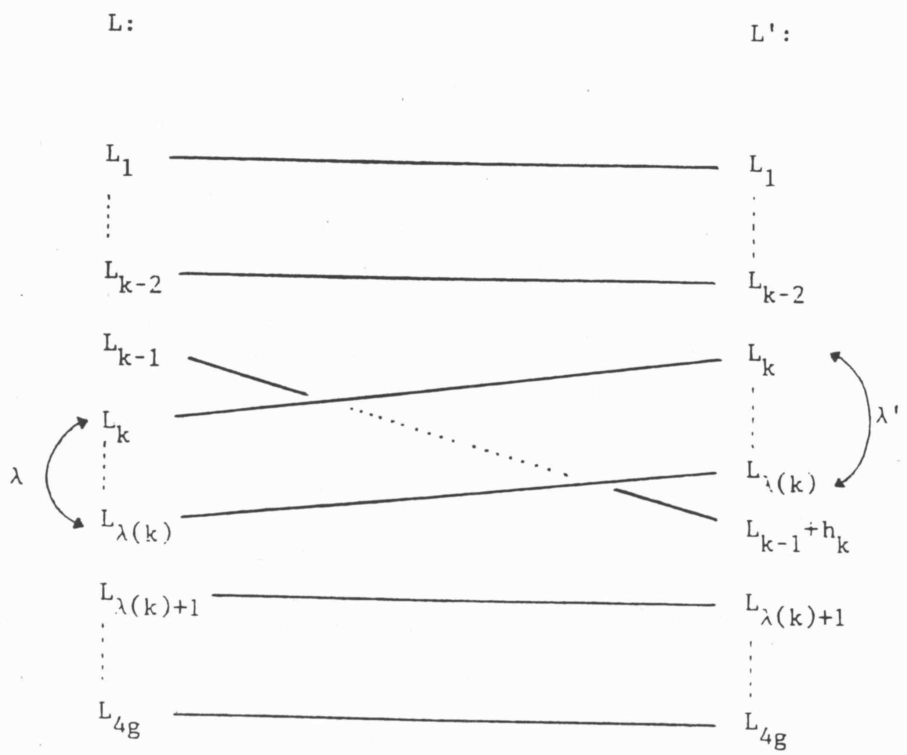
$$L'_i = L_{i+1} \qquad \text{for } i = k-1, \ldots, \lambda(k)-1 \quad ,$$

$$L'_{\lambda(k)} = L_{k-1} + h_k \quad ,$$

$$L'_i = L_i \qquad \text{for } i = \lambda(k)+1, \ldots, 4g$$

and $\lambda' = \lambda \circ \rho_{k-1,\lambda(k)}$.

Schematically such a jump is illustrated as follows and called a <u>crossing-over</u>.

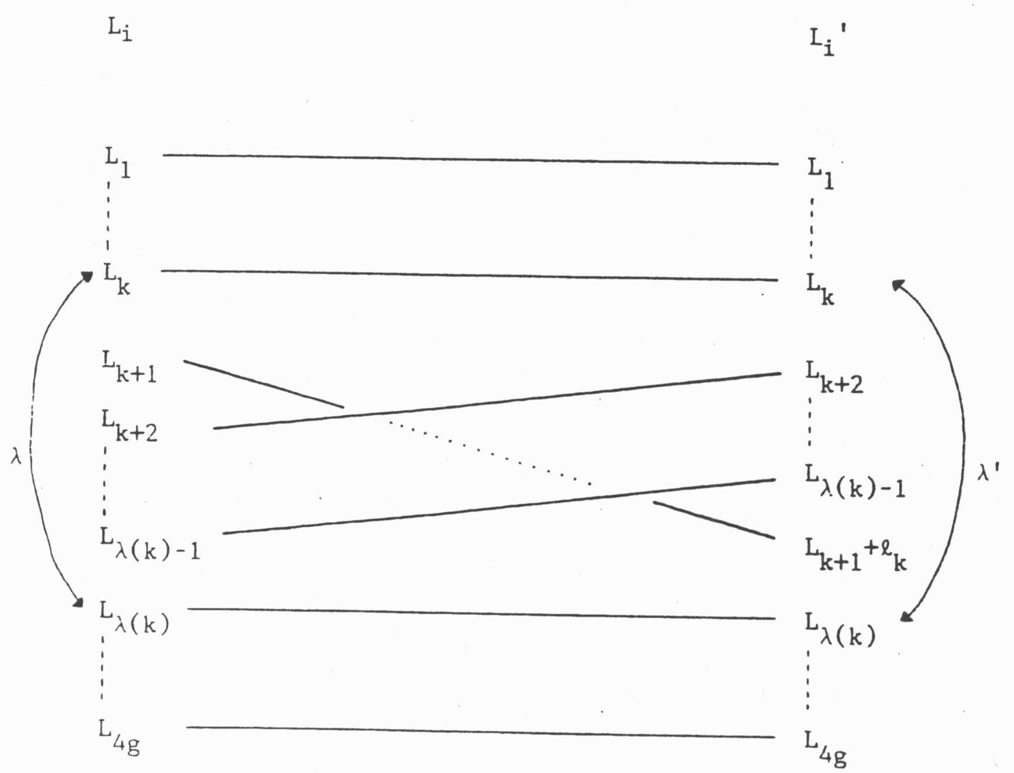


(ii) If $L_{k+1} \subseteq L_k$ and $k < \lambda(k)$, then $L \approx L' = (L'_1,\dots,L'_{4g};\lambda')$ where

$$
\begin{aligned}
L'_i &= L_i && \text{for } i = 1,\dots,k \quad , \\
L'_i &= L_{i+1} && \text{for } i = k+1,\dots,\lambda(k)-2 \quad , \\
L'_{\Lambda(k)-1} &= L_{k+1} + h_k && \text{for } \mathrm{Im}(S_{\lambda(k)}) - \mathrm{Im}(S_k) \quad , \\
L'_i &= L_i && \text{for } i = \lambda(k),\dots,4g
\end{aligned}
$$

and $\lambda' = \lambda \circ \rho_{k+1,\lambda(k)-1}$

We call this second type of a jump a <u>crossing-under</u>.



The equivalence relation generated by (i) and (ii) is denoted by $\simeq$, and the equivalence class of $L = (L_1,\dots,L_{4g};\lambda)$ by $\mathcal{L} = [L_1,\dots,L_{4g};\lambda]$. Each equivalence class contains only finitely many configurations.

(4.3.3) Proposition.

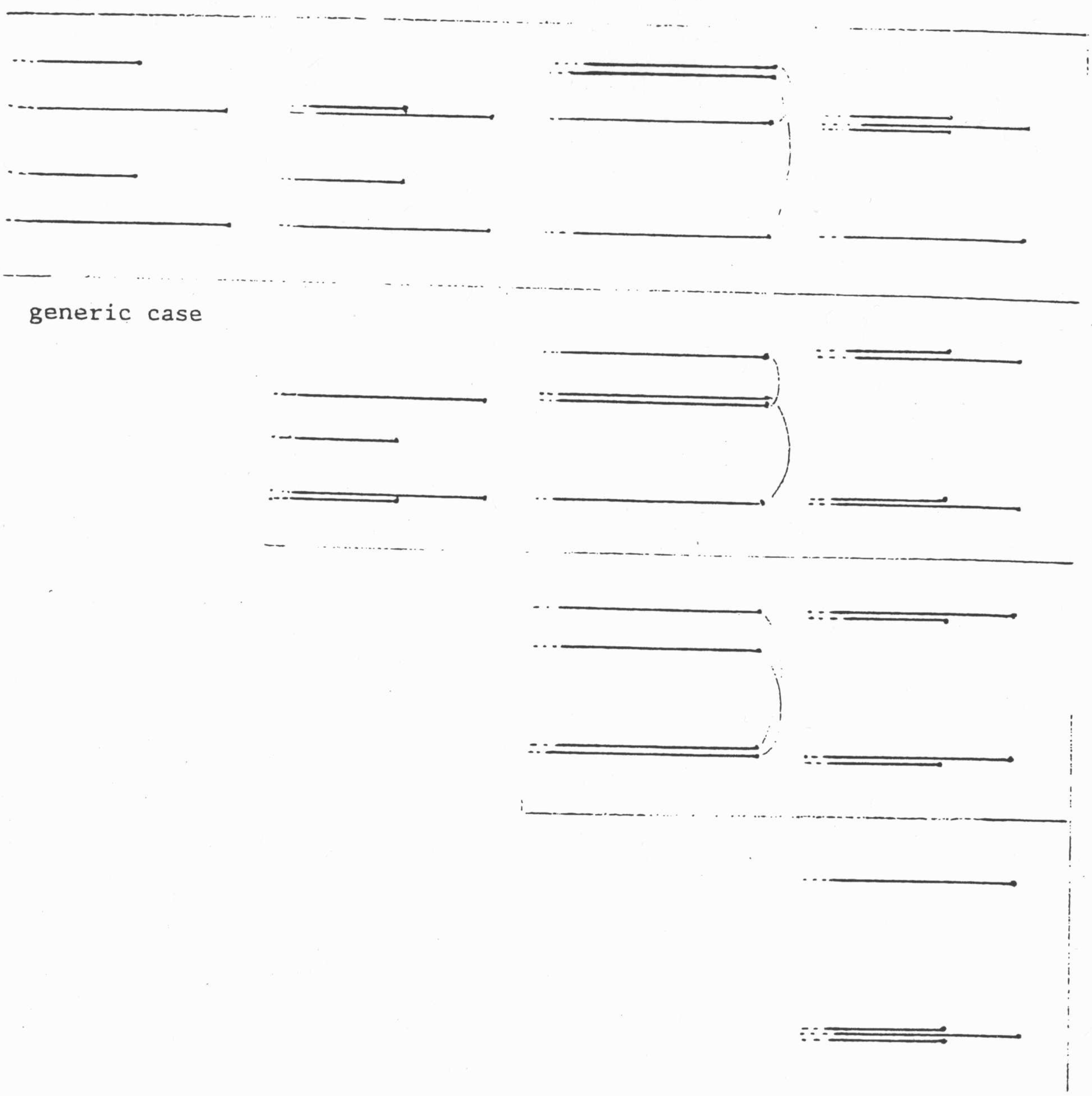
(i) *If* $L_1 \simeq L_2$ *are equivalent, then* $F(L_1) \cong F(L_2)$ *are homeomorphic.*

