LOCAL-GLOBAL COMPATIBILITY AND THE ACTION OF MONODROMY ON NEARBY CYCLES

ANA CARAIANI

Abstract

We strengthen the local-global compatibility of Langlands correspondences for GL_n *in the case when* n *is even and* $l \neq p$*. Let* L *be a CM field, and let* Π *be a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_L)$ *which is conjugate self-dual. Assume that* Π_∞ *is cohomological and not "slightly regular," as defined by Shin. In this case, Chenevier and Harris constructed an* l*-adic Galois representation* $R_l(\Pi)$ *and proved the local-global compatibility up to semisimplification at primes* v *not dividing* l*. We extend this compatibility by showing that the Frobenius semisimplification of the restriction of* $R_l(\Pi)$ *to the decomposition group at* v *corresponds to the image of* Π_v *via the local Langlands correspondence. We follow the strategy of Taylor and Yoshida, where it was assumed that* Π *is square-integrable at a finite place. To make the argument work, we study the action of the monodromy operator* N *on the complex of nearby cycles on a scheme which is locally étale over a product of strictly semistable schemes and we derive a generalization of the weight spectral sequence in this case. We also prove the Ramanujan–Petersson conjecture for* Π *as above.*

Contents

DUKE MATHEMATICAL JOURNAL
Vol. 161, No. 12, © 2012 DOI 10.1215/00127094-1723706
Received 7 December 2010. Revision received 12 January 2012.
2010 *Mathematics Subject Classification*. Primary 11R39; Secondary 11F70, 11F80, 14G35.

1. Introduction

In this paper, we strengthen the local-global compatibility of the Langlands correspondence.

THEOREM 1.1

Let $n \in \mathbb{Z}_{\geq 2}$ be an integer, and let L be any CM field with complex conjugation c. Let l be a prime of $\mathbb{Q}$, and let ι_l be an isomorphism $\iota_l : \bar{\mathbb{Q}}_l \to \mathbb{C}$. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ such that

- $\Pi^\vee \simeq \Pi \circ c$;
- *Π is cohomological for some irreducible algebraic representation Ξ of $\mathrm{GL}_n(L \otimes_{\mathbb{Q}} \mathbb{C})$.*

Let

$$R_l(\Pi) : \mathrm{Gal}(\bar{L}/L) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$$

be the Galois representation associated to Π by [Sh3] and [CH]. Let $p \neq l$, and let y be a place of L above p. Then we have the following isomorphism of Weil–Deligne representations

$$WD\big(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)}\big)^{F-\mathrm{ss}} \simeq \iota_l^{-1}\mathcal{L}_{n,L_y}(\Pi_y).$$

Here $\mathcal{L}_{n,L_y}(\Pi_y)$ is the image of Π_y under the local Langlands correspondence, where the geometric normalization is used.

In the process of proving Theorem 1.1, we also prove the Ramanujan–Petersson conjecture for Π as above.

THEOREM 1.2

Let $n \in \mathbb{Z}_{\geq 2}$ be an integer, and let L be any CM field. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ such that

- $\Pi^\vee \simeq \Pi \circ c$;
- *Π_∞ is cohomological for some irreducible algebraic representation Ξ of $\mathrm{GL}_n(L \otimes_{\mathbb{Q}} \mathbb{C})$.*

Then Π is tempered at any finite place of L.

The above theorems are already known when n is odd, or when n is even and Π is slightly regular, by the work of Shin [Sh3]. They are also known if Π is square-integrable at a finite place by the work of Harris and Taylor [HT] and Taylor and Yoshida [TY]. If n is even, then Chenevier and Harris [CH] construct a global $\mathrm{Gal}(\bar{L}/L)$-representation $R_l(\Pi)$ which is compatible with the local Langlands correspondence up to semisimplification. Theorem 1.2 was proven by Clozel [Cl2] at the places where Π is unramified. We extend the local-global compatibility up to Frobenius semisimplification by proving that both Weil–Deligne representations are pure. We use Theorem 1.2 to deduce that $\iota_l^{-1}\mathcal{L}_{n,\mathcal{L}_y}(\Pi_y)$ is pure. For the representation $WD(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)})$, our strategy is as follows: we find the Galois representation $R_l(\Pi)^{\otimes 2}$ in the cohomology of a system of Shimura varieties X_U associated to a unitary group which looks like

$$U(1,n-1)\times U(1,n-1)\times U(0,n)^{d-2}$$

at infinity. Following the same structure of argument as Taylor and Yoshida in [TY], we prove that the Weil–Deligne representation associated to

$$R_l(\Pi)^{\otimes 2}|_{\mathrm{Gal}(\bar{L}_y/L)}$$

is pure by explicitly computing the action of the monodromy operator N on the cohomology of the system of Shimura varieties. We use Theorem 1.2 at a crucial point in the computation. We conclude that $WD(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)})^{F-\mathrm{ss}}$ must also be pure.

We briefly outline our computation of the action of N on the Weil–Deligne representation associated to $R_l(\Pi)^{\otimes 2}|_{\mathrm{Gal}(\bar{L}_y/L_y)}$. First, we base change Π to a CM field F' such that there is a place $\mathfrak{p}$ of F' above the place y of L where $BC_{F'/L}(\Pi)_{\mathfrak{p}}$ has an Iwahori fixed vector. It suffices to study the Weil–Deligne representation corresponding to $\Pi^0 = BC_{F'/L}(\Pi)$ and to prove that it is pure. We then take a quadratic extension F of F' which is also a CM field and in which the place $\mathfrak{p}$ splits $\mathfrak{p}=\mathfrak{p}_1\mathfrak{p}_2$. We let $\sigma\in\mathrm{Gal}(F/F')$ be the automorphism which sends $\mathfrak{p}_1$ to $\mathfrak{p}_2$. We choose F and F' such that they contain an imaginary quadratic field E in which p splits. We take a $\mathbb{Q}$-group G which satisfies the following:

- G is quasi-split at all finite places;
- $G(\mathbb{R})$ has signature $(1,n-1)$ at two embeddings which differ by σ and $(0,n)$ everywhere else;
- $G(\mathbb{A}_E)\simeq \mathrm{GL}_1(\mathbb{A}_E)\times\mathrm{GL}_n(\mathbb{A}_F)$.

We let $\Pi^1 = BC_{F/F'}(\Pi^0)$. Then the Galois representation $R_l(\Pi^0)$ can be seen in the $\Pi^{1,\infty}$-part of the (base change of the) cohomology of a system of Shimura varieties associated to G. We let X_U be the inverse system of Shimura varieties associated to the group G. We let the level U vary outside $\mathfrak{p}_1\mathfrak{p}_2$ and be equal to the Iwahori subgroup at $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We construct an integral model of X_U which parameterizes abelian varieties with Iwahori level structure at $\mathfrak{p}_1$ and $\mathfrak{p}_2$. By abuse of notation, we will denote this integral model by X_U as well. The special fiber Y_U of X_U has a stratification by $Y_{U,S,T}$, where the $S,T\subseteq\{1,\dots,n\}$ are related to the Newton polygons of the p-divisible groups above $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We compute the completed strict local rings at closed geometric points of X_U and use this computation to show that X_U is locally étale over a product of strictly semistable schemes, which on the special fiber are closely related to the strata $Y_{U,S,T}$. If we let $\mathcal{A}_U$ be the universal abelian variety over X_U, then $\mathcal{A}_U$ has the same stratification and the same geometry as X_U.

Let ξ be an irreducible algebraic representation of G over $\bar{\mathbb{Q}}_l$, which determines nonnegative integers t_ξ, m_ξ and an endomorphism $a_\xi \in \mathrm{End}(\mathcal{A}_U^{m_\xi}/X_U) \otimes_{\mathbb{Z}} \mathbb{Q}$. We are interested in understanding the $\Pi^{1,\infty}$-part of

$$H^j(X_U, \mathcal{L}_\xi) = a_\xi H^{j+m_\xi}\big(\mathcal{A}_U^{m_\xi}, \bar{\mathbb{Q}}_l(t_\xi)\big).$$

Thus, we study the cohomology of the generic fiber $H^j(\mathcal{A}_U^{m_\xi}, \bar{\mathbb{Q}}_l)$, and we do so via the cohomology of the complex of nearby cycles $R\psi\bar{\mathbb{Q}}_l$ over the special fiber of $\mathcal{A}_U^{m_\xi}$. The key ingredients in studying the complex of nearby cycles together with the action of monodromy are logarithmic schemes, the Rapoport–Zink weight spectral sequence (see [RZ]) as constructed by Saito [Sa] (which on the level of complexes of sheaves describes the action of monodromy on the complex of nearby cycles for strictly semistable schemes), and the formula

$$(R\psi\bar{\mathbb{Q}}_l)_{X_1 \times X_2} \simeq (R\psi\bar{\mathbb{Q}}_l)_{X_1} \otimes^L (R\psi\bar{\mathbb{Q}}_l)_{X_2},$$

where X_1 and X_2 are semistable schemes. Using these ingredients, we deduce the existence of a spectral sequence relating terms of the form $H^j(\mathcal{A}_{U,S,T}^{m_\xi}, \bar{\mathbb{Q}}_l)$ (up to twisting and shifting) to the object we are interested in, $H^j(\mathcal{A}_U^{m_\xi}, \bar{\mathbb{Q}}_l)$. The cohomology of each stratum $H^j(\mathcal{A}_{U,S,T}^{m_\xi}, \bar{\mathbb{Q}}_l)$ is closely related to the cohomology of Igusa varieties. The next step is to compute the $\Pi^{1,\infty}$-part of the cohomology of certain Igusa varieties, for which we adapt the strategy of [Sh3, Theorem 6.1]. Using the result on Igusa varieties, we prove Theorem 1.2 and then we also make use of the classification of tempered representations. We prove that the $\Pi^{1,\infty}$-part of each $H^j(\mathcal{A}_{U,S,T}^{m_\xi}, \bar{\mathbb{Q}}_l)$ vanishes outside the middle dimension and thus that our spectral sequence degenerates at E_1. The E_1 page of the spectral sequence provides us with the exact filtration of the $\Pi^{1,\infty}$-part of

$$\varinjlim_{U^p} H^{2n-2}(X_U, \mathcal{L}_\xi),$$

which exhibits its purity.

2. An integral model

2.1. Shimura varieties

Let E be an imaginary quadratic field in which p splits, let c be the nontrivial element in $\mathrm{Gal}(E/\mathbb{Q})$, and choose a prime u of E above p. From now on, we assume that n is an even positive integer.

Let F_1 be a totally real field of finite degree over $\mathbb{Q}$, and let w be a prime of F_1 above p. Let F_2 be a quadratic totally real extension of F_1 in which w splits $w = w_1 w_2$. Let $d = [F_2 : \mathbb{Q}]$, and we assume that $d \geq 3$. Let $F = F_2.E$. Let $\mathfrak{p}_i$ be the prime of F above w_i and u for $i = 1, 2$. We denote by $\mathfrak{p}_i$ for $2 < i \leq r$ the rest of the primes which lie above the prime u of E. We choose embeddings $\tau_i : F \hookrightarrow \mathbb{C}$ with $i = 1, 2$ such that $\tau_2 = \tau_1 \circ \sigma$, where σ is the element of $\mathrm{Gal}(F/\mathbb{Q})$ which takes $\mathfrak{p}_1$ to $\mathfrak{p}_2$. In particular, this means that $\tau_E := \tau_1|_E = \tau_2|_E$ is well defined. By abuse of notation, we also denote by σ the Galois automorphism of F_2 taking w_1 to w_2.

We work with a Shimura variety corresponding to the PEL datum $(F, *, V, \langle\cdot,\cdot\rangle, h)$, where F is the CM field defined above and where $* = c$ is the involution given by complex conjugation. We take V to be the F-vector space F^n for some integer n. The pairing

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{Q}$$

is a nondegenerate Hermitian pairing such that $\langle f v_1, v_2\rangle = \langle v_1, f^* v_2\rangle$ for all $f \in F$ and $v_1, v_2 \in V$. The last element we need is an $\mathbb{R}$-algebra homomorphism $h : \mathbb{C} \to \mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R}$ such that the bilinear pairing

$$(v_1, v_2) \to \langle v_1, h(i) v_2\rangle$$

is symmetric and positive definite.

We define an algebraic group G over $\mathbb{Q}$ by

$$G(R) = \big\{(g, \lambda) \in \mathrm{End}_{F\otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \times R^\times \;\big|\; \langle g v_1, g v_2\rangle = \lambda \langle v_1, v_2\rangle\big\}$$

for any $\mathbb{Q}$-algebra R. For $\sigma \in \mathrm{Hom}_{E,\tau_E}(F, \mathbb{C})$, we let (p_σ, q_σ) be the signature at σ of the pairing $\langle\cdot,\cdot\rangle$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$. We claim that we can find a PEL datum as above such that $(p_\tau, q_\tau) = (1, n-1)$ for $\tau = \tau_1$ or τ_2 and $(p_\tau, q_\tau) = (0, n)$ otherwise and such that $G_{\mathbb{Q}_v}$ is quasi-split at every finite place v.

LEMMA 2.1
Let F be a CM field as above. For any embeddings $\tau_1, \tau_2 : F \hookrightarrow \mathbb{C}$ there exists a PEL datum $(F, *, V, \langle\cdot,\cdot\rangle, h)$ as above such that the associated group G is quasi-split at every finite place and has signature $(1, n-1)$ at τ_1 and τ_2 and $(0, n)$ everywhere else.

Proof
This lemma is standard and follows from computations in Galois cohomology found in [Cl1, Section 2], but see also [HT, Lemma 1.7]. The problem is that of constructing a global unitary similitude group with prescribed local conditions. It is enough to

consider the case of a unitary group G^0 over $\mathbb{Q}$, by taking it to be the algebraic group defined by $\ker(G(R) \to R^\times)$ sending $(g, \lambda) \mapsto \lambda$.

A group G defined as above has a quasi-split inner form over $\mathbb{Q}$ denoted G_n, defined as in Section 3 of [Sh3]. This inner form G_n is the group of similitudes which preserves the nondegenerate Hermitian pairing $\langle v_1, v_2 \rangle = v_1 \zeta \Phi^t v_2^c$ with $\Phi \in \mathrm{GL}_n(\mathbb{Q})$ having entries

$$\Phi_{ij} = (-1)^{i+1} \delta_{i,n+1-j}$$

and $\zeta \in F^*$ an element of trace 0. Let G' be the adjoint group of G_n^0. It suffices to show that the tuple of prescribed local conditions, classified by elements in $\bigoplus_v H^1(F_{2,v}, G')$, is in the image of the map

$$H^1(F_2, G') \to \bigoplus_v H^1(F_{2,v}, G'),$$

where the sum is taken over all places v of F_2. For n odd, [Cl1, Lemme 2.1] ensures that the above map is surjective, so there is no cohomological obstruction for finding the global unitary group. In the case we are interested in, n is even and the image of the above map is equal to the kernel of

$$\bigoplus_v H^1(F_{2,v}, G') \to \mathbb{Z}/2\mathbb{Z}.$$

We can use [Cl1, Lemme 2.2] to compute all the local invariants (i.e., the images of $H^1(F_{2,v}, G') \to \mathbb{Z}/2\mathbb{Z}$ for all places v). At the finite places, the sum of the invariants is $0 \pmod 2$ (this is guaranteed by the existence of the quasi-split inner form G_n of G, which has the same local invariants at finite places). At the infinite places τ_1 and τ_2 the invariants are $n/2 + 1 \pmod 2$, and at all other infinite places they are $n/2 \pmod 2$. The global invariant is $nd/2 + 2 \pmod 2$, where d is the degree of F_2 over $\mathbb{Q}$. Since d is even, the image in $\mathbb{Z}/2\mathbb{Z}$ is equal to $0 \pmod 2$, so the prescribed local unitary groups arise from a global unitary group. □

We choose the $\mathbb{R}$-homomorphism $h : \mathbb{C} \to \mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R}$ such that under the natural $\mathbb{R}$-algebra isomorphism $\mathrm{End}_F(V)_{\mathbb{R}} \simeq \prod_{\tau|_E = \tau_E} M_n(\mathbb{C})$ it equals

$$z \mapsto \left(\begin{pmatrix} z I_{p_\tau} & 0 \\ 0 & \bar{z} I_{q_\tau} \end{pmatrix}_\tau \right),$$

where τ runs over elements of $\mathrm{Hom}_{E,\tau_E}(F, \mathbb{C})$.