(ii) *In particular, if* $L_1 \simeq L_2$ *and one is regular, so is the other.* ∎

Hence we talk of regular (non-degenerate) classes $\mathcal{L} = [L_1,\dots,L_{4g};\lambda]$.
We write also $F(\mathcal{L})$ instead of $F(L)$.
Proof: Assume $L_1 \simeq L_2$ by a single crossing. The homeomorphism is the identity on each strip F^k which is not a line. On all others the gluing rules are invariant under crossings. ∎

The following table gives the four possible "types" of non-degenerate configurations for $g = 1$. The columns are complete equivalence classes.

(4.3.4)



4.4 Regularity of configurations.

Recall that a configuration $L = (L_1,\ldots,L_{4g};\lambda)$ of genus g is by the definition in 4.2 called regular (or non-degenerate) if the 2-complex $F(L)$ is a surface of genus g. We distinguish three cases of $F(L)$ not being a surface of genus g: (I) $z_o = \infty$ has no neighbourhood homeomorphic to a disc, (II) a point $z \neq \infty$ has no neighbourhood homeomorphic to a disc, (III) $F(L)$ is a surface of genus smaller than g. The cases do not exclude each other.

We begin with (I). Choose $R > 0$ such that $R > |S_i|$ for $i=1,\ldots,4g$, and consider the vertical

$$(4.4.1) \qquad V' = \{(z,k) \in F' \mid z \neq \infty,\ \mathrm{Re}(z) = -R,\ 0 \leqq k \leqq 4g\}$$

with its components $V'_k = V' \cap F^k$. If we glue these intervals according to (4.2.3) (2) and (3)

$$(z,k-1) \sim (z',\lambda(k)) \qquad \text{and}$$

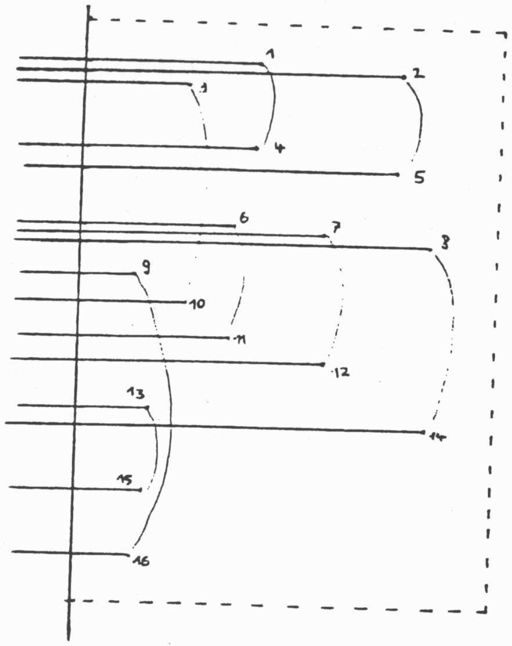
$$(z,k) \sim (z',\lambda(k)-1) \qquad \text{for} \quad z \in L_k\,, \quad z' \in L_{\lambda(k)}\,,$$

we obtain

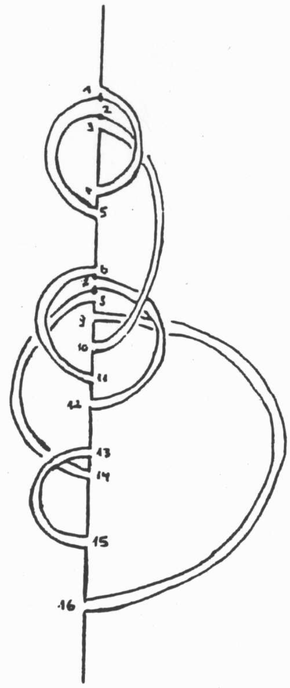
$$(4.4.2) \qquad V(L) := V'/\sim \quad .$$

$V(L)$ is independent of R as long as R is large enough. Furthermore, $V(L_1) \cong V(L_2)$ if $L_1 \approx L_2$.

(4.4.3)



L (g=4)



V(L)

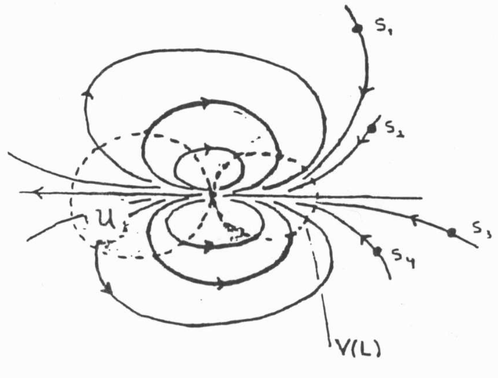
If V(L) is connected, then it is homeomorphic to $\mathbb{R}$; if it is disconnected, then it is the union of a line and circles and single points.

The set

(4.4.4) $\quad U = U'/\sim \;,\quad U' = \{(z,k) \in F^k \mid \max(|x|,|y| \geq R,\ z = x+iy,\ 0 \leq k \leq 4g\}$

is a neighbourhood of $z_o = \infty$ in F(L) , And U is a disc if and only if V(L) is connected.

(4.4.5)



But the connectivity of $V(L)$ depends only on λ, since the slits L_i are normalized to form a decreasing sequence, (4.1.2) (i). Let us call the pairing function λ connected if the algorithm

$$
\text{(4.4.6)} \qquad \ell_1 := 1 \ , \quad \ell_{i+1} := \lambda(i) \quad \text{for } i \text{ odd}
$$
$$
\ell_{i+1} := \ell_i + 1 \quad \text{for } i \text{ even}
$$

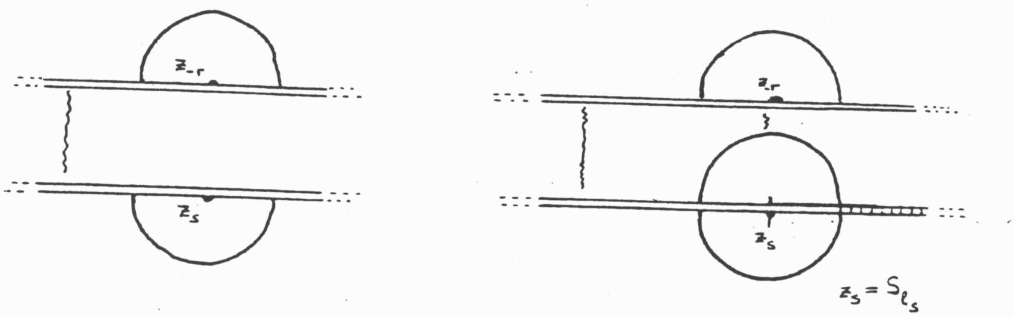
runs to produce a sequence of $8g$ numbers $\ell_1, \ldots, \ell_{8g}$ (which contains each of the numbers $1, \ldots, 4g$ exactly twice). This settles case (I).

For cases (II) and (III) assume $z \neq \infty$, $z \in L_k$; then $(z,k-1) \in F^{k-1}$ and $(z,k) \in F^k$ are two points in F'. Let us consider the identifications of $(z,k-1)$ and form the sequence (upwards and downwards) of indices ℓ_i and points z_i, starting with $\ell_o = k$, $z_o = z$:

$$
\begin{array}{lllll}
\text{if} & z_{-r+1} \in L_{\ell_{-r+1}-1} : & \ell_{-r} = \lambda(\ell_{-r+1}-1) , & z_{-r} \in L_{\ell_{-r}} , & \mathrm{Re}(z_{-r}) = \mathrm{Re}(z_o) \\
 & \vdots & \vdots & \vdots & \vdots \\
\text{if} & z_o \in L_{\ell_o -1} : & \ell_{-1} = \lambda(\ell_o - 1) , & z_{-1} \in L_{\ell_{-1}} , & \mathrm{Re}(z_{-1}) = \mathrm{Re}(z_o) \\
 & & & & \\
 & & \ell_o = k & z_o = z & \\
 & & & & \\
\text{if} & z_o \in L_{\ell_o} : & \ell_1 = \lambda(\ell_o) , & z_1 \in L_{\ell_1} , & \mathrm{Re}(z_1) = \mathrm{Re}(z_o) \\
\text{if} & z_1 \in L_{\ell_1+1} : & \ell_2 = \lambda(\ell_1+1) , & z_2 \in L_{\ell_2} , & \mathrm{Re}(z_2) = \mathrm{Re}(z_o) \\
\text{if} & z_2 \in L_{\ell_2+1} : & \ell_3 = \lambda(\ell_2+1) , & z_3 \in L_{\ell_3} , & \mathrm{Re}(z_3) = \mathrm{Re}(z_o) \\
 & \vdots & \vdots & \vdots & \vdots \\
\text{if} & z_{s-1} \in L_{\ell_{s-1}+1} : & \ell_s = \lambda(\ell_{s-1}+1) , & z_s \in L_{\ell_s} , & \mathrm{Re}(z_o) = \mathrm{Re}(z_o)
\end{array}
$$

This algorithm either terminates at both ends with statements $z_s \notin L_{\ell_s+1}$, $z_{-r} \notin L_{\ell_{-r}-1}$, or it is recurrent at both ends because $z_s = z_o$, $z_{-r} = z_o$. In the terminating case one finds neighbourhoods of $(z_{-r}, \ell_{-r}-1)$ in $F^{\ell_{-1}-1}$ and of (z_s, ℓ_s) in F^{ℓ_s} which (together with other half-discs in the case $z_s = S_{\ell_s}$ or $z_{-r} = S_{\ell_{-r}}$) form a neighbourhood of (z,k) in $F(L)$.