Now that we have defined the PEL datum, we can set up our moduli problem. Note that the reflex field of the PEL datum is $F' = F_1 \cdot E$. Let S/F' be a scheme, and let A/S be an abelian scheme of dimension dn. Suppose that we have an embedding

$i : F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Lie A is a locally free $\mathcal{O}_S$-module of rank dn with an action of F. We can decompose $\mathrm{Lie}\, A = \mathrm{Lie}^+ A \oplus \mathrm{Lie}^- A$, where $\mathrm{Lie}^+ A = \mathrm{Lie}\, A \otimes_{\mathcal{O}_S \otimes E} \mathcal{O}_{\mathcal{S}}$, and the map $E \hookrightarrow F' \to \mathcal{O}_{\mathcal{S}}$ is the natural map followed by the structure map. $\mathrm{Lie}^- A$ is defined in the same way using the complex conjugate of the natural map $E \hookrightarrow F'$. We ask that $\mathrm{Lie}^+ A$ be a free $\mathcal{O}_S$-module of rank 2 and that $\mathrm{Lie}^+ A \simeq \mathcal{O}_S \otimes_{F_1} F_2$ be an $\mathcal{O}_S$-module with an action of F_2.

Definition 2.2
If the conditions above are satisfied, we call the pair (A, i) *compatible*.

Remark
This is an adaptation to our situation of the notion of compatibility defined in [HT, Section III.1], which fulfills the same purpose as the determinant condition defined in [Ko2, p. 390].

For an open compact subgroup $U \subset G(\mathbb{A}^\infty)$, we consider the contravariant functor $\mathfrak{X}_U$ mapping

$$\begin{pmatrix} \text{Connected, locally Noetherian} \\ F'\text{-schemes with geometric point} \\ (S, s) \end{pmatrix} \to (\text{Sets}),$$

$$(S, s) \mapsto \{(A, \lambda, i, \bar{\eta})\}/\sim,$$

where

- A is an abelian scheme over S;
- $\lambda : A \to A^\vee$ is a polarization;
- $i : F \hookrightarrow \mathrm{End}^0(A) = \mathrm{End}\, A \otimes_{\mathbb{Z}} \mathbb{Q}$ is such that (A, i) is compatible and $\lambda \circ i(f) = i(f^*)^\vee \circ \lambda$, for all $f \in F$;
- $\bar{\eta}$ is a $\pi_1(S, s)$-invariant U-orbit of isomorphisms of Hermitian $F \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules

  $$\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s,$$

  which take the fixed pairing $\langle \cdot, \cdot \rangle$ on V to an $(\mathbb{A}^\infty)^\times$-multiple of the λ-Weil pairing on VA_s; here,

  $$VA_s = \big(\varprojlim A[N](k(s))\big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

  is the adelic Tate module.

We consider two quadruples as above equivalent if there is an isogeny between the abelian varieties which is compatible with the additional structures. If s' is a different geometric point of S, then there is a canonical bijection between $\mathfrak{X}_U(S, s)$

and $\mathfrak{X}_U(S,s')$. We can forget about the geometric points and extend the definition from connected to arbitrary locally Noetherian F'-schemes. When U is sufficiently small, this functor is representable by a smooth and quasi-projective variety X_U/F' of dimension $2n-2$ (this is explained in [Ko2, p. 391]). The variety X_U is a disjoint union of $|\ker^1(G,\mathbb{Q})|$ copies of the canonical model of the Shimura variety. As U varies, the inverse system of the X_U has a natural right action of $G(\mathbb{A}^\infty)$.

Let $\mathcal{A}_U$ be the universal abelian variety over X_U. The action of $G(\mathbb{A}^\infty)$ on the inverse system of the X_U extends to an action by quasi-isogenies on the inverse system of the $\mathcal{A}_U$. The following construction goes through as in [HT, Section III.2]. Let l be a rational prime (we impose no conditions on l yet, but we restrict to l different from p when we work with an integral model over the ring of integers in a p-adic field), and let ξ be an irreducible algebraic representation of G over $\mathbb{Q}_l^{ac}$. This defines a lisse $\mathbb{Q}_l^{ac}$-sheaf $\mathcal{L}_{\xi,l}$ over each X_U, and the action of $G(\mathbb{A}^\infty)$ extends to the inverse system of sheaves. The direct limit

$$H^i(X,\mathcal{L}_{\xi,l}) = \lim_{\rightarrow} H^i(X_U \times_{F'} \bar{F}', \mathcal{L}_{\xi,l})$$

is a (semisimple) admissible representation of $G(\mathbb{A}^\infty)$ with a continuous action of $\mathrm{Gal}(\bar{F}'/F')$. We can decompose it as

$$H^i(X,\mathcal{L}_{\xi,l}) = \bigoplus_{\pi} \pi \otimes R^i_{\xi,l}(\pi),$$

where the sum runs over irreducible admissible representations π of $G(\mathbb{A}^\infty)$ over $\mathbb{Q}_l^{ac}$. The $R^i_{\xi,l}(\pi)$ are finite-dimensional continuous representations of $\mathrm{Gal}(\bar{F}'/F')$ over $\mathbb{Q}_l^{ac}$. We suppress the l from $\mathcal{L}_{\xi.l}$ and $R^i_{\xi,l}(\pi)$ where it is understood from context. To the irreducible representation ξ of G we can associate as in [HT, Section III.2] nonnegative integers m_ξ and t_ξ and an idempotent $\epsilon_\xi \in \mathbb{Q}[S_{m_\xi}]$ (where S_{m_ξ} is the symmetric group on m_ξ letters). As in [TY, p. 476], define for each integer $N \geq 2$,

$$\epsilon(m_\xi,N) = \prod_{x=1}^{m_\xi} \prod_{y\neq 1} \frac{[N]_x - N}{N - N^y} \in \mathbb{Q}[(N^{\mathbb{Z}_{\geq 0}})^{m_\xi}],$$

where $[N]_x$ denotes the endomorphism generated by multiplication by N on the xth factor and y ranges from 0 to $2[F_2:\mathbb{Q}]n^2$ but excluding 1. Set

$$a_\xi = a_{\xi,N} = \epsilon_\xi P\big(\epsilon(m_\xi,N)\big),$$

which can be thought of as an element of $\mathrm{End}(\mathcal{A}_U^{m_\xi}/X_U) \otimes_{\mathbb{Z}} \mathbb{Q}$. Here $P(\epsilon(m_\xi,N))$ is the polynomial

$$P(X) = \big((X-1)^{4n-3}+1\big)^{4n-3}.$$

If we let $\mathrm{proj}: \mathcal{A}_U^{m_\xi} \to X_U$ be the natural projection, then $\epsilon(m_\xi, N)$ is an idempotent on each of the sheaves $R^j \mathrm{proj}_* \bar{\mathbb{Q}}_l(t_\xi)$, hence also on

$$H^i\big(X_U \times_{F'} \bar{F}', R^j \mathrm{proj}_* \bar{\mathbb{Q}}_l(t_\xi)\big) \Rightarrow H^{i+j}\big(\mathcal{A}_U^{m_\xi} \times_{F'} \bar{F}', \bar{\mathbb{Q}}_l(t_\xi)\big).$$

We get an endomorphism $\epsilon(m_\xi, N)$ of $H^{i+j}(\mathcal{A}_U^{m_\xi} \times_{F'} \bar{F}', \bar{\mathbb{Q}}_l(t_\xi))$, which is an idempotent on each graded piece of a filtration of length at most $4n-3$. In this case, $P(\epsilon(m_\xi, N))$ must be an idempotent on all of $H^{i+j}(\mathcal{A}_U^{m_\xi} \times_{F'} \bar{F}', \bar{\mathbb{Q}}_l(t_\xi))$. We have an isomorphism

$$H^i(X_U \times_{F'} \bar{F}', \mathcal{L}_\xi) \cong a_\xi H^{i+m_\xi}\big(\mathcal{A}_U^{m_\xi} \times_{F'} \bar{F}', \bar{\mathbb{Q}}_l(t_\xi)\big),$$

which commutes with the action of $G(\mathbb{A}^\infty)$.

2.2. An integral model for Iwahori level structure

Let $K = F_{\mathfrak{p}_1} \simeq F_{\mathfrak{p}_2}$, where the isomorphism is via σ, and denote by $\mathcal{O}_K$ the ring of integers of K and by π a uniformizer of $\mathcal{O}_K$.

Let $S/\mathcal{O}_K$ be a scheme, and let A/S be an abelian scheme of dimension dn. Suppose that we have an embedding $i: \mathcal{O}_F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Lie A is a locally free $\mathcal{O}_S$-module of rank dn with an action of F. We can decompose $\mathrm{Lie}\, A = \mathrm{Lie}^+ A \oplus \mathrm{Lie}^- A$, where $\mathrm{Lie}^+ A = \mathrm{Lie}\, A \otimes_{\mathbb{Z}_p \otimes \mathcal{O}_E} \mathcal{O}_{E,u}$. There are two natural actions of $\mathcal{O}_F$ on $\mathrm{Lie}^+ A$ via $\mathcal{O}_F \to \mathcal{O}_{F_{\mathfrak{p}_j}} \xrightarrow{\sim} \mathcal{O}_K$ composed with the structure map for $j = 1, 2$. These two actions differ by the automorphism $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. There is also a third action via the embedding i of $\mathcal{O}_F$ into the ring of endomorphisms of A. We ask that $\mathrm{Lie}^+ A$ be locally free of rank 2, that the part of $\mathrm{Lie}^+ A$ where the first action of $\mathcal{O}_F$ on $\mathrm{Lie}^+ A$ coincides with i be locally free of rank 1, and that the part where the second action coincides with i also be locally free of rank 1.

Definition 2.3

If the above conditions are satisfied, then we call (A, i) *compatible*. One can check that, for S/K, this notion of compatibility coincides with the one in Definition 2.2.

If p is locally nilpotent on S, then (A, i) is compatible if and only if

- $A[\mathfrak{p}_i^\infty]$ is a compatible, 1-dimensional Barsotti–Tate $\mathcal{O}_K$-module for $i = 1, 2$, and
- $A[\mathfrak{p}_i^\infty]$ is ind-étale for $i > 2$.

By a compatible Barsotti–Tate $\mathcal{O}_K$-module, we mean that the two actions on it by $\mathcal{O}_K$ via endomorphisms or via the structure map coincide.

We now define a few integral models for our Shimura varieties X_U. We can decompose $G(\mathbb{A}^\infty)$ as

$$G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \prod_{i=1}^r \mathrm{GL}_n(F_{\mathfrak{p}_i}).$$

For each i, let Λ_i be an $\mathcal{O}_{F_{\mathfrak{p}_i}}$-lattice in $F_{\mathfrak{p}_i}^n$ which is stable under $\mathrm{GL}_n(\mathcal{O}_{F_{\mathfrak{p}_i}})$ and self-dual with respect to $\langle\cdot,\cdot\rangle$. For each $\vec{m} = (m_1, \dots, m_r)$ and compact open subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$, we define the compact open subgroup $U^p(\vec{m})$ of $G(\mathbb{A}^\infty)$ as

$$U^p(\vec{m}) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^r \ker\bigl(\mathrm{GL}_{\mathcal{O}_{F_{\mathfrak{p}_i}}}(\Lambda_i) \to \mathrm{GL}_{\mathcal{O}_{F_{\mathfrak{p}_i}}}(\Lambda_i/\mathfrak{m}_{F_{\mathfrak{p}_i}}^{m_i}\Lambda_i)\bigr).$$

The corresponding moduli problem of sufficiently small level $U^p(\vec{m})$ over $\mathcal{O}_K$ is given by the functor

$$\begin{pmatrix} \text{Connected, locally Noetherian} \\ \mathcal{O}_K\text{-schemes with geometric point} \\ (S,s) \end{pmatrix} \to (\text{Sets}),$$

$$(S,s) \mapsto \bigl\{(A, \lambda, i, \bar{\eta}^p, \{\alpha_i\}_{i=1}^r)\bigr\}/\sim,$$

where

- A is an abelian scheme over S;
- $\lambda : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that (A,i) is compatible and $\lambda \circ i(f) = i(f^*)^\vee \circ \lambda, \forall f \in \mathcal{O}_F$;
- $\bar{\eta}^p$ is a $\pi_1(S,s)$-invariant U^p-orbit of isomorphisms of Hermitian $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules

  $$\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s,$$

  which take the fixed pairing $\langle\cdot,\cdot\rangle$ on V to an $(\mathbb{A}^{\infty,p})^\times$-multiple of the λ-Weil pairing on VA_s (here, V^pA_s is the adelic Tate module away from p);
- for $i = 1, 2$, $\alpha_i : \mathfrak{p}_i^{-m_i}\Lambda_i/\Lambda_i \to A[\mathfrak{p}_i^{m_i}]$ is a Drinfeld $\mathfrak{p}_i^{m_i}$-structure, that is, the set of $\alpha_i(x), x \in (\mathfrak{p}_i^{-m_i}\Lambda_i/\Lambda_i)$ forms a full set of sections of $A[\mathfrak{p}_i^{m_i}]$ in the sense of [KM, Section 1.8];
- for $i > 2$, $\alpha_i : (\mathfrak{p}_i^{-m_i}\Lambda_i/\Lambda) \xrightarrow{\sim} A[\mathfrak{p}_i^{m_i}]$ is an isomorphism of S-schemes with $\mathcal{O}_{F_{\mathfrak{p}_i}}$-actions;
- two tuples $(A, \lambda, i, \bar{\eta}^p, \{\alpha_i\}_{i=1}^r)$ and $(A', \lambda', i', (\bar{\eta}^p)', \{\alpha_i'\}_{i=1}^r)$ are equivalent if there is a prime-to-p isogeny $A \to A'$ taking $\lambda, i, \bar{\eta}^p, \alpha_i$ to $\gamma\lambda', i', (\bar{\eta}^p)', \alpha_i'$ for some $\gamma \in \mathbb{Z}_{(p)}^\times$.

This moduli problem is representable by a projective scheme over $\mathcal{O}_K$, which will be denoted $X_{U^p,\vec{m}}$. The projectivity follows from [L, Theorem 5.3.3.1, Remark 5.3.3.2]. If $m_1 = m_2 = 0$, this scheme is smooth as in [HT, Lemma III.4.1.2], since we can check smoothness on the completed strict local rings at closed geometric points and these are isomorphic to deformation rings for p-divisible groups (with level structure only at $\mathfrak{p}_i$ for $i > 2$, when the p-divisible group is étale). Moreover, if $m_1 = m_2 = 0$, the dimension of $X_{U^p,\vec{m}}$ is $2n - 1$.

When $m_1 = m_2 = 0$, we denote $X_{U^p,\vec{m}}$ by X_{U_0}. If $\mathcal{A}_{U_0}$ is the universal abelian scheme over X_{U_0}, we write $\mathcal{G}_i = \mathcal{A}_{U_0}[\mathfrak{p}_i^\infty]$ for $i = 1, 2$ and $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. Over a base where p is nilpotent, each of the $\mathcal{G}_i$ is a 1-dimensional compatible Barsotti–Tate $\mathcal{O}_K$-module.