(4.4.7)



In the recurrent case there is no disc neighbourhood for (z,k) in $F(L)$. Among the indices $\ell_{-r+1},\ldots,\ell_o,\ldots,\ell_{s-1}$ choose one with minimal $r = \mathrm{Re}(S_{\ell_j})$ and call this index m_1, and set $m_2 = \lambda(m_1)$. By a sequence of crossings one can now move L_{m_1} (or L_{m_2}) into the same position: this follows by induction on $r+s$; it is true for $r+s = 1$; and if $r+s > 1$ one can reduce the length of the recurrence circle by crossing $L_{\lambda(\ell_{s-1})}$ over/under the pair $L_{\lambda(\ell_s)}$, L_{ℓ_s} or by crossing $L_{\lambda(\ell_s)}$ over/under the pair $L_{\lambda(\ell_{s-1})}$, $L_{\ell_{s-1}}$ (whatever is possible). Thus we end with a configuration $L' = (L'_1,\ldots,L'_{4g};\lambda')$ equivalent to L with a pair of indices $n < \lambda'(n)$ such that

(4.4.8) $\quad L'_n = L'_{\lambda(n)}$ and

$$\mathrm{Re}(S_m) \leqq \mathrm{Re}(S_m) = \mathrm{Re}(S_{\lambda'(n)}) = r \qquad \text{for all } n < m < \lambda'(n) .$$

holds. This degenerary condition has 3 subcases:

(1) all m with $n < m < \lambda'(n)$ satisfy $L'_m \subsetneq L'_n = L'_{\lambda'(n)}$;

(2) there is some m with $n < m < \lambda'(n)$ satisfying $L'_m = L'_n = L'_{\lambda'(n)}$;

(3) there is some m with $n < m < \lambda'(n)$ satisfying $L'_m = L'_{\lambda'(m)} = L'_n = = L'_{\lambda'(n)}$.

The case (3) is really a special case of (2). (1) means an interval attached between two distinct points of $F(L')$; (2) means two points of $F(L')$ are identified, which are equal in subcase (3) ; thus $F(L')$ is still a surface, but of genus $g-1$ or smaller, since $F(L')$ is homeomorphic to the surface associated with the new configuration where L'_n , $L'_{\lambda'(n)}$, L'_m and $L'_{\lambda'(m)}$ have been deleted.

Thus we have

(4.4.9) Proposition. *A configuration L of genus g is non-degenerate if and only if the following conditions are satisfied:*

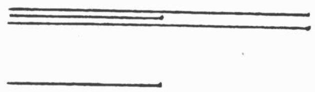
(i) $V(L)$ *is connected,*

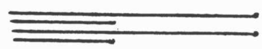
(ii) L *is not equivalent to some* L' *with a pair of indices* $n < \lambda'(n)$ *such that* $L'_k \subseteq L'_n = L'_{\lambda'(n)}$ *for all* k *such that* $n < k < \lambda'(n)$. ∎

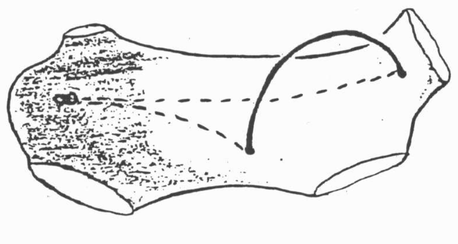
An immediate consequence is that $\lambda(k) \neq k-1$, $k+1$ for all k , and $\lambda(1) \neq 4g$ for the pairing function of a regular configuration. Furthermore, for each pair $n < \lambda(n)$ there exists another pair $m < \lambda(m)$ such that $n < m < \lambda(n) < \lambda(m)$ or $m < n < \lambda(m) < \lambda(n)$; we call two pairs interlocked in this way a quadrupel.

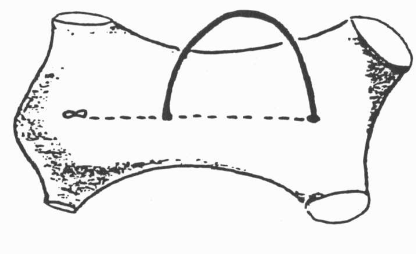
Assume (4.4.9) (ii) holds with several $k_1 < k_2 \dots$, i.e. $n < k_1 < k_2 < \dots < \lambda'(n$ and $L'_{k_1}, L'_{k_2}, \dots \subseteq L'_n = L'_{\lambda'(n)}$. Then either we can apply an equivalence relatio crossing L'_{k_2} over the pair L'_{k_1}, $L'_{\lambda'(k_1)}$ (if $\lambda'(k_1)$ is not among the k_i), or there is a nested system of situations as (4.4.9) (ii). In any case, we find an index r such that $\lambda'(r) = r+2$ and $L'_{r+1} \subseteq L'_r = L'_{r+2}$. Thus (4.4.9) (ii) is equivalent to

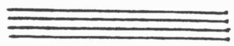
(4.4.10) L <u>is not equivalent to some</u> L' <u>containing one of the following subconfigurations:</u>

(1) 

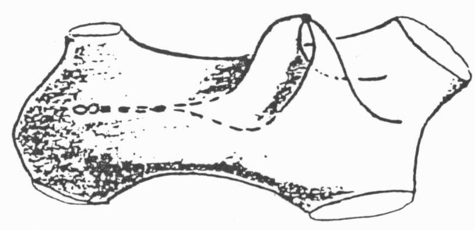
(2) 

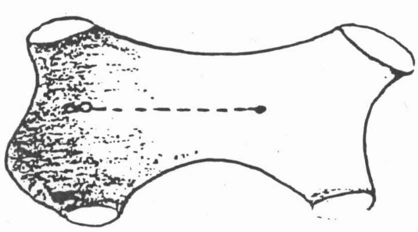




(3) 

(4) 





Of course, (2), (3) and (4) are subcases of (1), and (4) is a subcase of (3).

4.5 The space $PS\mathbb{C}(g)$.

A configuration $L = (L_1,\dots,L_{4g};\lambda)$ of slit pairs is determined by the sequence of endpoints $(S_1,\dots,S_{4g})$ together with the pairing function λ. This comprises a point in $C^{4g} \times \Sigma_{4g}$. Let $\mathrm{Conf}(g)$ denote the subspace consisting of all configurations, i.e. points satisfying (4.1.2), and $\mathrm{Reg\,Conf}(g)$ the subspace of all regular configurations (where Σ_{4g} is regarded as a discrete space),

(4.5.1) $$\mathrm{Reg\,Conf}(g) \subseteq \mathrm{Conf}(g) \subseteq C^{4g} \times \Sigma_{4g}$$

Since for $g \geq 2$ there always exist different λ satisfying (4.1.2) (ii), $\mathrm{Reg\,Conf}(g)$ and $\mathrm{Conf}(g)$ are in general not connected. We will use the maximal-coordinate-norm on these spaces, and define the metric on $\mathrm{Conf}(g)$ by

(4.5.2) $$d(L,L') = \max\{|S_i - S_i'| \mid i=1,\dots,4g\} + \max\{|\lambda(i)-\lambda'(i)| \mid i=1,\dots,4g\} .$$

The set of equivalence classes of non-degenerate configurations becomes the quotient space

(4.5.3) $$PS\mathbb{C}(g) = \mathrm{Reg\,Conf}(g)/\approx$$

and is called the <u>space of parallel slit domains</u>.

(4.5.4) $$\begin{array}{ccccc} \mathrm{Reg\,Conf}(g) & \subseteq & \mathrm{Conf}(g) & \subseteq & \mathbb{C}^{4g} \times \Sigma_{4g} \\ \downarrow & & \downarrow & & \\ PS\mathbb{C}(g) & \subseteq & \mathrm{Conf}(g)/\approx & & \end{array}$$

The metric on $PS\mathbb{C}(g)$ is

(4.5.5) $\quad d(\mathscr{L}, \mathscr{L}') = \min\{d(L,L') \mid L \in \mathscr{L}, L' \in \mathscr{L}'\}$.

Note that each class $\mathscr{L}$ contains only finitely many configurations.

We will see in the next chapter that $PS\mathbb{C}(g)$ is a manifold of dimension $6g$; it has also a cellular decomposition which we will study in detail somewhere else. The case $g = 0$ is special in-so-far $PS\mathbb{C}(0)$ consists of a single point represented by the empty configuration $\mathscr{L} = \emptyset$; and $F(\emptyset) = \overline{\mathbb{C}} = S^2$.

Here we will only prove the connectedness of $PS\mathbb{C}(g)$, which needs some preparations.

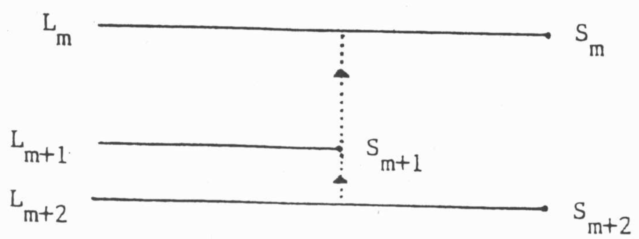
(4.5.6) Proposition. $PS\mathbb{C}(g)$ *is connected.*

The proof uses some standard paths in $PS\mathbb{C}(g)$ which are important for many later computations. To study them serves also the purpose to get familiar with the topology of $PS\mathbb{C}(g)$.

Rotating a slit inside a pair.

Assume $L = (L_1,\dots,L_{4g};\lambda)$ is a rgular configuration, and $\lambda(m) = m+2$ for some m . To start we also assume $Re(S_{m+1}) \leqq Re(S_m) = Re(S_{m+2})$.

(4.5.7)



As indicated in the figure, we define a path $L(t)$, $t \in [0,1]$, keeping all slits L_k fixed for $k \neq m+1$, and moving L_{m+1} along the dotted arrow:

(4.5.8)

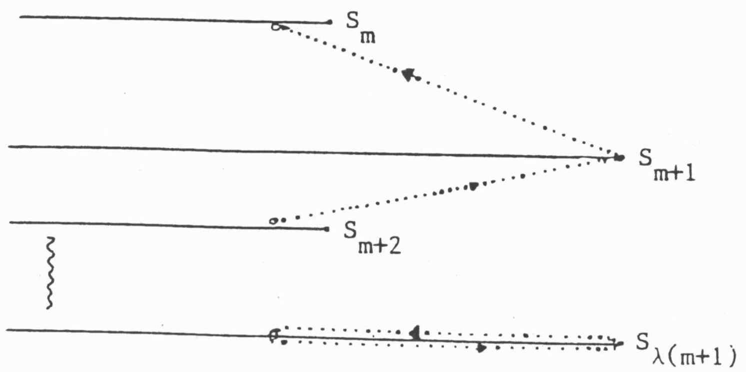
$$L_k(t) = \begin{cases} L_k & , \quad k \neq m+1 \,, \quad t \in [0,1] \,, \\ L_{m+1} + t\cdot h & , \quad k = m+1 \,, \quad 0 \leqq t \leqq \frac{h_1}{h} \,, \\ L_{m+1} - (1-t)h & , \quad k = m+1 \,, \quad \frac{h_1}{h} \leqq t \leqq 1 \,, \end{cases}$$

where $h_1 = \mathrm{Im}(S_m) - \mathrm{Im}(S_{m+1})$, and $h_2 = \mathrm{Im}(S_{m+1}) - \mathrm{Im}(S_{m+2})$, $h = h_1 + h_2$ are the heights between the slits. (Note that the regularity implies $h > 0$.) Because of the crossing-under

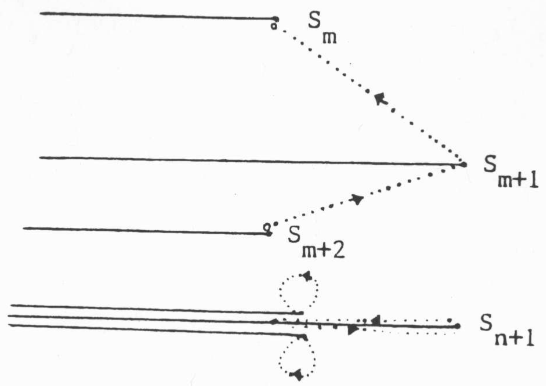
$$(L_1,\dots,L_m, L_{m+1}+h_1, L_{m+2},\dots,L_{4g}; \lambda) \approx (L_1,\dots,L_m, L_{m+1}-h_2, L_{m+2},\dots,L_{4g}; \lambda) \,,$$

this is a closed path, furthermore $L(t)$ is always regular, therefore a loop in $PS\mathbb{C}(g)$. Note, that λ remains unchanged throughout. Slightly more general, if $\mathrm{Re}(S_{m+1})$ is arbitrary, one can consider the path moving S_{m+1} to S_m , then jumping down to S_{m+2} , and moving it back to its old position. In this case, $S_{\lambda(m+1)}$ has to be moved also, keeping the same real part as S_{m+1} ; this might make it necessary to move other slits as well in order not to violat the regularity. The following two figures are better than more formulas.

(4.5.9)



(4.5.10)



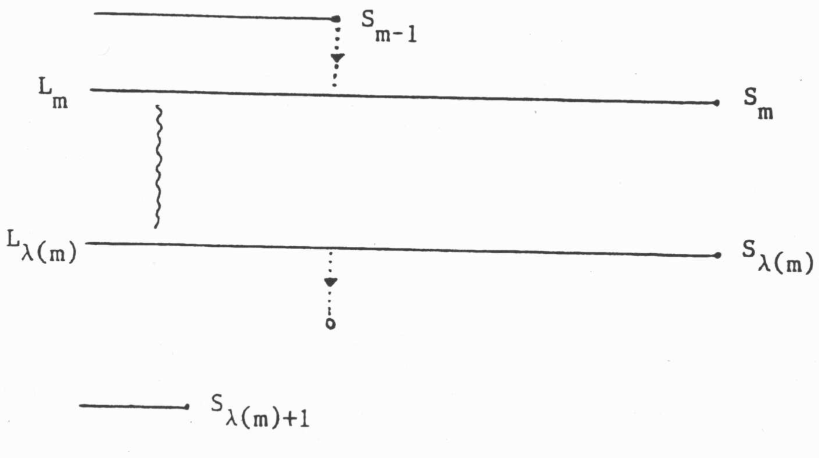
$\lambda(m) = n+1$

$\lambda(n) = n+2$

Moving a slit over a pair.

Assume $L = (L_1,\ldots,L_{4g};\cdot)$ is a regular configuration, and $L_{m-1} \not\leq L_m$, $L_{\lambda(m)+1} \not\leq L_{\lambda(m)}$ for some index m. Again, to simplify the situation, we assume also $\mathrm{Re}(S_{m-1}) \leq \mathrm{Re}(s_m)$.

(4.5.11)



The curly line indicates that there can be (and actually must be for reasons of regularity) more slits between L_m and $L_{\lambda(m)}$.

Now we move S_{m-1} down to L_m, cross-over to $L_{\lambda(m)}$, and more downward some distance:

$$(4.5.12)\qquad L_k(t) = \begin{cases} L_k & , \quad k < m-1 \ \text{ or } \ k > \lambda(m) \ , \quad 0 \leqq t \leqq 1 \ , \\ L_k & , \quad m \leqq k \leqq \lambda(m) \ , \quad 0 \leqq t \leqq \frac{h_1}{h} \ , \\ L_{k+1} & , \quad m-1 \leqq k \leqq \lambda(m)-1 \ , \quad \frac{h_1}{h} < t \leqq 1 \ , \\ L_{m-1} - t\cdot h & , \quad k = m-1 \ , \quad 0 \leqq t \leqq \frac{h_1}{h} \ , \\ L_{m-1} - t\cdot h - h_2 & , \quad k = \lambda(m) \ , \quad \frac{h_1}{h} \leqq t \leqq 1 \ , \end{cases}$$

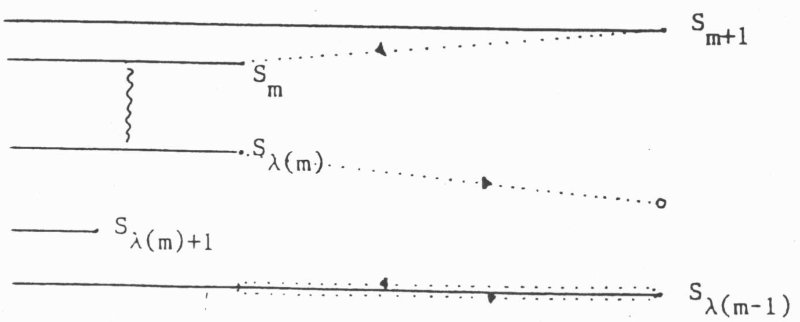
and

$$\lambda^t = \begin{cases} \lambda & , \quad 0 \leqq t \leqq \frac{h_1}{h} \ , \\ \lambda \circ \rho & , \quad \frac{h_1}{h} \leqq t \leqq 1 \ , \end{cases}$$

where $h_1 = \mathrm{Im}(S_{m-1}) - \mathrm{Im}(S_m)$, $h_2 = \mathrm{Im}(S_m) - \mathrm{Im}(S_{\lambda(m)})$, $h_3 = \mathrm{Im}(S_{\lambda(m)}) - \mathrm{Im}(S_{\lambda(m)+1})$, and $h = h_1 + h_3/2$, and $\rho = \rho_{m-1,\lambda(m)}$ is the (partial) cyclic permutation. Obviously, $t \longmapsto [L(t)] = [L_1(t),\dots,L_{4g}(t);\lambda^t]$ is a continuous path in $PS\mathbb{C}(g)$.