Let $\mathbb{F}$ be the residue field of $\mathcal{O}_K$. Let $\bar{X}_{U_0} = X_{U_0} \times_{\operatorname{Spec}\mathcal{O}_K} \operatorname{Spec}\mathbb{F}$ be the special fiber of X_{U_0}. We define a stratification on $\bar{X}_{U_0}$ in terms of $0 \leq h_1, h_2 < n - 1$. The scheme $\bar{X}_{U_0}^{[h_1,h_2]}$ will be the reduced closed subscheme of $\bar{X}_{U_0}$ whose closed geometric points s are those for which the maximal étale quotient of $\mathcal{G}_i$ has $\mathcal{O}_K$-height at most h_i. Let $\bar{X}_{U_0}^{(h_1,h_2)} = \bar{X}_{U_0}^{[h_1,h_2]} - (\bar{X}_{U_0}^{[h_1-1,h_2]} \cup \bar{X}_{U_0}^{[h_1,h_2-1]})$.

LEMMA 2.4
The scheme $\bar{X}_{U_0}^{(h_1,h_2)}$ is nonempty and smooth of pure dimension $h_1 + h_2$.

Proof
In order to see that this is true, note that the formal completion of $\bar{X}_{U_0}$ at any closed point is isomorphic to $\bar{\mathbb{F}}[[T_2, \ldots, T_n, S_2, \ldots, S_n]]$ since it is the universal formal deformation ring of a product of two 1-dimensional compatible Barsotti–Tate groups of height n each. (In fact, it is the product of the universal deformation rings for each of the two Barsotti–Tate groups.) Thus, $\bar{X}_{U_0}$ has dimension $2n - 2$ and, as in [HT, Lemma II.1.1], each closed stratum $\bar{X}_{U_0}^{[h_1,h_2]}$ has dimension at least $h_1 + h_2$. The lower bound on the dimension also holds for each open stratum $\bar{X}_{U_0}^{(h_1,h_2)}$. In order to get the upper bound on the dimension it suffices to show that the lowest stratum $\bar{X}_{U_0}^{(0,0)}$ is nonempty. Indeed, once we have a closed point s in any stratum $\bar{X}_{U_0}^{(h_1,h_2)}$, we can compute the formal completion $(\bar{X}_{U_0}^{(h_1,h_2)})_s^\wedge$ as in [HT, Lemma II.1.3] and find that the dimension is exactly $h_1 + h_2$. We start with a closed point of the lowest stratum $\bar{X}_{U_0}^{(0,0)} = \bar{X}_{U_0}^{[0,0]}$ and prove that this stratum has dimension 0. The higher closed strata $\bar{X}_{U_0}^{[h_1,h_2]} = \bigcup_{j_1 \leq h_1, j_2 \leq h_2} \bar{X}_{U_0}^{(j_1,j_2)}$ are nonempty, and it follows by induction on (h_1, h_2) that the open strata $\bar{X}_{U_0}^{(h_1,h_2)}$ are also nonempty.

It remains to see that $\bar{X}_{U_0}^{(0,0)}$ is nonempty. This can be done using Honda–Tate theory as in the proof of [HT, Corollary V.4.5], whose ingredients for Shimura varieties associated to more general unitary groups are supplied in [Sh1, Sections 8–12]. In our case, Honda–Tate theory exhibits a bijection between p-adic types over F (see [Sh1, Section 8] for the general definition) and pairs (A, i), where $A/\bar{\mathbb{F}}$ is an abelian variety of dimension dn and where $i : F \hookrightarrow \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The abelian variety A must also satisfy the following: $A[\mathfrak{p}_i^\infty]$ is ind-étale for $i > 2$, and $A[\mathfrak{p}_i^\infty]$ is 1-dimensional of étale height h_i for $i = 1, 2$. Note that the slopes of the p-divisible groups $A[\mathfrak{p}_i^\infty]$ are fixed for all i. All our p-adic types will be simple and given by pairs (M, η), where M is a CM field extension of F, and $\eta \in \mathbb{Q}[\mathfrak{P}]$, where $\mathfrak{P}$ is the set of places of M above p. The coefficients in η of places x of M above $\mathfrak{p}_i$ are related to the slope of the corresponding p-divisible group at $\mathfrak{p}_i$ as in [Sh3, Corollary 8.5]. More precisely, $A[x^\infty]$ has pure slope $\eta_x / e_{x/p}$. It follows that the coefficients of η at places x and x^c above p satisfy the compatibility

$$\eta_x + \eta_{x^c} = e_{x/p},$$

so to know η it is enough to specify $\eta_x \cdot x$ as x runs through places of M above u.

In order to exhibit a pair (A, i) with the right slope of $A[\mathfrak{p}_i^\infty]$ it suffices to exhibit its corresponding p-adic type. For this, we can simply take $M = F$ and $\eta_{\mathfrak{p}_i} = \frac{e_{\mathfrak{p}_i/p}}{n[F_{\mathfrak{p}_i} : \mathbb{Q}_p]} \cdot \mathfrak{p}_i$ for $i = 1, 2$ and $\eta_{\mathfrak{p}_i} = 0$ otherwise. The only facts remaining to be checked are that the associated pair (A, i) has a polarization λ which induces c on F and that the triple (A, i, λ) can be given additional structure to make it into a point on $\bar{X}_{U_0}^{(0,0)}$. First, we endow (A, i) with a polarization λ_0 for which the Rosati involution induces c on F using [Ko2, Lemma 9.2], and we use Lemma 5.5, an analogue of [HT, Lemma V.4.1], to construct an F-module W_0 together with a nondegenerate Hermitian pairing such that

$$W_0 \otimes \mathbb{A}^{\infty,p} \simeq V^p A \qquad \text{and} \qquad W_0 \otimes_{\mathbb{Q}} \mathbb{R} \simeq V \otimes_{\mathbb{Q}} \mathbb{R}$$

as Hermitian $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules (resp., $F \otimes_{\mathbb{Q}} \mathbb{R}$-modules). Then we use the difference (in the Galois cohomology sense) between W_0 and V as Hermitian F-modules over $\mathbb{Q}$ to find a polarization λ such that $V^p A$ with its λ-Weil pairing is equivalent to $V \otimes \mathbb{A}^{\infty,p}$ with its standard pairing, as in [HT, Lemma V.4.3]. Note that the argument is not circular, since the proof of Lemma 5.5 is independent of this section. ☐

The next lemma is an analogue of [TY, Lemma 3.1].

LEMMA 2.5
If $0 \le h_1, h_2 \le n - 1$, then the Zariski closure of $\bar{X}_{U_0}^{(h_1,h_2)}$ contains $\bar{X}_{U_0}^{(0,0)}$.

Proof
The proof follows exactly like the proof of [TY, Lemma 3.1]. Let x be a closed geometric point of $\bar{X}_{U_0}^{(0,0)}$. The main point is to note that the formal completion of $\bar{X}_{U_0} \times \operatorname{Spec} \bar{\mathbb{F}}$ at x is isomorphic to the equicharacteristic universal deformation ring of $\mathscr{G}_{1,x} \times \mathscr{G}_{2,x}$, so it is isomorphic to

$$\operatorname{Spf} \bar{\mathbb{F}}[[T_2, \dots, T_n, S_2, \dots, S_n]].$$

We can choose the T_i, the S_i, and formal parameters X on the universal deformation of $\mathcal{G}_{1,x}$ and Y on the universal deformation of $\mathcal{G}_{2,x}$ such that

$$[\pi](X) \equiv \pi X + \sum_{i=2}^{n} T_i X^{\#\mathbb{F}^{i-1}} + X^{\#\mathbb{F}^n} \pmod{X^{\#\mathbb{F}^n+1}}$$

and

$$[\pi](Y) \equiv \pi Y + \sum_{i=2}^{n} S_i X^{\#\mathbb{F}^{i-1}} + S^{\#\mathbb{F}^n} \pmod{S^{\#\mathbb{F}^n+1}}.$$

We get a morphism

$$\operatorname{Spec} \bar{\mathbb{F}}[[T_2, \dots, T_n, S_2, \dots, S_n]] \to \bar{X}_{U_0}$$

lying over $x : \operatorname{Spec} \bar{\mathbb{F}} \to \bar{X}_{U_0}$ such that if k denotes the algebraic closure of the field of fractions of

$$\operatorname{Spec} \bar{\mathbb{F}}[[T_2, \dots, T_n, S_2, \dots, S_n]]/(T_2, \dots, T_{n-h_1}, S_2, \dots, S_{n-h_2}),$$

then the induced map $\operatorname{Spec} k \to \bar{X}_{U_0}$ factors through $\bar{X}_{U_0}^{(h_1,h_2)}$. $\square$

For $i = 1, 2$, let $\mathrm{Iw}_{n,\mathfrak{p}_i}$ be the subgroup of matrices in $\mathrm{GL}_n(\mathcal{O}_K)$ which reduce modulo $\mathfrak{p}_i$ to $B_n(\mathbb{F})$ (here $B_n(\mathbb{F}) \subset \mathrm{GL}_n(\mathbb{F})$ is the Borel subgroup). We define an integral model for X_U, where $U \subseteq G(\mathbb{A}^\infty)$ is equal to

$$U^p \times U_p^{\mathfrak{p}_1,\mathfrak{p}_2}(\vec{m}) \times \mathrm{Iw}_{n,\mathfrak{p}_1} \times \mathrm{Iw}_{n,\mathfrak{p}_2} \times \mathbb{Z}_p^\times.$$

We define the following functor $\mathfrak{X}_U$ from connected locally Noetherian $\mathcal{O}_K$-schemes with a geometric point to sets sending

$$(S, s) \mapsto (A, \lambda, i, \bar{\eta}^p, \mathcal{C}_1, \mathcal{C}_2, \alpha_i),$$

where $(A, \lambda, i, \bar{\eta}^p, \alpha_i)$ is as in the definition of X_{U_0}, and, for $i = 1, 2$, where $\mathcal{C}_i$ is a chain of isogenies

$$\mathcal{C}_i : \mathcal{G}_{i,A} = \mathcal{G}_{i,0} \to \mathcal{G}_{i,1} \to \cdots \to \mathcal{G}_{i,n} = \mathcal{G}_{i,A}/\mathcal{G}_{i,A}[\mathfrak{p}_i]$$

of compatible Barsotti–Tate $\mathcal{O}_K$-modules each of degree $\#\mathbb{F}$ and with composite the canonical map $\mathcal{G}_{i,A} \to \mathcal{G}_{i,A}/\mathcal{G}_{i,A}[\mathfrak{p}_i]$.

LEMMA 2.6

If U^p is sufficiently small, the functor $\mathfrak{X}_U$ is represented by a scheme X_U which is finite over X_{U_0}. The scheme X_U has some irreducible components of dimension $2n - 1$.

Proof
The chains of isogenies $\mathcal{C}_i$ can be viewed as flags

$$0 = \mathcal{K}_{i,0} \subset \mathcal{K}_{i,1} \cdots \subset \mathcal{K}_{i,n} = \mathcal{G}_i[\mathfrak{p}_i],$$

where $\mathcal{K}_{i,j} = \ker(\mathcal{G}_{i,0} \to \mathcal{G}_{i,j})$. All the $\mathcal{K}_{i,j}$ are closed finite flat subgroup schemes with $\mathcal{O}_K$-action and $\mathcal{K}_{i,j}/\mathcal{K}_{i,j-1}$ of order $\#\mathbb{F}$. The representability can be proved in the same way as in [TY, Lemma 3.2] except in two steps. First, we note that the functor sending S to points of $X_{U_0}(S)$ together with flags $\mathcal{C}_1$ of $\mathcal{G}_1[\mathfrak{p}_1]$ is representable by a scheme X'_U over X_{U_0}. (If we let $\mathcal{H}_1$ denote the sheaf of Hopf algebras over X_{U_0} defining $\mathcal{G}_1[\mathfrak{p}_1]$, then X'_U will be a closed subscheme of the Grassmanian of chains of locally free direct summands of $\mathcal{H}_1$.) Then, we see in the same way that the functor sending S to points of $X'_U(S)$ together with flags $\mathcal{C}_2$ of $\mathcal{G}_2[\mathfrak{p}_2]$ is representable by a scheme X_U over X'_U. We also have that X_U is projective and finite over X_{U_0}. (Indeed, for each closed geometric point x of X_{U_0} there are finitely many choices of flags of $\mathcal{O}_K$-submodules of each $\mathcal{G}_{i,x}$.) On the generic fiber, the morphism $X_U \to X_{U_0}$ is finite étale and X_{U_0} has dimension $2n-1$, so X_U has some components of dimension $2n-1$. $\square$

We say that an isogeny $\mathcal{G} \to \mathcal{G}'$ of 1-dimensional compatible Barsotti–Tate $\mathcal{O}_K$-modules of degree $\#\mathbb{F}$ has *connected kernel* if it induces the zero map on $\operatorname{Lie}\mathcal{G}$. If we let $f = [\mathbb{F} : \mathbb{F}_p]$ and let $F : \mathcal{G} \to \mathcal{G}^{(p)}$ be the Frobenius map, then $F^f : \mathcal{G} \to \mathcal{G}^{(\#\mathbb{F})}$ is an isogeny of 1-dimensional compatible Barsotti–Tate $\mathcal{O}_K$-modules and has connected kernel. The following lemma appears as in [TY, Lemma 3.3].

LEMMA 2.7
Let W denote the ring of integers of the completion of the maximal unramified extension of K. Suppose that R is an Artinian local W-algebra with residue field $\bar{\mathbb{F}}$. Suppose that

$$\mathcal{C} : \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_g = \mathcal{G}_0/\mathcal{G}_0[\mathfrak{p}_i]$$

is a chain of isogenies of degree $\#\mathbb{F}$ of 1-dimensional compatible formal Barsotti–Tate $\mathcal{O}_K$-modules over R of $\mathcal{O}_K$-height g with composite equal to multiplication by π. If every isogeny has connected kernel, then R is an $\bar{\mathbb{F}}$-algebra and $\mathcal{C}$ is the pullback of a chain of isogenies of Barsotti–Tate $\mathcal{O}_K$-modules over $\bar{\mathbb{F}}$, with all isogenies isomorphic to F^f.

Now let $\bar{X}_U = X_U \times_{\operatorname{Spec} K} \operatorname{Spec} \mathbb{F}$ denote the special fiber of X_U. For $i = 1, 2$ and $1 \leq j \leq n$, let $Y_{i,j}$ denote the closed subscheme of $\bar{X}_U$ over which $\mathcal{G}_{i,j-1} \to \mathcal{G}_{i,j}$ has connected kernel. Note that, since each $\operatorname{Lie}\mathcal{G}_{i,j}$ is locally free of rank 1 over $\mathcal{O}_{X_U}$, we can pick a local basis for all of them. Then we can find locally $X_{i,j} \in \Gamma(X_U, O_{X_U})$ to represent the linear maps $\operatorname{Lie}\mathcal{G}_{i,j-1} \to \operatorname{Lie}\mathcal{G}_{i,j}$. Thus, each $Y_{i,j}$ is cut out locally in X_U by the equation $X_{i,j} = 0$.

PROPOSITION 2.8
Let s be a closed geometric point of X_U such that $\mathcal{G}_{i,s}$ has étale height h_i for $i = 1, 2$. Let W be the ring of integers of the completion of the maximal unramified extension of K. Let $\mathcal{O}^{\wedge}_{X_U,s}$ be the completion of the strict henselization of X_U at s, that is, the completed local ring of $X \times_{\operatorname{Spec}\mathcal{O}_K} \operatorname{Spec} W$ at s. Then

$$\mathcal{O}^{\wedge}_{X_U,s} \simeq W[[T_1, \dots, T_n, S_1, \dots, S_n]]/\Big(\prod_{i=h_1+1}^{n} T_i - \pi, \prod_{i=h_2+1}^{n} S_i - \pi\Big).$$

Assume that Y_{1,j_k} for $k = 1, \dots, n-h_1$ and $j_k \in \{1, \dots, n\}$ are distinct subschemes of X_U which contain s as a geometric point. We can choose the generators T_i such that the completed local ring $\mathcal{O}^{\wedge}_{Y_{1,j_k},s}$ is cut out in $\mathcal{O}^{\wedge}_{X_U,s}$ by the equation $T_{k+h_1} = 0$. The analogous statement is true for Y_{2,j_k} with $k = 1, \dots, n-h_2$ and $S_{k+h_2} = 0$.

Proof
First we prove that X_U has pure dimension $2n-1$ by using Deligne's homogeneity principle. We will follow closely the proof of [TY, Proposition 3.4.1]. The dimension of $\mathcal{O}^{\wedge}_{X_U,s}$ as s runs over geometric points of X_U above $\bar{X}^{(0,0)}_{U_0}$ is constant, say it is equal to m. Then we claim that $\mathcal{O}^{\wedge}_{X_U,s}$ has dimension m for every closed geometric point of X_U. Indeed, assume that the subset of X_U, where $\mathcal{O}^{\wedge}_{X_U,s}$ has dimension different from m, is nonempty. Then this subset is closed, so its projection to X_{U_0} is also closed and so it must contain some $\bar{X}^{(h_1,h_2)}_{U_0}$ (since the dimension of $\mathcal{O}^{\wedge}_{X_U,s}$ only depends on the stratum of X_{U_0} that s is above). By Lemma 2.5, the closure of $\bar{X}^{(h_1,h_2)}_{U_0}$ contains $\bar{X}^{(0,0)}_{U_0}$, which is a contradiction. Thus, X_U has pure dimension m and by Lemma 2.6, $m = 2n-1$.

The completed local ring $\mathcal{O}^{\wedge}_{X_U,s}$ is the universal deformation ring for tuples $(A, \lambda, i, \bar{\eta}^p, \mathcal{C}_1, \mathcal{C}_2, \alpha_i)$ deforming $(A_s, \lambda_s, i_s, \bar{\eta}^p_s, \mathcal{C}_{1,s}, \mathcal{C}_{2,s}, \alpha_{i,s})$. Deforming the abelian variety A_s is the same as deforming its p-divisible group $A_s[p^\infty]$ by Serre–Tate theory and $A_s[p^\infty] = A_s[u^\infty] \times A_s[(u^c)^\infty]$. The polarization λ together with $A[u^\infty]$ determine $A[(u^c)^\infty]$, so it suffices to deform $A_s[u^\infty]$ as an $\mathcal{O}_F$-module together with the level structure. At primes other than $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the p-divisible group is étale, so the deformation is uniquely determined. Moreover, $A[(\mathfrak{p}_1\mathfrak{p}_2)^\infty]$ decomposes as $A[\mathfrak{p}_1^\infty] \times A[\mathfrak{p}_2^\infty]$ (because $\mathcal{O}_F \otimes_{\mathcal{O}_{F'}} \mathcal{O}_{F'_{\mathfrak{p}_1\mathfrak{p}_2}} \simeq \mathcal{O}_{F,\mathfrak{p}_1} \times \mathcal{O}_{F,\mathfrak{p}_2}$), so it suffices to consider deformations of the chains

$$\mathscr{C}_{i,s} : \mathscr{G}_{i,s} = \mathscr{G}_{i,0} \to \mathscr{G}_{i,1} \to \cdots \to \mathscr{G}_{i,n} = \mathscr{G}_{i,s}/\mathscr{G}_{i,s}[\mathfrak{p}_i]$$

for $i = 1, 2$ separately.

Let $\mathscr{G} \simeq \Sigma \times (K/\mathscr{O}_K)^h$ be a p-divisible $\mathscr{O}_K$-module over $\bar{\mathbb{F}}$ of dimension 1 and total height n. Let

$$\mathscr{C} : \mathscr{G} = \mathscr{G}_0 \to \mathscr{G}_1 \to \cdots \to \mathscr{G}_n = \mathscr{G}/\mathscr{G}[\pi]$$

be a chain of isogenies of degree $\#\mathbb{F}$. Since we are working over $\bar{\mathbb{F}}$, the chain $\mathscr{C}$ splits into a formal part and an étale part. Let $\mathscr{C}^0$ be the chain obtained from $\mathscr{C}$ by restricting it to the formal part

$$\tilde{\Sigma} \to \tilde{\Sigma}_1 \to \cdots \to \tilde{\Sigma}_n = \tilde{\Sigma}/\tilde{\Sigma}[\pi].$$

Let $J \subseteq \{1, \ldots, n\}$ be the subset of indices j for which $\mathscr{G}_{j-1} \to \mathscr{G}_j$ has connected kernel. (The cardinality of J is $n - h$.) Also assume that the chain $\mathscr{C}^{\mathrm{et}}$ consists of

$$\mathscr{G}_j^{\mathrm{et}} = (K/\pi^{-1}\mathscr{O}_K)^j \oplus (K/\mathscr{O}_K)^{h-j}$$

for all $j \in J$ with the obvious isogenies between them.

We claim that the universal deformation ring of $\mathscr{C}$ is isomorphic to

$$W[[T_1, \ldots, T_n]]/\Big(\prod_{j \in J} T_j - \pi\Big).$$

We will follow the proof of [D, Proposition 4.5]. To see the claim, we first consider deformations of $\mathscr{G}$ without level structure. By [D, Proposition 4.5], the universal deformation ring of Σ is

$$R^0 \simeq W[[X_{h+1}, \ldots, X_h]]/(X_{h+1}, \ldots, X_n - \pi).$$

Let $\tilde{\Sigma}$ be the universal deformation of Σ. By considering the connected-étale exact sequence, we see that the deformations of $\mathscr{G}$ are classified by extensions of the form

$$0 \to \tilde{\Sigma} \to \tilde{\mathscr{G}} \to (K/\mathscr{O}_K)^h \to 0.$$

Thus, the universal deformations of $\mathscr{G}$ are classified by elements of $\mathrm{Hom}(T\mathscr{G}, \tilde{\Sigma})$, where $T\mathscr{G}$ is the Tate module of $\mathscr{G}$. The latter ring is noncanonically isomorphic to

$$R \simeq W[[X_1, \ldots, X_n]]/\Big(\prod_{j \in J} X_j - \pi\Big).$$

Let S be the universal deformation ring for deformations of the chain $\mathscr{C}$, and let S^0 be the universal deformation ring for the chain $\mathscr{C}^0$. Let

$$\tilde{\mathscr{C}} : \tilde{\mathscr{G}} = \tilde{\mathscr{G}}_0 \to \tilde{\mathscr{G}}_1 \to \cdots \to \tilde{\mathscr{G}}_n = \tilde{\mathscr{G}}/\tilde{\mathscr{G}}[\pi]$$

be the universal deformation of $\mathscr{C}$ which corresponds when restricted to the formal part to the universal chain

$$\tilde{\Sigma} \to \tilde{\Sigma}_1 \to \cdots \to \tilde{\Sigma}_n = \tilde{\Sigma}/\tilde{\Sigma}[\pi].$$

Each deformation $\tilde{\mathscr{G}}_j$ of $\mathscr{G}_j$ is defined by a connected-étale exact sequence

$$0 \to \tilde{\Sigma}_j \to \tilde{\mathscr{G}}_j \to (K/\mathscr{O}_K)^h \to 0,$$

so it is defined by an element $f_j \in \mathrm{Hom}(T\mathscr{G}_j, \tilde{\Sigma}_j)$. We will explore the compatibilities between the $\mathrm{Hom}(T\mathscr{G}_j, \tilde{\Sigma}_j)$ as j ranges from 0 to n. If $j \in J$, then $\tilde{\mathscr{G}}_{j-1} \to \tilde{\mathscr{G}}_j$ has connected kernel, so $T\mathscr{G}_{j-1} \simeq T\mathscr{G}_j$. The isogeny $\tilde{\Sigma}_{j-1} \to \tilde{\Sigma}_j$ determines a map $\mathrm{Hom}(T\mathscr{G}_{j-1}, \tilde{\Sigma}_{j-1}) \to \mathrm{Hom}(T\mathscr{G}_j, \tilde{\Sigma}_j)$, which determines the extension $\tilde{\mathscr{G}}_j$. Thus, in order to know the extension classes of $\tilde{\mathscr{G}}_j$ it suffices to focus on the case $j \notin J$.

Let $(e_j)_{j\in J}$ be a basis of $\mathscr{O}_K^h$, which we identify with $T\mathscr{G}_j$ for each j. We claim that it suffices to know $f_j(e_j) \in \tilde{\Sigma}_j$ for each $j \notin J$. Indeed, if $j \notin J$, then we know that $\tilde{\Sigma}_{j-1} \simeq \tilde{\Sigma}_j$, and we also have a map $T\mathscr{G}_{j-1} \to T\mathscr{G}_j$ sending

$$e_{j'} \mapsto e_{j'} \quad \text{for } j' \neq j \text{ and } e_j \mapsto \pi e_j.$$

Thus, for $i \neq j$, we can identify $f_{j-1}(e_i) \in \tilde{\Sigma}_{j-1}$ with $f_j(e_i) \in \tilde{\Sigma}_j$. Hence if we know $f_j(e_j)$, then we also know $f_{j'}(e_j)$ for all $j' > j$. Thus we know $f_n(e_j)$, but recall that f_n corresponds to the extension

$$0 \to \tilde{\Sigma}/\tilde{\Sigma}[\pi] \to \tilde{\mathscr{G}}/\tilde{\mathscr{G}}[\pi] \to (K/\pi^{-1}\mathscr{O}_K)^h \to 0,$$

which is isomorphic to the extension

$$0 \to \tilde{\Sigma} \to \tilde{\mathscr{G}} \to (K/\mathscr{O}_K)^h \to 0.$$

Therefore, we also know $f_0(e_j)$ and by extension all $f_{j'}(e_j)$ for $j' < j$. This proves the claim that the only parameters needed to construct all the extensions $\tilde{\mathscr{G}}_j$ are the elements $f_j(e_j) \in \tilde{\Sigma}_j$ for all $j \notin J$.

We have a map $S^0 \otimes_{R^0} R \to S$ induced by restricting the Iwahori level structure to the formal part. From the discussion above, we see that this map is finite and that S is obtained from $S^0 \otimes_{R^0} R$ by adjoining for each $j \in J$ a root T_j of

$$f(T_j) = X_j$$

in $\tilde{\Sigma}$, where $f : \tilde{\Sigma} \to \tilde{\Sigma}$ is the composite of the isogenies $\tilde{\Sigma}_j \to \tilde{\Sigma}_{j+1} \to \cdots \to \tilde{\Sigma}_n$. If we quotient S by all the T_j for $j \notin J$, we are left only with deformations of the chain $\mathscr{C}^0$, since all of the connected-étale exact sequences will split. Thus $S/(T_j)_{j\notin J} \simeq S^0$.

Now, the formal part $\tilde{\mathcal{C}}^0$ can be written as a chain

$$\tilde{\Sigma} = \tilde{\Sigma}_0 \to \cdots \to \tilde{\Sigma}_j \to \cdots \to \tilde{\Sigma}/\tilde{\Sigma}[\pi]$$

of length $n-h$. Choose bases e_j for $\operatorname{Lie}\mathcal{G}_j$ over S^0 as j runs over J such that

$$e_n = e_j \quad \text{for the largest } j \in J$$

maps to

$$e_0 = e_j \quad \text{for the smallest } j \in J$$

under the isomorphism $\mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[\pi] \xrightarrow{\sim} \mathcal{G}_0$ induced by π. Let $T_j \in S^0$ represent the linear map $\operatorname{Lie}\tilde{\Sigma}_{j'} \to \operatorname{Lie}\tilde{\Sigma}_j$, where j' is the largest element of J for which $j' < j$. Then

$$\prod_{j\in J} T_j = \pi.$$

Moreover, $S^0/(T_j)_{j\in J} = \bar{\mathbb{F}}$ by Lemma 2.7. (See also the proof of [TY, Proposition 3.4].) Hence we have a surjection

$$W[[T_1,\dots,T_n]]/\Big(\prod_{j=h_1+1}^{n} T_j - \pi\Big) \twoheadrightarrow S,$$

which by dimension reasons must be an isomorphism.

Applying the preceding argument to the chains $\mathcal{C}_{1,s}$ and $\mathcal{C}_{2,s}$, we conclude that

$$\mathcal{O}^{\wedge}_{X_U,s} \simeq W[[T_1,\dots,T_n,S_1,\dots,S_n]]/\Big(\prod_{i=h_1+1}^{n} T_i - \pi, \prod_{i=h_2+1}^{n} S_i - \pi\Big).$$

Moreover, the closed subvariety Y_{1,j_k} of X_U is exactly the locus where $\mathcal{G}_{j_k-1} \to \mathcal{G}_{j_k}$ has connected kernel, so if s is a geometric point of Y_{1,j_k}, then $\mathcal{O}^{\wedge}_{Y_{1,j_k},s}$ is cut out in $\mathcal{O}^{\wedge}_{X_U,s}$ by the equation $T_{k+h_1} = 0$. (Indeed, by our choice of the parameters T_{k+h_1} with $1 \le k \le n-h_1$, the condition that $\mathcal{G}_{1,j_k-1} \to \mathcal{G}_{1,j_k}$ has connected kernel is equivalent to $T_{k+h_1} = 0$.) □

For $S, T \subseteq \{1,\dots,n\}$ nonempty, let

$$Y_{U,S,T} = \Big(\bigcap_{i\in S} Y_{1,i}\Big) \cap \Big(\bigcap_{j\in T} Y_{2,j}\Big).$$

Then $Y_{U,S,T}$ is smooth over $\operatorname{Spec}\mathbb{F}$ of pure dimension $2n - \#S - \#T$ (we can check smoothness on completed local rings) and it is also proper over $\operatorname{Spec}\mathbb{F}$, since $Y_{U,S,T} \hookrightarrow \bar{X}_U$ is a closed immersion and $\bar{X}_U$ is proper over $\operatorname{Spec}\mathbb{F}$. We also define

$$Y^0_{U,S,T} = Y_{U,S,T} \setminus \Big(\Big(\bigcup_{S' \supsetneq S} Y_{U,S',T} \Big) \cup \Big(\bigcup_{T' \supsetneq T} Y_{U,S,T'} \Big) \Big).$$

Note that the inverse image of $\bar{X}_U^{(h_1,h_2)}$ with respect to the finite flat map $\bar{X}_U \to \bar{X}_{U_0}$ is

$$\bigcup_{\substack{\#S=n-h_1 \\ \#T=n-h_2}} Y^0_{U,S,T}.$$

Note that, when we consider the Shimura variety X_{U_i}, with U_i having Iwahori level structure at only one of the primes $\mathfrak{p}_i$ for $i = 1, 2$, this will be flat over X_{U_0}, since it can be checked that it is a finite map between regular schemes of the same dimension (for the same reason as in the setting of [TY]). The morphism $X_U \to X_{U_0}$ is the fiber product of the morphisms $X_{U_i} \to X_{U_0}$ for $i = 1, 2$, so it is flat as well.

LEMMA 2.9
The Shimura variety X_U is locally étale over

$$X_{r,s} = \operatorname{Spec} \mathcal{O}_K[X_1, \ldots, X_n, Y_1, \ldots, Y_n] / \Big(\prod_{i=1}^{r} X_i - \pi, \prod_{j-1}^{s} Y_j - \pi \Big)$$

with $1 \le r, s \le n$.