The following figure shows the general situation.

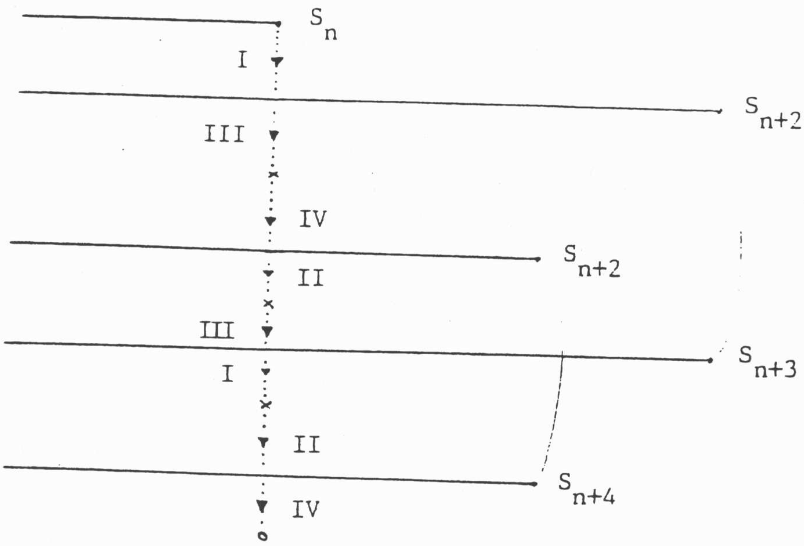
(4.5.13)



Moving a slit through a quadrupel.

Let $L = (L_1,\dots,L_{4g};\lambda)$ be a regular configuration with quadruple $\lambda(n+1) = n+3$ $\lambda(n+2) = n+4$ for some index n.

(4.5.14)

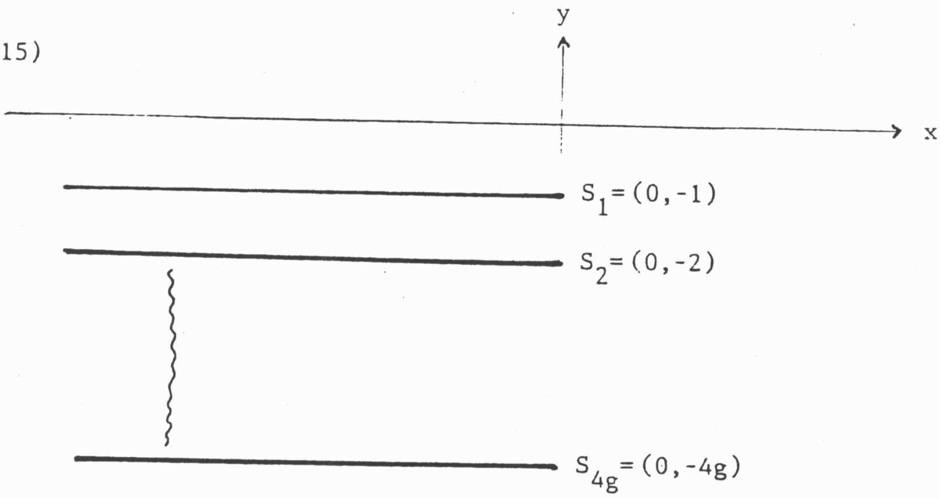


The figure indicates how L_n is moved to the other side of $L_{n+1},\dots,L_{n+4}$. All slits except L_n stay fixed (only changing their index). The path is a composition of four paths: (I) moves L_n over the pair L_{n+1}, L_{n+3} ; (II) is a half-rotation in the pair L_{n+2}, L_{n+4} ; (III) is a half-rotation in the pair L_{n+1}, L_{n+3} ; and (IV) finally is a move over the pair L_{n+2}, L_{n+4}.

Retracting a configuration into a generic one.

Given a regular configuration $L = (L_1,\dots,L_{4g};\lambda)$ we can move successively all slits apart from eachother. In the first move L_1 is kept fixed and $L_2,\dots,L_{4g}$ are simultanuously moved downwards or upwards, till the height between L_1 and L_2 is 1. In the next step, L_1 and L_2 are kept fixed, and $L_3,\dots,L_{4g}$ are moved till the height between L_2 and L_3 is also 1. And so on, till all slits are disjoint and two successive slits have height distance 1. Note that this path depends on the configuration, not on the equivalence class, i.e. thi: is a path in RegConf(g) . On the other hand, there are no crossings involved, and therefore λ remains unchanged. In two final steps we can move all slits horizontally to make $Re(S_k) = 0$ for all k , and then move all slits up or dowr (without changing their distances) to normalize to S_1 being the point $(0,-1)$ in the plane. The final result therefore depends just on λ .

(4.5.15)



In other words, we have

(4.5.16) <u>Proposition.</u> <u>The components of</u> Reg Conf(g) <u>correspond bijectively to the connected pairing functions; each component is contractible.</u> ∎

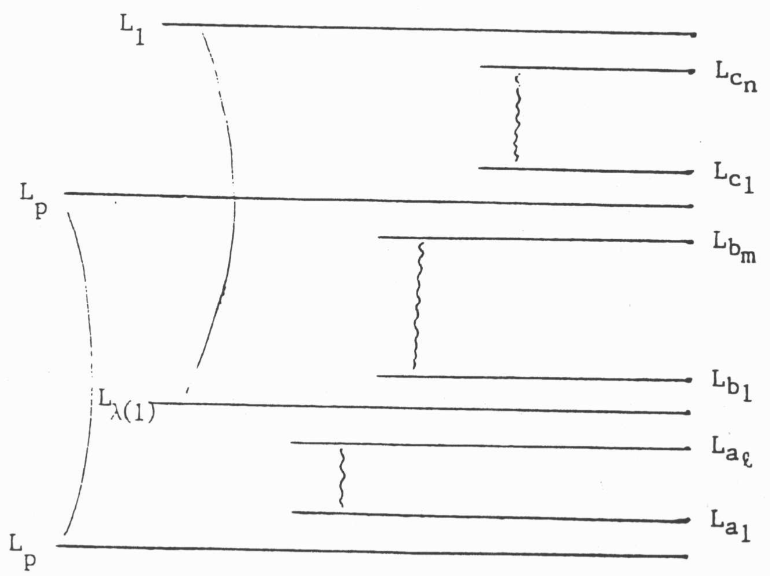
The number of components for $g = 0,1,2,3$ is $1,1,21$ and 1485 , respectively. In terms of a cell decomposition the interior of such a component in Reg Conf(g becomes a top dimensional "cell" in $PS\mathbb{C}(g)$.

Proof of Proposition (4.5.16):

Let $\mathcal{L} = [L_1,\dots,L_{4g};\lambda]$ be a regular parallel slit domain. To prove the connectivity of $PS\mathbb{C}(g)$ we can assume that the representing configuration $L = (L_1,\dots,L_{4g};\lambda)$ has been moved into a normalized position (4.5.15), i.e. $S_k = (0,-k)$.

Since $\mathcal{L}$ is non-degenerate, λ is connected. Thus there must be some index p such that $1 < p < \lambda(1)$; for the same reason, there must be some such p with $\lambda(1) < \lambda(p)$ in addition. So choose, e.g. the smallest p such that $1 < p < \lambda(1) < \lambda(p)$. Let c_n, $c_{n-1},\dots,c_1$ be the indices of all slits between L_1 and L_p (numbered upwards), and $b_m,\dots,b_1$ those of the slits between L_p and $L_{\lambda(1)}$, and $a_\ell,\dots,a_1$ those of the slits between $L_{\lambda(1)}$ and $L_{\lambda(p)}$.

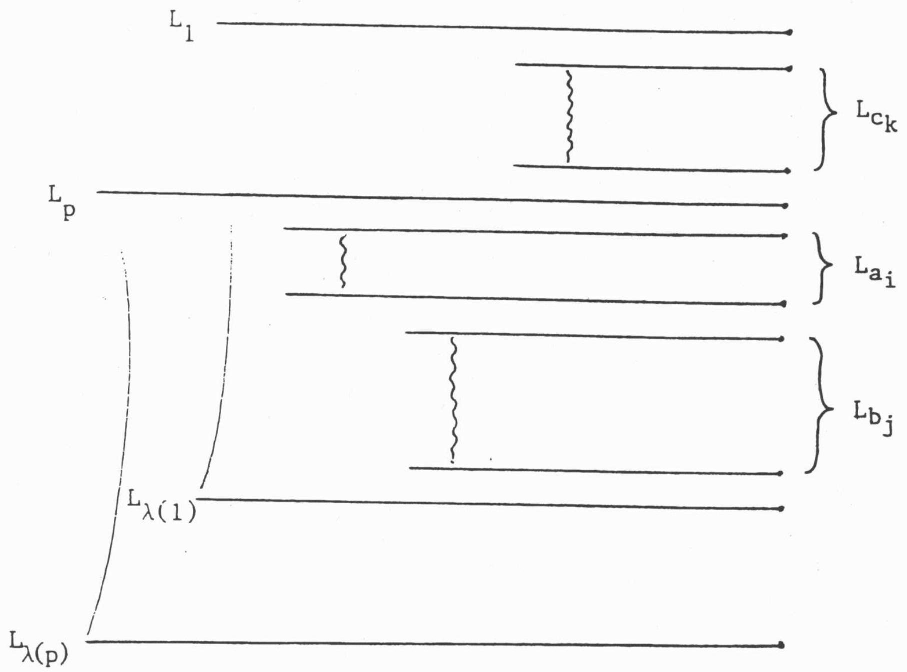
(4.5.17)



We will now "empty" the quadruple, compartment by compartment.

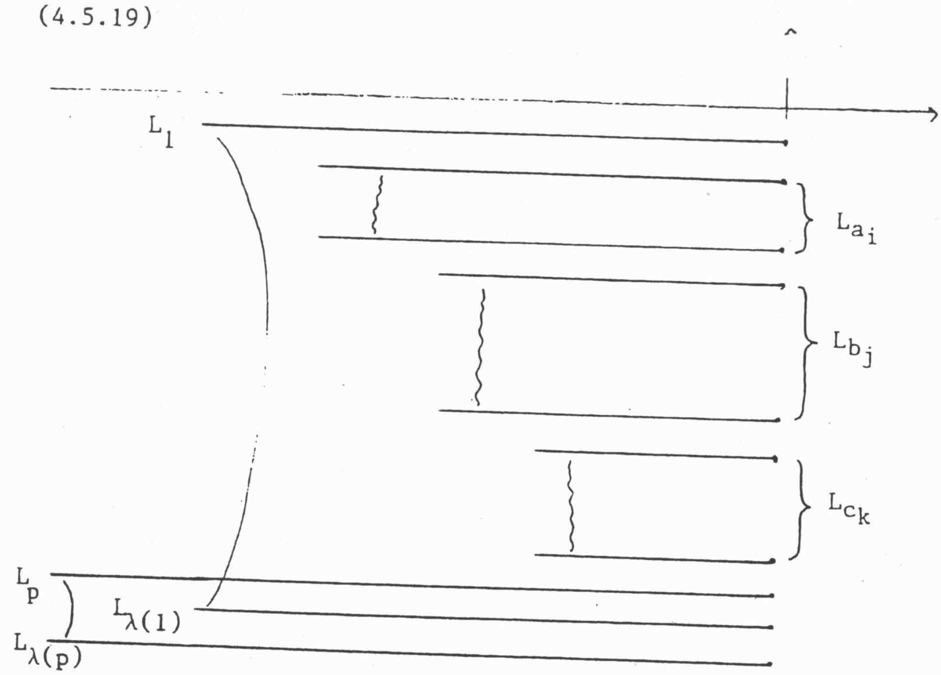
(I) First, we make a half-rotation downwards of the L_{a_i} inside the pair L_p, $L_{\lambda(p)}$, starting with L_{a_1}. This leads to the following situation (where we - incorrectly - keep the old indices of the slits).

(4.5.18)



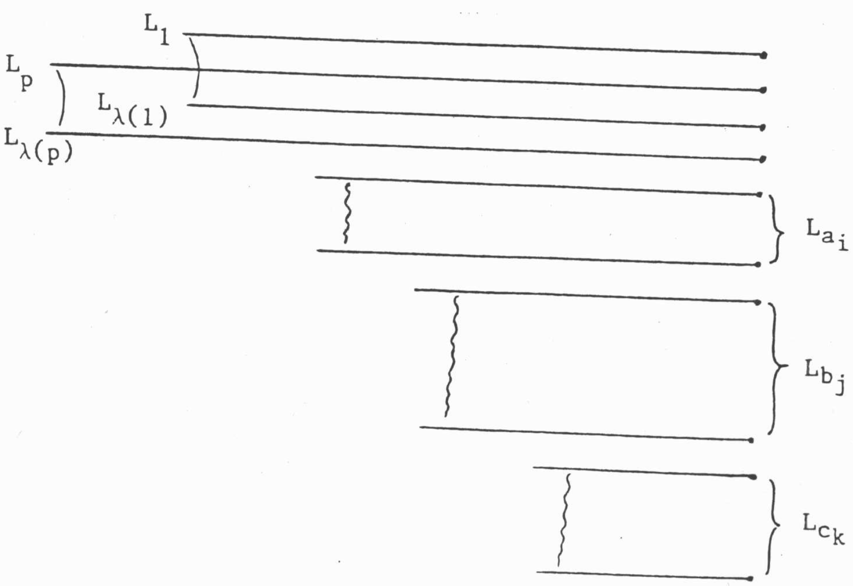
(II) Then, we make a half-rotation downward of the L_{b_j} and L_{a_i}, inside the pair L_1, $L_{\lambda(1)}$, starting with L_{b_1}.

(4.5.19)



(III) In the last step, all L_{c_k}, L_{b_j}, L_{a_i} make a more downwards over the pair L_p, $L_{\lambda(p)}$, starting with L_{c_1}.

(4.5.20)



In the resulting configuration we have a configuration $L' = (L'_1,\dots,L'_{4g};\lambda')$ - with correct indices - which starts with a quadruple $1,2,3,4$, i.e. $\lambda'(1) = 3$, $\lambda'(2) = 4$, and in which the remaining slits $L'_5,\dots,L'_{4g}$ form a subconfiguration of genus $g-1$. L' is so decomposed into two independent configurations. By induction on the genus, the assertion follows. ∎

By this method, any $\mathcal{L} \in PS\mathbb{C}(g)$ is moved to the standard

$$(4.5.21) \qquad \mathcal{L}^o = [L^o_1,\dots,L^o_{4g};\lambda^o] \quad ,$$

where $S^o_k = (0,-k)$ for $k = 1,\dots,4g$, and $\lambda^o = (13)(24)(57)(68)\dots(4k+1\ 4k+3)(4k+2\ 4k+4)\dots(4g-3\ 4g-1)(4g-2\ 4g)$. This $\mathcal{L}^o$ will be used as the basepoint in $PS\mathbb{C}(g)$.

4.6 $F(\mathscr{L})$ as a Riemann surface.

Let $\mathscr{L} = [L_1,\dots,L_{4g};\lambda]$ be a regular configuration class of slit pairs. Then $F(\mathscr{L})$ is not only a surface, but a Riemann surface.

(4.6.1) Proposition. Let $\mathscr{L}$ be in $PS\mathbb{C}(g)$. Then $F(\mathscr{L})$ is a Riemann surface.

Proof. To define charts for a holomorphic atlas on $F(\mathscr{L}) = F'/\sim$ we need only consider those components of $F' = \coprod_{k=o}^{4g-1} F^k$ which are not degenerated into lines. For $k = 1,\dots 4g-1$ let $h_k = \operatorname{Im}(S_k) - \operatorname{Im}(S_{k+1})$ be the height of F^k , and set $h = \min\{h_k \mid h_k > 0\}$. If $z \neq \infty$ is an interior point of F^k , we use $\operatorname{int}(F^k) \to \mathbb{C}$ as a chart.

If $z \neq \infty$ lies on some L_k , let $k_1 < k_2 < \dots < k_r$ be sequence of all indices such that there is some $z_{k_i} \in \mathbb{C}$ with $\operatorname{Re}(z_{k_i}) = \operatorname{Re}(z)$ and $(z,k) \sim (z_{k_i},k_i)$ in F' . With

$$U(z_{k_i}) = \{\zeta \in \mathbb{C} \mid |z_{k_i}-\zeta| < h/2\} \qquad (i = 1,\dots,r)$$

set

$$U'_{k_i} = \{(\zeta,k_i) \mid \zeta \in U(z_{k_i}) \cap F^{k_i}\} \qquad (i = 1,\dots,r)$$

are either open intervals (if F^{k_i} is degenerated into a line), or decompose into one or two upper and lower disjoint half-discs (if F^{k_i} has positive height h_{k_i}) .

Those U'_{k_i} which are intervals can be ignored, since they are identified with boundary diameters of half-discs. There remains an even number $2s$ of half-discs; their boundary radii are identified in pairs according to the pairing function λ . The resulting disc

$$U = \left(\coprod_{i=1}^{r} U'_{k_i}\right)/\sim$$

is a neighbourhood of z in $F(\mathcal{L}) = F'/\sim$. The function $(\zeta, k_i) \to (\zeta - z_{k_i})$ is well-defined and continuous; it covers the disc $\mathbb{D}_h = \{u \in \mathbb{C} \mid |u| < h/2\}$ precisely s times. Thus by choosing a branch of $\sqrt[s]{}$ we have a chart for z.

Finally, if $z = \infty$, we consider

$$U_R = \{\zeta \in \overline{\mathbb{C}} \mid |\zeta| > |s_i| \text{ for } i=1,\dots,4g\}$$

and set

$$U'_k = \{(\zeta,k) \mid \zeta \in U_R \cap F^k\}$$

for $k = o,1,\dots 4g$. As before, the U'_k are intervals, the union of two (spherical) triangles, or extended upper or lower half-planes. Their boundary is identified to form a disc

$$U = \left(\coprod_{k=o}^{4g} U'_k\right)/\sim$$

and the function $(\zeta,k) \longrightarrow \frac{1}{\zeta}$ is a homeomorphism onto $\mathbb{D}_{\frac{1}{R}} = \{u \in \mathbb{C} \mid |u| < \frac{1}{R}\}$.