Proof
Let x be a closed point of X_U. The completion of the strict henselization of X_U at x $\mathcal{O}^\wedge_{X_U,x}$ is isomorphic to

$$\mathcal{O}_{r,s} = W[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]] / \Big(\prod_{i=1}^{r} X_i - \pi, \prod_{j=1}^{s} Y_j - \pi \Big)$$

for certain $1 \le r, s \le n$. We show that there is an open affine neighborhood U of x in X such that U is étale over $X_{r,s}$. Note that there are local equations $T_i = 0$ with $1 \le i \le r$ and $S_j = 0$ with $1 \le j \le s$ which define the closed subschemes $Y_{1,i}$ with $1 \le i \le r$ and $Y_{2,j}$ with $1 \le j \le s$ passing through x. Moreover, the parameters T_i and S_j satisfy

$$\prod_{i=1}^{r} T_i = u\pi \qquad \text{and} \qquad \prod_{j=1}^{s} S_i = u'\pi$$

with u and u' units in the local ring $\mathcal{O}_{X_U,x}$. We will explain why this is the case for the T_i. In the completion of the strict henselization $\mathcal{O}^\wedge_{X_U,x}$ both T_i and X_i cut out the completion of the strict henselization $\mathcal{O}^{\wedge}_{Y_{1,i},x}$, which means that T_i and X_i differ by a unit. Taking the product of the T_i, we find that $\prod_{i=1}^{r} T_i = u\pi$ for $u \in \mathcal{O}^{\wedge}_{X_U,x}$ a unit in the completion of the strict henselization of the local ring. At the same time, in an open neighborhood of x, the special fiber of X is a union of the divisors corresponding to $T_i = 0$ for $1 \le i \le r$, so that $\prod_{i=1}^{r} T_i$ belongs to the ideal of $\mathcal{O}_{X_U,x}$ generated by π. We conclude that u is actually a unit in the local ring $\mathcal{O}_{X_U,x}$, not only in $\mathcal{O}^{\wedge}_{X_U,x}$. In a neighborhood of x, we can change one of the T_i by u^{-1} and one of the S_i by $(u')^{-1}$ to ensure that

$$\prod_{i=1}^{r} T_i = \pi \qquad \text{and} \qquad \prod_{j=1}^{s} S_i = \pi.$$

We now adapt the argument used in the proof of [Y, Proposition 4.8] to our situation. We first construct an unramified morphism f from a neighborhood of x in X_U to $\operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n, Y_1, \dots, Y_n]$. We can do this simply by sending the X_i to the T_i for $i = 1, \dots, r$ and the Y_j to the S_j for $j = 1, \dots, s$. The rest of the X_i and Y_j can be sent to parameters in a neighborhood of x which approximate the remaining parameters in $\mathcal{O}^{\wedge}_{X_U,x}$ modulo the square of the maximal ideal. Then f will be formally unramified at the point x. By [GD2, Corollaire 18.4.7], we see that, when restricted to an open affine neighborhood $\operatorname{Spec} A$ of x in X, $f|_{\operatorname{Spec} A}$ can be decomposed as a closed immersion $\operatorname{Spec} A \to \operatorname{Spec} B$ followed by an étale morphism $\operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n, Y_1, \dots, Y_n]$. The closed immersion translates into the fact that $A \simeq B/I$ for some ideal I of B. The inverse image of I in $W[X_1, \dots, X_n, Y_1, \dots, Y_n]$ is an ideal J which contains $\prod_{i=1}^{r} X_i - \pi$ and $\prod_{j=1}^{s} Y_j - \pi$. The morphism f factors through the morphism $g : \operatorname{Spec} A \to \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n, Y_1, \dots, Y_n]/J$ which is étale. Moreover, J is actually generated by $\prod_{i=1}^{r} X_i - \pi$ and $\prod_{j=1}^{s} Y_j - \pi$, since g induces an isomorphism on completed strict local rings

$$W[[X_1, \dots, X_n, Y_1, \dots, Y_n]]/J \xrightarrow{\sim} \mathcal{O}_{r,s}.$$

This completes the proof of the lemma. □

Let $\mathcal{A}_U$ be the universal abelian variety over the integral model X_U. Let ξ be an irreducible representation of G over $\bar{\mathbb{Q}}_l$ for a prime number $l \neq p$. The sheaf $\mathcal{L}_\xi$ extends to a lisse sheaf on the integral models X_{U_0} and X_U. Also, $a_\xi \in \operatorname{End}(\mathcal{A}_U^{m_\xi}/X_U) \otimes_{\mathbb{Z}} \mathbb{Q}$ extends as an étale morphism on $\mathcal{A}_U^{m_\xi}$ over the integral model. We have

$$H^j(X_U \times_{F'} \bar{F}'_{\mathfrak{p}}, \mathcal{L}_\xi) \simeq a_\xi H^{j+m_\xi}\big(\mathcal{A}_U^{m_\xi} \times_{F'} \bar{F}'_{\mathfrak{p}}, \bar{\mathbb{Q}}_l(t_\xi)\big),$$

and we can compute the latter via the nearby cycles $R\psi\bar{\mathbb{Q}}_l$ on $\mathcal{A}_U^{m_\xi}$ over the integral model of X_U. Note that $\mathcal{A}_U^{m_\xi}$ is smooth over X_U, so $\mathcal{A}_U^{m_\xi}$ is locally étale over

$$X_{r,s,m} := \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n, Y_1, \dots, Y_n, Z_1, \dots, Z_m] / \Big(\prod_{j=1}^{r} X_{i_j} - \pi, \prod_{j=1}^{s} Y_{i_j} - \pi \Big)$$

for some nonnegative integer m.

3. Sheaves of nearby cycles

In this section, we start to understand the complex of nearby cycles on a scheme $X/\mathcal{O}_K$ which has the same geometric properties as our Iwahori level Shimura variety X_U. We work with $K/\mathbb{Q}_p$, a finite extension with ring of integers $\mathcal{O}_K$ which has uniformizer π and residue field $\mathbb{F}$. Let $I_K = \operatorname{Gal}(\bar{K}/K^{\mathrm{ur}}) \subset G_K = \operatorname{Gal}(\bar{K}/K)$ be the inertia subgroup of K. Let Λ be either one of $\mathbb{Z}/l^r\mathbb{Z}$, $\mathbb{Z}_l$, $\mathbb{Q}_l$, or $\bar{\mathbb{Q}}_l$ for $l \neq p$ prime. Let $X/\mathcal{O}_K$ be a scheme such that X is locally étale over

$$X_{r,s,m} = \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n, Y_1, \dots, Y_n, Z_1, \dots, Z_m] / \Big(\prod_{j=1}^{r} X_j - \pi, \prod_{j=1}^{s} Y_j - \pi \Big).$$

Let Y be the special fiber of X. Assume that Y is a union of closed subschemes $Y_{1,j}$ with $j \in \{1, \dots, n\}$ which are cut out locally by one equation and that this equation over $X_{r,s,m}$ corresponds to $X_j = 0$. Similarly, assume that Y is a union of closed subschemes $Y_{2,j}$ with $j \in \{1, \dots, n\}$ which are cut out over $X_{r,s,m}$ by $Y_j = 0$.

Let $j : X_K \hookrightarrow X$ be the inclusion of the generic fiber, and let $i : Y \hookrightarrow X$ be the inclusion of the special fiber. Let $S = \operatorname{Spec} \mathcal{O}_K$, with generic point η and closed point s. Let $\bar{K}$ be an algebraic closure of K, with ring of integers $\mathcal{O}_{\bar{K}}$. Let $\bar{S} = \operatorname{Spec} \mathcal{O}_{\bar{K}}$, with generic point $\bar{\eta}$ and closed point $\bar{s}$. Let $\bar{X} = X \times_S \bar{S}$ be the base change of X to $\bar{S}$, with generic fiber $\bar{j} : X_{\bar{\eta}} \hookrightarrow \bar{X}$ and special fiber $\bar{i} : X_{\bar{s}} \hookrightarrow \bar{X}$. The sheaves of nearby cycles associated to the constant sheaf Λ on X_K are sheaves $R^k\psi\Lambda$ on $X_{\bar{s}}$ defined for $k \geq 0$ as

$$R^k\psi\Lambda = \bar{i}^* R^k \bar{j}_* \Lambda$$

and they have continuous actions of I_K.

PROPOSITION 3.1
The action of I_K on $R^k\psi\Lambda$ is trivial for any $k \geq 0$.

The proof of this proposition is based on endowing X with a logarithmic structure, showing that the resulting log scheme is log smooth over $\operatorname{Spec} \mathcal{O}_K$ (with the canonical log structure determined by the special fiber), and then using the explicit computation of the action of I_K on the sheaves of nearby cycles that was done by Nakayama in [Na, Theorem 3.5].

3.1. Log structures

Definition 3.2
A log structure on a scheme Z is a sheaf of monoids M together with a morphism $\alpha : M \to \mathcal{O}_Z$ such that α induces an isomorphism $\alpha^{-1}(\mathcal{O}_Z^*) \simeq \mathcal{O}_Z^*$. A scheme endowed with a log structure is a log scheme. A morphism of log schemes $(Z_1, M_1) \to (Z_2, M_2)$ consists of a pair (f, h), where $f : Z_1 \to Z_2$ is a morphism of schemes and $h : f^*M_2 \to M_1$ is a morphism of sheaves of monoids.

From now on, we regard $\mathcal{O}_Z^*$ as a subsheaf of M via α^{-1}, and we define $\bar{M} := M/\mathcal{O}_Z^*$.

Given a scheme Z and a closed subscheme V with complement U, there is a canonical way to associate to V a log structure. If $j : U \hookrightarrow X$ is an open immersion, we can simply define $M = j_*((\mathcal{O}_X \mid U)^*) \cap \mathcal{O}_X \to \mathcal{O}_X$. This amounts to formally "adjoining" the sections of $\mathcal{O}_X$ which are invertible outside V to the units $\mathcal{O}_X^*$. The sheaf $\bar{M}$ will be supported on V.

If P is a monoid, then the scheme $\operatorname{Spec}\mathbb{Z}[P]$ has a canonical log structure associated to the natural map $P \to \mathbb{Z}[P]$. A chart for a log structure on Z is given by a monoid P and a map $Z \to \operatorname{Spec}\mathbb{Z}[P]$ such that the log structure on Z is pulled back from the canonical log structure on $\operatorname{Spec}\mathbb{Z}[P]$. A chart for a morphism of log schemes $Z_1 \to Z_2$ is a triple of maps $Z_1 \to \operatorname{Spec}\mathbb{Z}[Q]$, $Z_2 \to \operatorname{Spec}\mathbb{Z}[P]$, and $P \to Q$ such that the first two maps are charts for the log structures on Z_1 and Z_2 and such that the obvious diagram is commutative.

For more background on log schemes, the reader should consult [I1] and [K].

For a scheme over $\mathcal{O}_K$, we let j denote the open immersion of its generic fiber and we let i denote the closed immersion of its special fiber into the scheme. We endow $S = \operatorname{Spec}\mathcal{O}_K$ with the log structure given by $N = j_*(K^*) \cap \mathcal{O}_K \hookrightarrow \mathcal{O}_K$. The sheaf $\bar{N}$ is trivial outside the closed point and is isomorphic to a copy of $\mathbb{N}$ over the closed point. Another way to describe the log structure on S is by pullback of the canonical log structure via the map

$$S \to \operatorname{Spec}\mathbb{Z}[\mathbb{N}],$$

where $1 \mapsto \pi \in \mathcal{O}_K$.

We endow X with the log structure given by $M = j_*(\mathcal{O}_{X_K}^*) \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$. It is easy to check that the only sections of $\mathcal{O}_X$ which are invertible outside the special fiber but not invertible globally are those given locally by the images of the X_i for $1 \le i \le r$ and the Y_j for $1 \le j \le s$. On étale neighborhoods U of X which are étale over $X_{r,s,m}$, this log structure is given by the chart

$$U \to X_{r,s,m} \to \operatorname{Spec}\mathbb{Z}[P_{r,s}],$$

where

$$P_{r,s} := (\mathbb{N}^r \oplus \mathbb{N}^s)/\big((1,\dots,1,0,\dots,0) = (0,\dots,0,1,\dots,1)\big).$$

The map $X_{r,s,m} \to \operatorname{Spec}\mathbb{Z}[P_{r,s}]$ can be described as follows: the element with 1 only in the kth place $(0,\dots,0,1,0,\dots,0) \in P_{r,s}$ maps to X_k if $k \le r$ and to Y_{k-r} if $k \ge r+1$. Note that the log structure on X is trivial outside the special fiber, so X is a *vertical* log scheme.

The map $X \to S$ induces a map of the corresponding log schemes. Étale locally, this map has a chart subordinate to the map of monoids $\mathbb{N} \to P_{r,s}$ such that

$$1 \mapsto (1,\dots,1,0,\dots,0) = (0,\dots,0,1,\dots,1)$$

to reflect the relations $X_1,\dots,X_r = Y_1,\dots,Y_s = \pi$.

LEMMA 3.3
The map of log schemes $(X,M) \to (S,N)$ is log smooth.

Proof
The map of monoids $\mathbb{N} \to P_{r,s}$ induces a map on groups $\mathbb{Z} \to P_{r,s}^{\mathrm{gp}}$, which is injective and has torsion-free cokernel $\mathbb{Z}^{r+s-2}$. Since the map of log schemes $(X,M) \to (S,N)$ is given étale locally by charts subordinate to such maps of monoids, by [K, Theorem 3.5] the map $(X,M) \to (S,N)$ is log smooth. $\square$

3.2. Nearby cycles and log schemes
There is a generalization of the functor of nearby cycles to the category of log schemes.

Recall that $\mathcal{O}_{\bar{K}}$ is the integral closure of $\mathcal{O}_K$ in $\bar{K}$ and that $\bar{S} = \operatorname{Spec}\mathcal{O}_{\bar{K}}$ with generic point $\bar{\eta}$ and closed point $\bar{s}$. The canonical log structure associated to the special fiber (given by the inclusion $\bar{j}_*(\bar{K}^*) \cap \mathcal{O}_{\bar{K}} \hookrightarrow \mathcal{O}_{\bar{K}}$) defines a log scheme $\tilde{S}$ with generic point $\bar{\eta}$ and closed point $\tilde{s}$. Note that $\tilde{s}$ is a log geometric point of $\tilde{S}$, so it has the same underlying scheme as $\bar{s}$. The Galois group G_K acts on $\tilde{s}$ through its tame quotient. Let $\tilde{X} = X \times_S \tilde{S}$ in the category of log schemes, with special fiber $X_{\tilde{s}}$ and generic fiber $X_{\bar{\eta}}$. Note that, in general, the underlying scheme of $X_{\tilde{s}}$ is not the same as that of $X_{\bar{s}}$. This is because $X_{\tilde{s}}$ is the fiber product of $X_{\bar{s}}$ and $\tilde{s}$ in the category of integral and saturated log schemes and saturation corresponds to normalization, so it changes the underlying scheme.

The sheaves of log nearby cycles are sheaves on $X_{\tilde{s}}$ defined by

$$R^k\psi^{\log}\Lambda = \tilde{i}^* R^k \tilde{j}_*\Lambda,$$

where $\tilde{i}, \tilde{j}$ are the obvious maps and where the direct and inverse images are taken with respect to the Kummer étale topology. Theorem 3.2 of [Na] states that when

X/S is a log smooth scheme, we have $R^0\psi^{\log}\Lambda \cong \Lambda$ and $R^p\psi^{\log}\Lambda = 0$ for $p > 0$. Let

$$\tilde{\epsilon} : \tilde{X} \to \bar{X},$$

which restricts to $\epsilon : X_{\bar{\eta}} \to X_{\bar{\eta}}$, be the morphism that simply forgets the log structure. Note that we have $\bar{j}_*\epsilon_* = \tilde{\epsilon}_*\tilde{j}_*$, by commutativity of the square

$$\begin{tikzcd} X_{\bar{\eta}} \arrow[r, "\tilde{j}"] \arrow[d, "\epsilon"'] & \tilde{X} \arrow[d, "\tilde{\epsilon}"'] \\ X_{\bar{\eta}} \arrow[r, "\bar{j}"] & \bar{X} \end{tikzcd}$$

We also have $\bar{i}^*R\tilde{\epsilon}_*\mathscr{F} \simeq R\tilde{\epsilon}_*\tilde{i}^*\mathscr{F}$ for every Kummer étale sheaf $\mathscr{F}$, by strict base change (see [I1, Proposition 6.3]). We deduce that

$$\bar{i}^*\bar{j}_*\epsilon_* = \tilde{\epsilon}_*\tilde{i}^*\tilde{j}_*,$$

so the corresponding derived functors must satisfy a similar relation. When we write this out, using $R\psi^{\log}\Lambda \cong \Lambda$ by Nakayama's result and $R\epsilon_*\Lambda \cong \Lambda$ because the log structure is vertical and so ϵ is an isomorphism, we get

$$R^k\psi^{\mathrm{cl}}\Lambda = R^k\tilde{\epsilon}_*(\Lambda \mid X_{\tilde{s}}).$$

Therefore, it suffices to figure out what the sheaves $R^k\tilde{\epsilon}_*\Lambda$ look like and how I_K acts on them, where $\tilde{\epsilon} : X_{\tilde{s}} \to X_{\bar{s}}$. This has been done in general by Nakayama, thus deriving an [GDK, Théorème I.3.3]-type formula for log smooth schemes. We describe his argument below and specialize to our particular case.