Since all functions involved are holomorphic, the coordinate change between two charts with non-empty intersection is holomorphic. ∎

The surface $F(\mathcal{L})$ will always be considered with this atlas as a Riemann surface. As a distinguished point we take $P = \infty$, and as tangential direction X we take the inverse image of $-\frac{\partial}{\partial \xi}$ under the chart $(\zeta,k) \longrightarrow \frac{1}{\zeta}$, $\zeta = \xi + i\eta$.

Note that the triple $Z = [F(\mathcal{L}),X]$ represents a point in $\vec{\mathfrak{M}}(g)$.

4.7 The harmonic function $h_{\mathcal{L}}$.

When the surface $F(\mathcal{L})$ is constructed by identifying points in $F' = \coprod_k F^k$, and $F^k \subseteq \bar{\mathbb{C}}$, only points with equal real part are identified. Hence the function

(4.7.1)
$$h = h_{\mathcal{L}} : F(\mathcal{L}) - \infty \longrightarrow \mathbb{R}$$
$$h(z) = \mathrm{Re}(z)$$

is well-defined and continuous. It has one pole at $z_o = \infty$, which is a simple dipole of the form

(4.7.2)
$$h(\zeta) = \mathrm{Re}(\tfrac{1}{z}) = \frac{x}{x^2+y^2}$$

for the local parameter $\zeta = \frac{1}{z}$, $z = x + iy$ around z_o . Locally, as a function of the local parameters on $Z(\mathcal{L}) = F(\mathcal{L})$, h is the real part of a holomorphic function; thus h is harmonic,

(4.7.3)
$$\Delta h = 0 \ , \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

for each local parameter $z = x + iy$.

The gradient flow $\mathrm{grad}(h)$ of h has, in local parameters, the lines $\mathrm{Im}(z) = \text{constant}$ as flow lines. The zeroes of $\mathrm{grad}(h)$ are precisely the end points S_i of all L_i in $\mathcal{L}$: there are $4g$ of them, but on $F(\mathcal{L})$ they are identified in pairs $S_i - S_{\lambda(i)}$; counted with their multiplicities there are $2g$ zeroes.

4.8 The support of a configuration.

The positions of the slits L_i in a (regular) configuration class $\mathcal{L} = [L_1,\dots,L_{4g};\lambda]$ are no longer well-defined because of the jumps enforced by the equivalence relation. But the real parts of the endpoints $S_i = S(L_i)$ are. Therefore the two functions

$$(4.8.1) \qquad a^- : PSC(g) \longrightarrow \mathbb{R}$$

$$a^-(\mathcal{L}) = \min\{Re(S_i) \mid 1 \leq i \leq 4g\} ,$$

and

$$a^+ : PSC(g) \longrightarrow \mathbb{R}$$

$$a^+(\mathcal{L}) = \max\{Re(S_i) \mid 1 \leq i \leq 4g\}$$

are well-defined and continuous. We have

$$(4.8.2) \qquad a^-(\mathcal{L}) \leq a^+(\mathcal{L}) \qquad \text{for all } \mathcal{L} \in PS\mathbb{C}(g) ,$$

and equality can occur, e.g. for $\mathcal{L} = \mathcal{L}_o$ the base point in $PS\mathbb{C}(g)$.

Although these jumps occur a slit can only jump to another slit; hence the set of real numbers which occur as imaginary parts of endpoints S_i is still invariant (only the multiplicities are not). We can therefore define

$$(4.8.3) \qquad b^- : PS\mathbb{C}(g) \longrightarrow \mathbb{R}$$

$$b^-(\mathcal{L}) = \min\{Im(S(L_i)) \mid 1 \leq i \leq 4g\} ,$$

and

$$b^+ : PS\mathbb{C}(g) \longrightarrow \mathbb{R}$$

$$b^+(\mathcal{L}) = \max\{Im(SL_i)) \mid 1 \leq i \leq 4g\} .$$

Both functions are continuous. In contrast to (4.8.2) we now have

(4.8.4) $$b^-(\mathscr{L}) < b^+(\mathscr{L}) \quad \text{for all } \mathscr{L} \in PS\mathbb{C}(g) ,$$

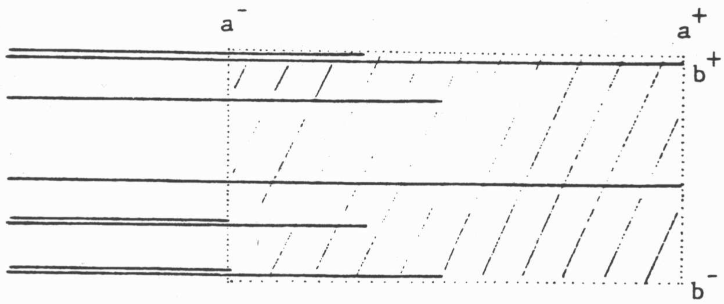
since all L_i contained in the same horizontal would contradict the regularity. (As one can easily see $F(\mathscr{L})$ would be a surface of genus 0, or a surface of genus 0 with some intervals attached like handles.)

The rectangle

(4.8.5) $$\mathrm{supp}(\mathscr{L}) = \{z = x+iy \in \mathbb{C} \mid a^-(\mathscr{L}) \leq x \leq a^+(\mathscr{L}), b^-(\mathscr{L}) \leq y \leq b^+(\mathscr{L})\}$$

is called the <u>support</u> of $\mathscr{L}$: there are no slits to the right, above or below it, and to the left there are at least no new slits. In view of (4.8.2) $\mathrm{supp}(\mathscr{L})$ may reduce to a vertical interval.

(4.8.6)



4.9. <u>The action of $\mathrm{Sim}(\mathbb{C})$ on $PS\mathbb{C}(g)$</u> .

Let $\mathrm{Sim}(\mathbb{C})$ be the group of similarities of the complex plane $\mathbb{C}$, i.e. the subgroup of $GL(2,\mathbb{C})$ consisting of matrices

(4.9.1) $$M = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} , \quad a \in \mathbb{R} , \quad a > 0 , \quad b \in \mathbb{C} .$$

The associated Möbius transformations $M(z) = az+b$ are the only automorphisms of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ fixing ∞ and mapping horizontal lines to horizontal lines. As a space $\mathrm{Sim}(\mathbb{C})$ is the product $\mathbb{C} \times \mathbb{R}_+$, hence contractible; as a group $Sim(\mathbb{C})$ is a semi-direct product of the (multiplicative) group $\mathbb{R}_+$ and the (additive) group $\mathbb{C}$, $Sim(\mathbb{C}) = \mathbb{C} \times \mathbb{R}_+$.

$Sim(\mathbb{C})$ acts on $PS\mathbb{C}(g)$ by

(4.9.2) $$M \cdot \mathcal{L} = M \cdot [L_1,\dots,L_{4g};\lambda] = [M(L_1),\dots,M(L_{4g});\lambda] .$$

Obviously $M\cdot L = (M(L_1),\dots,M(L_{4g});\lambda)$ is again a configuration; it is regular precisely if L is; and $M\cdot L \simeq M\cdot L'$ precisely if $L \simeq L'$.

(4.9.3) <u>Proposition</u>. $Sim(\mathbb{C})$ <u>acts freely on</u> $PS\mathbb{C}(g)$. <u>Moreover, the orbit projection</u> $PS\mathbb{C}(g) \longrightarrow Sim(\mathbb{C})\backslash PS\mathbb{C}(g)$ <u>is a trivial</u> $Sim(\mathbb{C})$-<u>bundle</u>.

<u>Proof</u>. For any $\mathcal{L} \in PS\mathbb{C}(g)$ we use the lower left corner of $supp(\mathcal{L})$,

$$t(\mathcal{L}) = t = a^-(\mathcal{L}) + ib^-(\mathcal{L}) \in \mathbb{C} ,$$

and its height

$$h(\mathcal{L}) = h = b^+(\mathcal{L}) - b^-(\mathcal{L}) \in \mathbb{R} ,$$

to define a continuous map $m : PS\mathbb{C}(g) \longrightarrow Sim(\mathbb{C})$, by

$$m(\mathcal{L}) = \begin{pmatrix} \frac{1}{h} & t \\ 0 & 1 \end{pmatrix} .$$

Applying $m(\mathcal{L})$ normalizes the support to lie in the upper right quarter-plane with $a^-(m(\mathcal{L})) = 0 = b^-(m(\mathcal{L}))$ and to have height 1. This induces a product decomposition

(4.9.4) $$PS\mathbb{C}(g) \xrightarrow{\cong} Sim(\mathbb{C})\backslash PS\mathbb{C}(g) \times Sim(\mathbb{C}) . \quad ■$$

(4.9.5) <u>Corollary</u>. $PS\mathbb{C}(g) \longrightarrow Sim(\mathbb{C})\backslash PS\mathbb{C}(g)$ <u>is a homotopy equivalence</u>. ∎

The functions $a^{\pm}$, $b^{\pm}$ are equivariant with respect to $Sim(\mathbb{C})$.

The reason for our interest in the action by $Sim(\mathbb{C})$ is the fact that $F(\mathcal{L})$ and $F(M\cdot\mathcal{L})$ are conformally equivalent.