LEMMA 3.4 ([Na, Theorem 3.5])
We hold that I_K acts on $R^p\epsilon_*(\Lambda \mid X_{\tilde{s}})$ through its tame quotient.

Proof
Let $S^t = \operatorname{Spec}\mathcal{O}_{K^t}$ be endowed with the canonical log structure (here $K^t \subset \bar{K}$ is the maximal extension of K which is tamely ramified). The closed point s^t with its induced log structure is a universal Kummer étale cover of s, and I_K acts on it through its tame quotient I^t. Moreover, the projection $\tilde{s} \to s^t$ is a limit of universal Kummer homeomorphisms, and it remains so after base change with X (see [I1, Theorem 2.8]). Thus, every automorphism of $X_{\tilde{s}}$ comes from a unique automorphism of X_{s^t}, on which I_K acts through I^t. $\square$

Now we have the commutative diagram

$$\begin{tikzcd} X_{\tilde{s}}^{\log} \arrow[r, "\alpha"] \arrow[d, "\epsilon"'] & X_{\bar{s}}^{\log} \arrow[d, "\epsilon"] \\ X_{\tilde{s}}^{\mathrm{cl}} \arrow[r, "\beta"] & X_{\bar{s}}^{\mathrm{cl}} \end{tikzcd}$$

where the objects in the top row are log schemes and the objects in the bottom row are their underlying schemes. The morphisms labeled ϵ are forgetting the log structure, and we have $\tilde{\epsilon} = \epsilon \circ \alpha = \beta \circ \epsilon$. We can use either of these decompositions to compute $R^k \tilde{\epsilon}_* \Lambda$. For example, we have $R\tilde{\epsilon}_* \Lambda = R\beta_* R\epsilon_* \Lambda$, which translates into having a spectral sequence

$$R^{n-k} \beta_* R^k \epsilon_* \Lambda \Rightarrow R^n \tilde{\epsilon}_* \Lambda.$$

We know that $R^k \epsilon_* \Lambda = \bigwedge^k \bar{M}_{\mathrm{rel}}^{\mathrm{gp}} \otimes \Lambda(-k)$, where

$$\bar{M}_{\mathrm{rel}}^{\mathrm{gp}} = \mathrm{coker}(\bar{N}^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}})/\text{torsion}.$$

Recall that N is the log structure on $\mathcal{O}_K$ associated to its special fiber. The map of log schemes $(X, M) \to (\mathcal{O}_K, N)$ induces a map from the (pullback of) N to M. We form $\bar{M}_{\mathrm{rel}}^{\mathrm{gp}}$ using this map. The formula for $R^k \epsilon_* \Lambda$ follows from [KN, Theorem 2.4], as explained in [Na, Section 3.6]. Theorem 2.4 of [KN] is a statement about log schemes over $\mathbb{C}$, but the same proof also applies to the case of log schemes over a field of characteristic p, as explained in [I1].

On the other hand, at a geometric point $\bar{x}$ of $X_{\bar{s}}^{\mathrm{cl}}$, we have $(\beta_* \mathcal{F})_{\bar{x}} \cong \mathcal{F}[E_{\bar{x}}]$ for a sheaf $\mathcal{F}$ of Λ-modules on $X_{\tilde{s}}^{\mathrm{cl}}$, where $E_{\bar{x}}$ is the cokernel of the map of log inertia groups

$$I_x \to I_s.$$

Indeed, $\beta^{-1}(\bar{x})$ consists of $\#\mathrm{coker}(I_x \to I_s)$ points, which follows from the fact that $X_{\tilde{s}}^{\mathrm{cl}}$ is the normalization of $(X_{\bar{s}} \times_{\bar{s}} \tilde{s})^{\mathrm{cl}}$. The higher derived functors $R^{n-k} \beta_* \mathcal{F}$ are all trivial since β_* is exact. Therefore, the spectral sequence becomes

$$\wedge^k \bar{M}_{\mathrm{rel},\bar{x}}^{\mathrm{gp}} \otimes \Lambda[E_{\bar{x}}] \otimes \Lambda(-k) = (R^k \tilde{\epsilon}_* \Lambda)_{\bar{x}}.$$

The tame inertia acts on the stalks of these sheaves through $I^t \cong I_s \mapsto \Lambda[I_s] \to \Lambda[E_{\bar{x}}]$.

In our particular case, it is easy to compute $R^k \tilde{\epsilon}_* \Lambda$ globally. Let

$$\hat{\mathbb{Z}}'(1) = \varprojlim_{(m,p)=1} \mu_m.$$

We have

$$I_x = \mathrm{Hom}\big(\bar{M}_x^{\mathrm{gp}}, \hat{\mathbb{Z}}'(1)\big)$$

and

$$I_s = \mathrm{Hom}\big(\bar{N}_s^{\mathrm{gp}}, \hat{\mathbb{Z}}'(1)\big).$$

The map of inertia groups is induced by the map $\bar{N}_s^{\mathrm{gp}} \to \bar{M}_x^{\mathrm{gp}}$, which is determined by $1 \mapsto (1, \dots, 1, 0, \dots, 0)$, where the first n terms are nonzero. Any homomorphism of $\bar{N}_s^{\mathrm{gp}} \cong \mathbb{Z} \to \hat{\mathbb{Z}}'(1)$ can be obtained from some homomorphism $\bar{M}_x^{\mathrm{gp}} \to \hat{\mathbb{Z}}'(1)$. Thus $E_{\bar{x}}$ is trivial for all log geometric points $\bar{x}$, and I^t acts trivially on the stalks of the sheaves of nearby cycles.

Moreover, in our situation we can check that β is an isomorphism, which follows from the fact that $X_{\bar{s}}$ is reduced, which can be checked étale locally. Indeed, if X is reduced, then the underlying scheme of $X_{\tilde{s}}^{\log}$ is the same as $X_{\bar{s}}$, since $X_{\tilde{s}}^{\log}$ is defined as the inverse limit over $n \in \mathbb{N}$ prime to p of fiber products of fs log schemes

$$(X_{\bar{s}}, M) \times_{(\bar{\mathbb{F}}, N), \gamma_n} (\bar{\mathbb{F}}, N),$$

where γ_n is the identity on the underlying schemes and is multiplication by n on the nontrivial part of the log structures. The underlying scheme of a fiber product of fs log schemes is not usually the same as the fiber product of underlying schemes. The reason for this is that the log structure on the fiber product does not a priori need to be saturated, so we may need to introduce additional units. However, it can be checked that if X is reduced, then the product log structure is already saturated. Thus we have the global isomorphism

$$R^k \tilde{\epsilon}_* \Lambda \simeq \wedge^k \bar{M}_{\mathrm{rel}}^{\mathrm{gp}} \otimes \Lambda(-k).$$

The above discussion also allows us to determine the sheaves of nearby cycles. Indeed, we have $R^k \psi \Lambda \simeq \wedge^k \bar{M}_{\mathrm{rel}}^{\mathrm{gp}} \otimes \Lambda(-k)$, and $\bar{M}_{\mathrm{rel}}^{\mathrm{gp}}$ can be computed explicitly on neighborhoods. If U is a neighborhood of X with U étale over $X_{r,s}$, then the log structure on U is induced from the log structure on $X_{r,s}$. Let $J_1, J_2 \subseteq \{1, \dots, n\}$ be sets of indices with cardinalities r and s, respectively, corresponding to sets of divisors $Y_{1,i}$ and $Y_{2,j}$ which intersect U.

PROPOSITION 3.5
For $i = 1, 2$ and $j = 1, \dots, n$, let $a_j^i : Y_{i,j} \hookrightarrow Y$ denote the closed immersion. Then we have the following isomorphism of sheaves on U

$$R^k \psi \Lambda(k)|_U \simeq \wedge^k \Big[\Big(\Big(\bigoplus_{j \in J_1} a^1_{j*}\Lambda\Big)/\Lambda\Big) \oplus \Big(\Big(\bigoplus_{j \in J_2} a^2_{j*}\Lambda\Big)/\Lambda\Big)\Big]\Big|_U,$$

where for $i = 1, 2$ *we are quotienting by the canonical diagonal map*

$$\Lambda \to \bigoplus_{j \in J_i} a^i_{j*}\Lambda.$$

Proof

This follows from the fact that on U a chart for the log structure $M|_U$ can be given by the map

$$U \to X_{r,s,m} \to \operatorname{Spec} \mathbb{Z}[P_{r,s}]$$

as explained in Section 3.1, so that $\bar{M}^{\mathrm{gp}}_{\mathrm{rel}}|_U$ can be identified with $\big((\bigoplus_{j \in J_1} a^1_{j*}\mathbb{Z})/\mathbb{Z}\big) \oplus \big((\bigoplus_{j\in J_2} a^2_{j*}\mathbb{Z})/\mathbb{Z}\big)|_U$. □

We can now define a global map of sheaves

$$\wedge^k \Big[\Big(\bigoplus_{j=1}^n a^1_{j*}\Lambda\Big) \oplus \Big(\bigoplus_{j=1}^n a^2_{j*}\Lambda\Big)\Big] \to \wedge^k \bar{M}^{\mathrm{gp}}_{\mathrm{rel}} \otimes_{\mathbb{Z}} \Lambda \simeq R^k\psi\Lambda(k).$$

It is enough to describe a global map of sheaves $a^i_{j*}\mathbb{Z} \to \bar{M}^{\mathrm{gp}}_{\mathrm{rel}}$. Locally, on neighborhoods U, we map section $1 \in a^i_{j*}\mathbb{Z}(U)$ to the image in $\bar{M}^{\mathrm{gp}}_{\mathrm{rel}}(U)$ of a generator for the local equation defining $Y_{i,j}$ (this is independent of the choice of generator). These local maps over neighborhoods U glue to give a global map, since the two images of 1 in $M(U \times_X U')$ differ by units, so they are identified once we pass to $\bar{M}^{\mathrm{gp}}_{\mathrm{rel}}(U \times_X U')$. We see from the local description in Proposition 3.5 that the above map of sheaves is surjective and that the kernel is generated by images of the two diagonal maps $\Lambda \to \bigoplus_{j=1}^n a^i_{j*}\Lambda$ for $i = 1, 2$.

COROLLARY 3.6

There is a global isomorphism

$$\wedge^k \Big[\Big(\Big(\bigoplus_{j\in J_1} a^1_{j*}\Lambda\Big)/\Lambda\Big) \oplus \Big(\Big(\bigoplus_{j\in J_2} a^2_{j*}\Lambda\Big)/\Lambda\Big)\Big] \simeq R^k\psi\Lambda(k).$$

Let $\mathcal{L}_1 = \big(\bigoplus_{j=1}^n a^1_{j*}\Lambda\big)/\Lambda$, and let $\mathcal{L}_2 = \big(\bigoplus_{j=1}^n a^2_{j*}\Lambda\big)/\Lambda$. From the above corollary, we see that $R^k\psi\Lambda(k)$ can be decomposed as $\sum_{l=0}^k \wedge^l\mathcal{L}_1 \otimes \wedge^{k-l}\mathcal{L}_2$. If X was actually a product of strictly semistable schemes $X = X_1 \times_S X_2$, then the sheaves $\wedge^l\mathcal{L}_1$ and $\wedge^{k-l}\mathcal{L}_2$ would have an interpretation as pullbacks of the sheaves of nearby cycles $R^l\psi\Lambda$ and $R^{k-l}\psi\Lambda$ associated to X_1 and X_2, respectively. Corollary 3.6 would then look like a Künneth-type formula computing the sheaves of nearby cycles for a product of strictly semistable schemes. In fact, in such a situation, the computation of the sheaves of nearby cycles reflects the stronger relation between the actual complexes of nearby cycles

$$R\psi\Lambda_{X_1\times_S X_2} \simeq R\psi\Lambda_{X_1} \otimes^L_{\bar{s}} R\psi\Lambda_{X_2},$$

which takes place in the derived category of constructible sheaves of Λ-modules on $(X_1 \times_S X_2)_{\bar{s}}$. This result was proved in [I2] for a product of schemes of finite type. The isomorphism is stated in the case when Λ is torsion; however, the analogue morphism for Λ a finite extension of $\mathbb{Z}_l$ or $\mathbb{Q}_l$ can be defined by passage to the limit (see the formalism in [E]) and it will still be an isomorphism. Here we would like to give a different proof of this result in the case of the product of two strictly semistable schemes. We will use log schemes, specifically Nakayama's computation of log vanishing cycles for log smooth schemes.

Recall that the scheme S has generic point η and closed point s. We will freely use the notation $\bar{S}, \tilde{S}$ and $\bar{s}, \tilde{s}$, and also the corresponding notation for a scheme X fixed in the beginning of this section. We first need a preliminary result.

LEMMA 3.7
Let X_1 be a strictly semistable scheme over S. Then the sheaves $R^k\psi\Lambda$ are flat over Λ.

Proof
By [Sa, Proposition 1.1.2.1], we have an exact sequence of sheaves on $X_{1,\bar{s}}$

$$0 \to R^k\psi\Lambda \to i^*R^{k+1}j_*\Lambda(1) \to R^{k+1}\psi\Lambda(1) \to 0.$$

We will prove by induction on k that $R^{n-k}\psi\Lambda$ is flat over Λ. Indeed, $R^n\psi\Lambda = 0$, so the induction hypothesis is true for $k = 0$. For the induction step, note that we can compute $i^*R^{n-k+1}j_*\Lambda$ using log étale cohomology. Since X_1 is strictly semistable, it can be endowed with the canonical log structure M_1 associated to the special fiber. If $a^1_i : Y_{1,i} \hookrightarrow Y$ are the closed embeddings, then we have

$$\bar{M}_1^{\mathrm{gp}} = (M_1/\mathcal{O}^*_{X_1})^{\mathrm{gp}} \simeq \bigoplus_{i=1}^{n} a^1_{i*}\mathbb{Z}. \tag{1}$$

By [Na, Proposition 2.0.2, Theorem 0.2 (purity for log smooth morphisms)], we can compute $i^*R^{n-k+1}j_*\Lambda$ in the same way that we have computed $R^k\psi\Lambda$ above, getting

$$i^*R^{n-k+1}j_*\Lambda \simeq \wedge^{n-k+1}(\bar{M}_1^{\mathrm{gp}}) \otimes_{\mathbb{Z}} \Lambda(-n+k-1),$$

which is flat over Λ by 1. In the short exact sequence

$$0 \to R^{n-k}\psi\Lambda \to i^* R^{n-k+1} j_*\Lambda(1) \to R^{n-k+1}\psi\Lambda(1) \to 0$$

the middle term is flat, the right term is flat by the induction hypothesis, and so the left term must be flat as well. □

PROPOSITION 3.8

Let X_1 and X_2 be strictly semistable schemes over S. Then we have the following equality in the derived category of constructible $\Lambda[I_s]$-modules on $(X_1 \times_S X_2)_s$

$$R\psi(\Lambda_{X_{1,\eta}}) \otimes_s^L R\psi(\Lambda_{X_{2,\eta}}) \simeq R\psi(\Lambda_{(X_1 \times_S X_2)_\eta}),$$

where the external tensor product of complexes is obtained by taking $pr_1^* \otimes pr_2^*$ and where the superscript L refers to left derived tensor product.