(4.9.6) <u>Proposition</u>. $M \in Sim(\mathbb{C})$ <u>induces a conformal map</u>

$$C_M : F(\mathcal{L}) \xrightarrow{\cong} F(M\cdot\mathcal{L})$$

<u>which respects the distinguished point and tangential direction</u>. ∎

We will denote the orbit space by

$$(4.9.7) \qquad \mathfrak{P}\mathfrak{S}\mathbb{C}(g) = PS\mathbb{C}(g)/Sim(\mathbb{C}) \quad .$$

Note that $\mathfrak{P}\mathfrak{S}\mathbb{C}(g)$ is homotopy equivalent to $PS\mathbb{C}(g)$ since

$$(4.9.8) \qquad PS\mathbb{C}(g) \xrightarrow{\cong} \mathfrak{P}\mathfrak{S}\mathbb{C}(g) \times Sim(\mathbb{C})$$

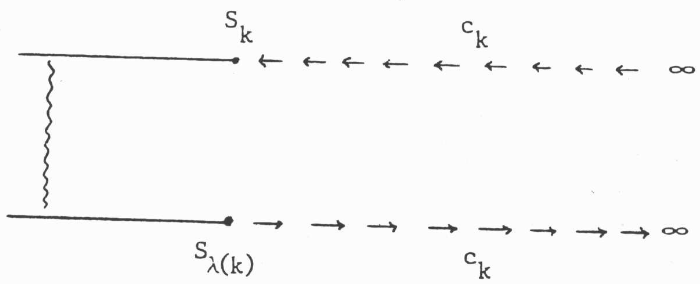
is a homeomorphism; therefore

$$(4.9.9) \qquad \dim \mathfrak{P}\mathfrak{S}\mathbb{C}(g) = 6g - 3 \quad .$$

4.10 The canonical homology basis of $F(L)$.

Let $L = (L_1,\dots,L_{4g};\lambda)$ be a configuration. The basepoint of $F(L)$ is $z_o = \infty$. Define the curve c_k $(k = 1,\dots,4g)$ to be the curve from z_o to S_k along the horizontal $\mathrm{Im}(z) = \mathrm{Im}(S_k)$, and then from $S_{\lambda(k)}$ back to z_o along the horizontal $\mathrm{Im}(z) = \mathrm{Im}(S_{\lambda(k)})$.

(4.10.1)



Since $S_k \sim S_{\lambda(k)}$ in $F(L)$ this is a closed curve. We have

$$(4.10.2) \qquad c_{\lambda(k)} = c_k^{-1} \quad .$$

Of course, the curves and their homotopy classes depend on L and not only on the equivalence class of L . They constitute a marking of the (Riemann) surface $F(L)$.

These curves generate the fundamental group. As elements of $H_1F(L)$ they form a basis (plus negatives). If $L = L_o$ is the (standard) basepoint in $PS\mathbb{C}(g)$, then $a_i = c_i$ $(i = 1,\dots,g)$ and $b_i = c_{g+1}$ $(i = 1,\dots,g)$ form a symplectic basis with respect to the cup product $H^1 \times H^1 \longrightarrow H^2 \cong \mathbb{Z}$, or to the intersection product $\# : H_1 \times H_1 \longrightarrow \mathbb{Z}$.

The curves c_k can be used to formulate another regularity criterion: L is regular if and only if none of the curves $c_1,\dots,c_{4g}$ is in $F(L)$ freely homotopic to a constant curve. As can be seen in the figures (4.2.5) (3) the pinching of such a curve degenerates the surfaces. This characterization is already in [Shiffman 1939; p.862].

4.11 The canonical polygon of $F(\mathcal{L})$.

Let $L = (L_1,\dots,L_{4g};\lambda)$ be a non-degenerate configuration. Recall that the strips F^k of 4.2 all have $z_o = \infty$ in common. We consider only proper strips of positive height and disregard those degenerated into lines. If we delete z_o then $F^k - z_o$ is a closed subset of $\mathbb{C}$. For $k=1,\dots,4g-1$ we attach two ends $-\infty$, $+\infty$ to obtain a compact $\overline{F}^k$; for $k=0$, $4g$ we attach one end, called $+\infty$ in both cases. To make all $\overline{F}^k$ disjoint, we denote points as usual by (z,k). In the disjoint union

$$(4.11.1) \qquad \overline{F}' = \coprod \overline{F}^k$$

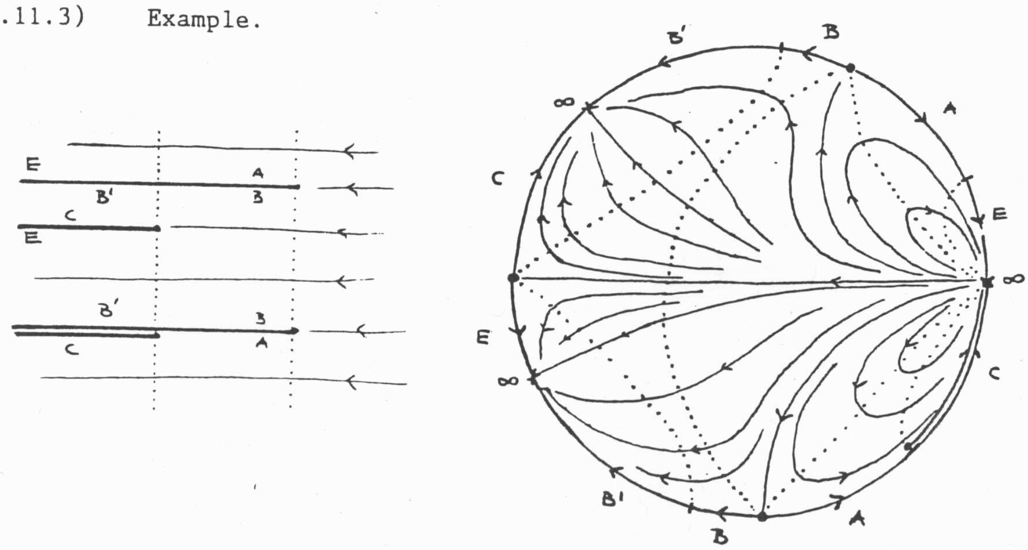
we identify points by

$$(4.11.2) \qquad \text{(i)} \quad (+\infty,k) \sim (+\infty,k') \qquad \text{for all } k, k' \ ,$$

$$\text{(ii)} \quad (z,k-1) \sim (z,k) \qquad \text{for } z \in L_k\,, \ \mathrm{Re}(z) \geq \mathrm{Re}(S_k) \ .$$

Then $\overline{F}(L) = \overline{F}'/\sim$ is a schlicht surface with one boundary component. It is therefore a disc. The boundary is partitioned into at most $8g$ arcs paired by λ.

(4.11.3) Example.



4.12 The canonical rectangulation of $F(\mathcal{L})$.

Let L be a regular configuration, $L = (L_1, \dots, L_{4g}; \lambda)$. Consider the (not necessarily different) horizontals and verticals

$$(4.12.1) \qquad H_i = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = \operatorname{Im}(S_i)\} \quad , \qquad i = 1, \dots, 4g \quad ,$$

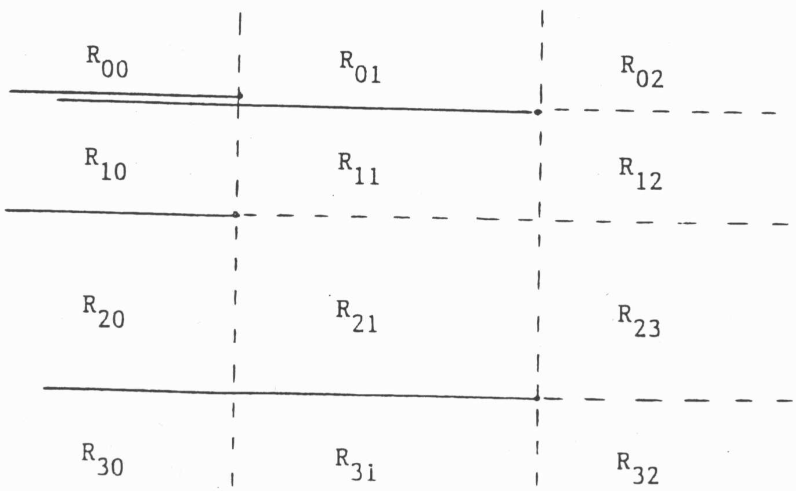
$$V_i = \{z \in \mathbb{C} \mid \operatorname{Re}(z) = \operatorname{Re}(S_i)\} \quad , \qquad i = 1, \dots, 4g \quad .$$

The horizontals H_i , used to build $F(L)$, lead already to a decomposition of $F(L)$ into biangles F^k . Now we decompose further

$$(4.12.2) \qquad R_{ij} = \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{l} \operatorname{Im}(S_i) \leq \operatorname{Im}(z) \leq \operatorname{Im}(S_{i+1}) \\ \operatorname{Re}(S_j) \leq \operatorname{Re}(z) \leq \operatorname{Re}(S_{j+1}) \end{array} \right\}$$

for $0 \leq i,j \leq 4g+1$, where the expressions $\operatorname{Im}(S_i)$, $\operatorname{Re}(S_j)$ are to be disregarded for $i,j = 0$ or $i,j = 4g+1$. Some of these R_{ij} are triangles, four are biangles, all others are rectangles, or degenerate into points or lines.

(4.12.3)



The union of the bounded R_{ij} is the support $\operatorname{supp}(L)$ of the configuration.