Proof

We have seen from the above discussion that in the case of a log smooth scheme with vertical log structure, the complex of vanishing cycles depends only on the special fiber endowed with the canonical log structure. In other words, for $i = 1, 2$, we have $R\psi\Lambda_{X_{i,\eta}} \simeq R\tilde{\epsilon}_{i,*}\Lambda_{X_{i,s}}$ as complexes on $X_{i,s}$, where $\tilde{\epsilon}_i : \tilde{X}_{i,\tilde{s}} \to \bar{X}_{i,\bar{s}}$ is the identity morphism on the underlying schemes and forgets the log structure. Analogously, we also have $R\psi\Lambda_{(X_1 \times_S X_2)_\eta} = R\tilde{\epsilon}_*\Lambda$, where

$$\tilde{\epsilon} : (\tilde{X}_1 \times_{\tilde{S}} \tilde{X}_2)_{\tilde{s}} \to (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)_{\bar{s}}$$

is the morphism which forgets the log structure. (Here we have used the fact that the fiber product of log smooth schemes with vertical log structure is log smooth with vertical log structure and that the underlying scheme of the fiber product of log schemes $\tilde{X}_1 \times_{\tilde{S}} \tilde{X}_2$ is just $\bar{X}_1 \times_{\bar{S}} \bar{X}_2$; the latter holds since the induced log structure on $\bar{X}_1 \times_{\bar{S}} \bar{X}_2$ is saturated.) Therefore, it suffices to prove that we have an isomorphism

$$R\tilde{\epsilon}_*\Lambda_{(\tilde{X}_1 \times_{\tilde{S}} \tilde{X}_2)_{\tilde{s}}} \simeq R\tilde{\epsilon}_{1,*}\Lambda_{\tilde{X}_{1,\tilde{s}}} \otimes_{\bar{s}}^L R\tilde{\epsilon}_{2,*}\Lambda_{\tilde{X}_{2,\tilde{s}}}$$

in the derived category of constructible sheaves of $\Lambda[I_s]$-modules on $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)_{\bar{s}}$.

It is enough to show that the Künneth map

$$\mathcal{C} = R\tilde{\epsilon}_{1,*}\Lambda_{\tilde{X}_{1,s}} \otimes_{\bar{s}}^L R\tilde{\epsilon}_{2,*}\Lambda_{\tilde{X}_{2,\tilde{s}}} \to R\tilde{\epsilon}_*\Lambda_{(\tilde{X}_1 \times_{\tilde{S}} \tilde{X}_2)_{\tilde{s}}} = \mathcal{D},$$

which is defined as in [AGV, XVII, Équation 5.4.1.4], induces an isomorphism on the cohomology of the two complexes above, for then the map itself will be a quasi-isomorphism. The cohomology of the product complex can be computed using a Künneth formula as $H^n(\mathcal{C}) = \bigoplus_{k=0}^n R^k\tilde{\epsilon}_{1,*}\Lambda \otimes_{\bar{s}} R^{n-k}\tilde{\epsilon}_{2,*}\Lambda$. In general, the Künneth formula involves a spectral sequence with terms

$$E_2^{l,n-l} = \sum_{k=0}^{n-l} \mathrm{Tor}_l^{\Lambda[I_s]}(R^k\tilde{\epsilon}_{1,*}\Lambda, R^{n-l-k}\tilde{\epsilon}_{2,*}\Lambda) \Rightarrow H^n(\mathcal{C})$$

(see [GD1, XVII, Équation 6.5.4.2] for a statement using homology). In our case, the cohomology sheaves $R^k\tilde{\epsilon}_{i,*}\Lambda$ are flat Λ-modules with trivial I_s-action by Lemmas 3.1 and 3.7, so for $l > 0$ all the $E_2^{l,n-l}$ terms vanish. (Alternatively, one can prove the formula for $H^n(\mathcal{C})$ by taking flat resolutions for both of the factor complexes and using the fact that the cohomology sheaves of the flat complexes are flat as well.)

In order to prove that the induced map $H^n(\mathcal{C}) \to H^n(\mathcal{D})$ is an isomorphism, it suffices to check that it induces an isomorphism on stalks at geometric points. Let x be a geometric point of $X_1 \times_S X_2$ above the geometric point $\bar{s}$ of S. The point x will project to geometric points x_1 and x_2 of X_1 and X_2. From [I1] it follows that there is an isomorphism on stalks

$$R^k\tilde{\epsilon}_{i,*}\Lambda_{x_i} \simeq H^k(J_i, \Lambda)$$

for $0 \leq k \leq n$ and $i = 1, 2$, where J_i is the relative log inertia group

$$\ker\bigl(\pi_1^{\log}(X_i, x_i) \to \pi_1^{\log}(S, s)\bigr).$$

A similar statement holds for the stalks at x

$$R^n\tilde{\epsilon}_*\Lambda_x \simeq H^n(J, \Lambda),$$

where J is the relative log inertia group $\ker(\pi_1^{\log}(X, x) \to \pi_1^{\log}(S, s))$. Directly from the definition of the log fundamental group we can compute $J = J_1 \times J_2$. We have the following commutative diagram

$$\begin{tikzcd} H^n(\mathcal{C})_x \arrow[r] \arrow[d, "\cong"] & H^n(\mathcal{D})_x \arrow[d, "\cong"] \\ \bigoplus_{k=0}^{n} H^k(J_1, \Lambda) \otimes_\Lambda H^{n-k}(J_2, \Lambda) \arrow[r] & H^n(J_1 \times J_2, \Lambda) \end{tikzcd}$$

where the bottom arrow is the Künneth map in group cohomology and is also an isomorphism. (Again, the Künneth spectral sequence

$$E_2^{l,n-l} = \sum_{k=0}^{n-l} \mathrm{Tor}_l^{\Lambda}\bigl(H^k(J_1, \Lambda), H^{n-k}(J_2, \Lambda)\bigr)$$

degenerates at E_2, and all terms outside the vertical line $l = 0$ vanish because these cohomology groups are flat Λ-modules.) Therefore, the top arrow $H^n(\mathcal{C})_x \to$

$H^n(\mathcal{D})_x$ has to be an isomorphism for all geometric points x of X, which means that it comes from a global isomorphism of sheaves on X. $\square$

4. The monodromy filtration

4.1. Overview of the strictly semistable case

In this section, we explain a way of writing down explicitly the monodromy filtration on the complex of nearby cycles $R\psi\Lambda$ in the case of a strictly semistable scheme. Our exposition follows that of [Sa], which constructs the monodromy filtration using perverse sheaves. We let $\Lambda = \mathbb{Z}/l^r\mathbb{Z}, \mathbb{Z}_l, \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$. In fact, the proofs use $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$; then the results extend to $\Lambda = \mathbb{Z}_l, \mathbb{Q}_l, \bar{\mathbb{Q}}_l$.

Let $X_1/\mathcal{O}_K$ be a strictly semistable scheme of relative dimension $n-1$ with generic fiber $X_{1,\eta}$ and special fiber $Y_1 = X_{1,s}$. Let $R\psi\Lambda = \bar{i}^*R\bar{j}_*\Lambda$ be the complex of nearby cycles over $Y_{1,\bar{\mathbb{F}}}$. Let $D_1, \dots, D_m$ be the irreducible components of Y_1, and for each index set $I \subseteq \{1, \dots, m\}$ let $Y_I = \bigcap_{i\in I} D_i$, and let $a_I : Y_I \to Y_1$ be the immersion. The scheme Y_I is smooth of dimension $n-1-k$ if $\#I = k+1$. For all $0 \le k \le m-1$, we set

$$Y_1^{(k)} = \bigsqcup_{I\subseteq\{1,\dots,m\},\#I=k+1} Y_I$$

and we let $a_k : Y_1^{(k)} \to Y_1$ be the projection. We identify $a_{k*}\Lambda = \wedge^{k+1}a_{0*}\Lambda$.

We work in the derived category of bounded complexes of constructible sheaves of Λ-modules on $Y_{1,\bar{\mathbb{F}}}$. We denote this category by $D_c^b(Y_{1,\bar{\mathbb{F}}}, \Lambda)$.

Let $\partial[\pi]$ be the boundary of π with respect to the Kummer sequence obtained by applying i^*Rj_* to the exact sequence of étale sheaves on $X_{1,\eta}$,

$$0 \to \Lambda(1) \to \mathcal{O}^*_{X_{1,\eta}} \to \mathcal{O}^*_{X_{1,\eta}} \to 0$$

for $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$. Taking an inverse limit over r and tensoring, we get an element $\partial[\pi] \in i^*R^1j_*\Lambda(1)$ for $\Lambda = \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$. Let $\theta : \Lambda_{Y_1} \to i^*R^1j_*\Lambda(1)$ be the map sending 1 to $\partial[\pi]$. Let $\delta : \Lambda_{Y_1} \to a_{0*}\Lambda$ be the canonical map. The following result appears as [Sa, Corollary 1.3].

PROPOSITION 4.1

We have the following.

1. There is an isomorphism of exact sequences

$$\begin{tikzcd}
\Lambda_{Y_1} \arrow[r, "\delta"] \arrow[d] & a_{0*}\Lambda \arrow[r, "\delta\wedge"] \arrow[d] & \cdots \arrow[r, "\delta\wedge"] \arrow[d] & a_{n-1*}\Lambda \arrow[r] \arrow[d] & 0 \\
\Lambda_{Y_1} \arrow[r, "\theta"] & i^*R^1j_*\Lambda(1) \arrow[r, "\theta\cup"] & \cdots \arrow[r, "\theta\cup"] & i^*R^nj_*\Lambda(n) \arrow[r] & 0
\end{tikzcd}$$

where the first vertical arrow is the identity and all the other vertical arrows are isomorphisms.

2. *For* $k \geq 0$*, we have an exact sequence*

$$0 \to R^k\psi\Lambda \to i^*R^{k+1}j_*\Lambda(1) \to \cdots \to i^*R^n j_*\Lambda(n-k) \to 0,$$

where all the horizontal maps are induced from $\theta\cup$.

Note 4.1(A)

We note the following.

1. The vertical isomorphisms in the first part of Proposition 4.1 come from the Kummer sequence corresponding to each of the D_i for $i = 1, \ldots, m$. The maps $\theta_i : \Lambda_{D_i} \to i^*R^1 j_*\Lambda(1)$ are defined by sending 1 to $\partial[\pi_i]$, where π_i is the generator of the ideal defining D_i and ∂ is the connecting differential in the Kummer sequence. The isomorphism $a_{0*}\Lambda \xrightarrow{\sim} i^*R^1 j_*\Lambda(1)$ is the direct sum of the θ_i for $i = 1, \ldots, m$.

2. Putting together the two isomorphisms, we get a quasi-isomorphism of complexes

$$R^k\psi\Lambda(k)[-k] \xrightarrow{\sim} [a_{k*}\Lambda \to \cdots \to a_{n-1*}\Lambda \to 0], \tag{2}$$

where $R^k\psi\Lambda(k)$ is put in degree k and $a_{n-1*}\Lambda$ is put in degree $n-1$.

LEMMA 4.2

The complex $a_{l*}\Lambda[-l]$ *is a* $-(n-1)$*-shifted perverse sheaf for all* $0 \leq l \leq n-1$*, and so also is the complex* $R^k\psi\Lambda(k)[-k]$ *for all* $0 \leq k \leq n-1$.

Proof

Since $Y_1^{(l)}$ is smooth of dimension $n-1-l$, we know that $\Lambda[-l]$ is a $-(n-1)$-shifted perverse sheaf on $Y_1^{(l)}$. The map $a_l : Y^{(l)} \to Y$ is finite, and since the direct image for a finite map is exact for the perverse t-structure, we deduce that $a_{l*}\Lambda[-l]$ is a $-(n-1)$-shifted perverse sheaf on Y. This is true for each $0 \leq l \leq n-1$. The complex $R^k\psi\Lambda(k)[-k]$ is a successive extension of terms of the form $a_{l*}\Lambda[-l]$ (as objects in the triangulated category $D_c^b(Y_{\bar{\mathbb{F}}}, \Lambda)$). Because the category of $-(n-1)$-shifted perverse sheaves is stable under extensions, we conclude that $R^k\psi\Lambda(k)[-k]$ is also a $-(n-1)$-shifted perverse sheaf. □

Assume that $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$. Let $\mathcal{L} \in D_c^b(Y_{1,\bar{\mathbb{F}}}, \Lambda)$ be represented by the complex

$$\cdots \to \mathcal{L}^{k-1} \to \mathcal{L}^k \to \mathcal{L}^{k+1} \to \cdots.$$

Definition 4.3

We define $\tau_{\leq k}\mathcal{L}$ to be the standard truncation of $\mathcal{L}$ represented by the complex

$$\cdots \to \mathscr{L}^{k-1} \to \ker(\mathscr{L}^k \to \mathscr{L}^{k+1}) \to 0.$$

Then $\tau_{\leq k}$ is a functor on $D_c^b(Y_{1,\bar{\mathbb{F}}}, \Lambda)$. We also define $\tilde{\tau}_{\leq k}\mathscr{K}$ to be represented by the complex

$$\cdots \to \mathscr{L}^{k-1} \to \mathscr{L}^k \to \operatorname{im}(\mathscr{L}^k \to \mathscr{L}^{k+1}) \to 0.$$

For every k, we have a quasi-isomorphism $\tau_{\leq k}\mathscr{L} \xrightarrow{\sim} \tilde{\tau}_{\leq k}\mathscr{L}$, which is given degree by degree by the inclusion map.

COROLLARY 4.4
The complex $R\psi\Lambda$ is a $-(n-1)$-shifted perverse sheaf and the truncations $\tau_{\leq k}R\psi\Lambda$ make up a decreasing filtration of $R\psi\Lambda$ by $-(n-1)$-shifted perverse sheaves.

Proof
Since the cohomology of $R\psi\Lambda$ vanishes in degrees greater than $n-1$, we have $R\psi\Lambda \simeq \tau_{\leq n-1}R\psi\Lambda$, so it suffices to prove by induction that each $\tau_{\leq k}R\psi\Lambda$ is a $-(n-1)$-shifted perverse sheaf. For $k=0$, we have $\tau_{\leq 0}R\psi\Lambda \simeq R^0\psi\Lambda$, which is a $-(n-1)$-shifted perverse sheaf by Lemma 4.2. For $k \geq 1$, we have a distinguished triangle

$$(\tau_{\leq k-1}R\psi\Lambda, \tau_{\leq k}R\psi\Lambda, R^k\psi\Lambda[-k]),$$

and assuming that $\tau_{\leq k-1}R\psi\Lambda$ is a $-(n-1)$-shifted perverse sheaf, we conclude that $\tau_{\leq k}R\psi\Lambda$ is as well. The distinguished triangles become short exact sequences in the abelian category of perverse sheaves, from which we deduce that the $\tau_{\leq k}R\psi\Lambda$ make up a decreasing filtration of $R\psi\Lambda$ and that the graded pieces of this filtration are the $R^k\psi\Lambda[-k]$. □

Note 4.5
For $\Lambda = \mathbb{Z}_l, \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$, we still have standard truncation functors $\tau_{\leq k}$ which give us a distinguished triangle

$$(\tau_{\leq k-1}R\psi\Lambda, \tau_{\leq k}R\psi\Lambda, R^k\psi\Lambda[-k]),$$

but the $\tau_{\leq k}$ are defined differently. With the new definition, the proof and results of Corollary 4.4 still go through for $\Lambda = \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$.

The complex $R\psi\Lambda$ has an action of I_s which acts trivially on the cohomology sheaves $R^k\psi\Lambda$. From this, it follows that the action of I_s factors through the action of its tame pro-l-quotient. Let T be a generator of the pro-l part of the tame inertia (i.e., such that $t_l(T)$ is a generator of $\mathbb{Z}_l(1)$, where $t_l : I_l \to \mathbb{Z}_l(1)$ is the tame inertial character). We are interested in understanding the action of T on $R\psi\Lambda$. For $\Lambda = \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$, we are interested in understanding the action of $N = \log T$ by recovering its monodromy filtration (convolution of the kernel and image filtrations). However, the monodromy filtration of N is the same as the monodromy filtration of $\nu := T - 1$, so we will explain how to compute the latter.

We have seen that T acts trivially on the $R^k\psi\Lambda$, which means that ν sends $\tau_{\leq k} R\psi\Lambda \to \tilde{\tau}_{\leq k-1} R\psi\Lambda \xrightarrow{\sim} \tau_{\leq k-1} R\psi\Lambda$. We get an induced map

$$\bar{\nu} : R^k\psi\Lambda[-k] \to R^{k-1}\psi\Lambda[-k+1].$$

We record [Sa, Lemma 2.5(4)].

LEMMA 4.6
The map $\bar{\nu}$ and the isomorphisms of item 2 of Note 4.1(A) make a commutative diagram

$$\begin{tikzcd}
R^{k+1}\psi\Lambda[-(k+1)] \arrow[r, "\sim"] \arrow[d, "\bar{\nu}"] & \big[0 \arrow[r] \arrow[d] & a_{k+1*}\Lambda\big(-(k+1)\big) \arrow[r, "\delta\wedge"] \arrow[d, "\otimes t_l(T)"] & \cdots \\
R^k\psi\Lambda[-k] \arrow[r, "\sim"] & \big[a_{k*}\Lambda(-k) \arrow[r, "\delta\wedge"] & a_{k+1*}\Lambda(-k) \arrow[r, "\delta\wedge"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\delta\wedge"] & a_{n-1*}\Lambda\big(-(k+1)\big)\big] \arrow[d] \\
{} \arrow[r, "\delta\wedge"] & a_{n-1*}\Lambda(-k)\big]
\end{tikzcd}$$

where the sheaves $a_{n-1*}\Lambda(-(k+1))$ and $a_{n-1*}\Lambda(-k)$ are put in degree $n-1$.

Note 4.7
When $\Lambda = \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$, the monodromy operator $N = \log T$ is defined and it induces a map

$$\bar{N} : R^k\psi\Lambda[-k] \to R^{k-1}\psi\Lambda[-k+1].$$

This map coincides with $\bar{\nu}$, since $\log T \equiv T - 1 \pmod{(T-1)^2}$ and $(T-1)^2$ sends $\tau_{\leq k} R\psi\Lambda \to \tau_{\leq k-2} R\psi\Lambda$.

From the preceding commutative diagram, it is easy to see that the map $\bar{\nu}$ is injective, since we can just compute the cone of the map of complexes on the right.

In general, to compute the kernel and cokernel of a map of perverse sheaves, we have to compute the cone C of that map, then the perverse truncation $\tau^{p}_{\geq 0} C$ will be the cokernel and $\tau^{p}_{\leq -1} C[-1]$ will be the kernel (see the proof of [BBD, Théorème 1.3.6]). It is straightforward to check that the cone of $\bar{\nu}$ is quasi-isomorphic to $a_{k*}\Lambda(-k)[-k]$, which is a $-(n-1)$-shifted perverse sheaf. We deduce that $\bar{\nu}$ has kernel 0 and cokernel $a_{k*}\Lambda(-k)[-k]$.

The fact that $\bar{\nu}$ is injective means that the canonical filtration $\tau_{\leq k} R\psi\Lambda$ coincides with the kernel filtration of ν on $R\psi\Lambda$ and that the $R^k\psi\Lambda[-k]$ for $0 \leq k \leq n-1$ are the graded pieces of the kernel filtration. Moreover, the graded pieces of the induced image filtration of ν on the $R^k\psi\Lambda$ are $a_{k+h*}\Lambda(-h)[-(k+h)]$ for $0 \leq h \leq n-1-k$. This information suffices to reconstruct the graded pieces of the monodromy filtration on $R\psi\Lambda$.

PROPOSITION 4.8
There is an isomorphism

$$\bigoplus_{h-k=r} a_{(k+h)*}\Lambda(-h)[-(k+h)] \to \mathrm{Gr}^M_r R\psi\Lambda.$$

This isomorphism, together with the spectral sequence associated to the monodromy filtration, induces the weight spectral sequence (see [Sa, Corollary 2.2.4]).

4.2. The product of strictly semistable schemes

Let X_1 and X_2 be strictly semistable schemes of relative dimension $n-1$ over $\mathcal{O}_K$, and let $\Lambda = \mathbb{Z}/l^r\mathbb{Z}, \mathbb{Z}_l, \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$ (we will be more specific about Λ where it is important). Let $R\psi\Lambda_{X_i}$ be the complex of nearby cycles on $X_{i,\bar{s}}$ for $i = 1, 2$, and let $R\psi\Lambda_{X_1\times X_2}$ be the complex of nearby cycles on $(X_1 \times_S X_2)_{\bar{s}}$. By Proposition 3.8, we have

$$R\psi\Lambda_{X_1\times X_2} \simeq R\psi\Lambda_{X_1} \otimes_\Lambda R\psi\Lambda_{X_2},$$

and notice that this isomorphism is compatible with the action of the inertia I in G_K. From Proposition 3.1, the action of I is trivial on the cohomology sheaves of $R\psi\Lambda_{X_1\times X_2}$, so only the pro-$l$ part of I acts nontrivially on $R\psi\Lambda_{X_1\times X_2}$. Let T be a generator of the pro-l part of I, and set $\nu = T - 1$. Let ν, ν_1, ν_2 denote the action of ν on $R\psi\Lambda_{X_1\times X_2}$, $R\psi\Lambda_{X_1}$, and $R\psi\Lambda_{X_2}$, respectively. Since the above isomorphism is compatible with the action of T, we deduce that T acts on $R\psi\Lambda_{X_1} \otimes_{\Lambda[I]} R\psi\Lambda_{X_2}$ via $T \otimes T$. From this, we conclude that ν acts on $R\psi\Lambda_{X_1} \otimes_{\Lambda[I]} R\psi\Lambda_{X_2}$ as $\nu_1 \otimes 1 + 1 \otimes \nu_2 + \nu_1 \otimes \nu_2$.

As in the proof of Proposition 3.5, we have a decomposition

$$R^k\psi\Lambda \simeq \bigoplus_{l=0}^{k} R^l\psi\Lambda_{X_1} \otimes R^{k-l}\psi\Lambda_{X_2}.$$

We will see that ν induces a map

$$\bar{\nu}: R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-k+1]$$

which acts on $R^l\psi\Lambda_{X_1} \otimes_\Lambda R^{k-l}\psi\Lambda_{X_2}[-k]$ by $\bar{\nu}_1 \otimes 1 + 1 \otimes \bar{\nu}_2$. First we prove a few preliminary results.

For $i = 1, 2$ and $0 \le l \le n$, define the following schemes.

- Let $Y_i/\mathbb{F}$ be the special fiber of X_i.
- Let $D_{i,1}, \dots, D_{i,m_i}$ be the irreducible components of X_i.
- For $J \subseteq \{1, \dots, m_i\}$, let $Y_{i,J}$ be $\bigcap_{j\in J} D_{i,j}$, and let $a^i_J : Y_{i,J} \to Y_i$ be the immersion. Note that if the cardinality of J is $h+1$, then the scheme $Y_{i,J}$ is smooth of dimension $n-h-1$.
- For all $0 \le h \le m_i - 1$, set $Y_i^{(h)} = \bigsqcup_{\#J=h+1} Y_{i,J}$, and let $a^i_l : Y_i^{(h)} \to Y_i$ be the projection.

Then for each $i = 1, 2$, we have a resolution of $R^h\psi\Lambda_{X_i}[-h]$ in terms of the sheaves $a^i_{j*}\Lambda$

$$R^h\psi\Lambda_{X_i}[-h] \xrightarrow{\sim} [a^i_{l*}\Lambda(-h) \to \cdots \to a^i_{n-1*}\Lambda(-h)],$$

where $a^i_{n-1*}\Lambda(-h)$ is put in degree $n-1$.

Now let $Y/\mathbb{F}$ be the special fiber of $X_1 \times X_2$. Let

$$Y_{J_1,J_2} = \bigcap_{j_1\in J_1, j_2\in J_2} (D_{j_1} \times_{\mathbb{F}} D_{j_2}).$$

Set $Y^{(h_1,h_2)} = \bigsqcup_{\#J_1=h_1+1, \#J_2=h_2+1} Y_{J_1,J_2}$, and let $a_{h_1,h_2} : Y^{(h_1,h_2)} \to Y$ be the projection. The scheme $Y^{(h_1,h_2)}$ is smooth of dimension $2n-2-h_1-h_2$. Note that $Y^{(h_1,h_2)} = Y_1^{(h_1)} \times Y_2^{(h_2)}$ and that $a_{h_1,h_2*}\Lambda \simeq a^1_{h_1*}\Lambda \otimes a^2_{h_2*}\Lambda$, where the tensor product of sheaves is an external tensor product.

LEMMA 4.9

We have the following resolution of $R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k]$ as the complex

$$a_{h,k-h*}\Lambda(-k) \to a_{h,k-h+1*}\Lambda(-k) \oplus a_{h+1,k-h*}\Lambda(-k) \to \cdots \to a_{n-1,n-1*}\Lambda(-k),$$

where the sheaf $a_{n-1,n-1*}\Lambda(-k)$ is put in degree $2n-2$. The general term of the complex which appears in degree $h_1 + h_2$ is

$$\bigoplus_{\substack{h_1\ge h\\ h_2\ge k-h}} a_{h_1,h_2*}\Lambda(-k).$$

For each h_1, h_2, the complexes $a_{h_1,h_2*}\Lambda(-k)[-h_1-h_2]$ are $-(2n-2)$-shifted perverse sheaves, so the complex $R^k\psi\Lambda_{X_1\times X_2}[-k]$ is also a $-(2n-2)$-shifted perverse sheaf.

Proof
Each of the complexes $R^h\psi\Lambda_{X_1}$ and $R^{k-h}\psi\Lambda_{X_2}$ have resolutions in terms of $a^1_{h_1*}\Lambda(-h)$ and $a^2_{h_2*}\Lambda(-k+h)$, respectively, where $h\le h_1\le n-1$ and $k-h\le h_2\le n-1$. We form the double complex associated to the product of these resolutions, and the single complex associated to it is a resolution of $R^h\psi\Lambda_{X_1}\otimes R^{k-h}\psi\Lambda_{X_2}[-k]$ of the following form:

$$\begin{aligned} &a^1_{h*}\Lambda(-h)\otimes a^2_{k-h*}\Lambda(-k+h)\\ &\qquad\to a^1_{h+1*}\Lambda(-h)\otimes a^2_{k-h*}\Lambda(-k+h)\oplus a^1_{h*}\Lambda(-h)\otimes a^2_{k-h+1*}\Lambda(-k+h)\\ &\qquad\qquad\to\cdots\to a^1_{n-1*}\Lambda(-h)\otimes a^2_{n-1*}\Lambda(-k+h). \end{aligned}$$

In the above complex, the sheaf $a^1_{n-1*}\Lambda(-h)\otimes a^2_{n-1*}\Lambda(-k+h)$ is put in degree $2n-2$. Now we use the formula

$$a_{h_1,h_2*}\Lambda(-k)=a^1_{h_1*}\Lambda(-h)\otimes a^1_{h_2*}\Lambda(-k+h)$$

to conclude the first part of the lemma. The complex $a_{h_1,h_2*}\Lambda(-k)[-h_1-h_2]$ is the direct image via a_{h_1,h_2*} of the complex $\Lambda(-k)[-h_1-h_2]$ on $Y^{(h_1,h_2)}$. Since $Y^{(h_1,h_2)}$ is smooth of dimension $2n-2-h_1-h_2$, we know that $\Lambda(-k)[-h_1-h_2]$ is a $-(2n-2)$-shifted perverse sheaf, so its direct image under the finite map a_{h_1,h_2*} is also a $-(2n-2)$-shifted perverse sheaf. We have just seen that each $R^h\psi\Lambda_{X_1}\otimes R^{k-h}\psi\Lambda_{X_2}$ can be obtained from successive extensions of factors of the form $a_{h_1,h_2*}\Lambda(-k)\times[-h_1-h_2]$, and since the category of $-(2n-2)$-shifted perverse sheaves is stable under extensions, we deduce that $R^h\psi\Lambda_{X_1}\otimes R^{k-h}\psi\Lambda_{X_2}[-k]$ is a $-(2n-2)$-shifted perverse sheaf. Now $R^k\psi\Lambda_{X_1\times X_2}[-k]=\bigoplus_{h=0}^k R^h\psi\Lambda_{X_1}\otimes R^{k-h}\psi\Lambda_{X_2}[-k]$, so it is also a $-(2n-2)$-shifted perverse sheaf. □

COROLLARY 4.10
We hold that $R\psi\Lambda_{X_1\times X_2}$ is a $-(2n-2)$-shifted perverse sheaf. The standard truncation $\tau_{\le k}R\psi\Lambda_{X_1\times X_2}$ is a filtration by $-(2n-2)$-shifted perverse sheaves, and the graded pieces of this filtration are the $R^k\psi\Lambda_{X_1\times X_2}[-k]$.

Proof
The proof is exactly the same as that of Corollary 4.4. It suffices to show that each $\tau_{\le k}R\psi\Lambda$ is a $-(2n-2)$-shifted perverse sheaf, and we can do this by induction, using the distinguished triangle

$$(\tau_{\le k-1} R\psi\Lambda_{X_1\times X_2}, \tau_{\le k} R\psi\Lambda_{X_1\times X_2}, R^k\psi\Lambda_{X_1\times X_2}[-k]).$$

Once everything is proved to be in an abelian category, the distinguished triangle becomes a short exact sequence, and we get a filtration on $R\psi\Lambda_{X_1\times X_2}$ with its desired graded pieces. □

Now we can deduce that there is a map

$$\bar{\nu} : R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-(k-1)].$$

Indeed, since T acts trivially on the cohomology sheaves of $R\psi\Lambda_{X_1\times X_2}$, we deduce that ν sends $\tau_{\le k} R\psi\Lambda_{X_1\times X_2}$ to $\tau_{\le k-1} R\psi\Lambda_{X_1\times X_2}$, which induces $\bar{\nu}$. It remains to check that this induced map $\bar{\nu}$ restricted to $R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}$ is the same map as $\bar{\nu}_1 \otimes 1 + 1 \otimes \bar{\nu}_2$, sending

$$\begin{aligned} &R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k] \\ &\qquad \to (R^{h-1}\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_1} \oplus R^h\psi\Lambda_{X_1} \otimes R^{k-h-1}\psi\Lambda_{X_2})[-(k-1)]. \end{aligned}$$

First notice that, for each $0 \le h \le k \le n-1$, the complex $\tau_{\le h} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h} R\psi\Lambda_{X_2}$ is a $-(2n-2)$-shifted perverse sheaf because it is the external tensor product of $-(n-1)$-shifted perverse sheaves on X_1 and on X_2 (see [BBD, Proposition 4.2.8]). Let

$$\tau_{\le h-1} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h} R\psi\Lambda_{X_2} + \tau_{\le h} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h-1} R\psi\Lambda_{X_2}$$

be the image of

$$\begin{aligned} &\tau_{\le h-1} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h} R\psi\Lambda_{X_2} \oplus \tau_{\le h} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h-1} R\psi\Lambda_{X_2} \\ &\qquad\qquad \to \tau_{\le k-1} R\psi\Lambda_{X_1\times X_2}. \end{aligned}$$

We have a commutative diagram of $-(2n-2)$-shifted perverse sheaves

$$\begin{array}{ccc} \tau_{\le h} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h} R\psi\Lambda_{X_2} & \longrightarrow & \tau_{\le k} R\psi\Lambda_{X_1\times X_2} \\ \Big\downarrow {\scriptstyle \nu_1\otimes 1 + 1\otimes\nu_2 + \nu_1\otimes\nu_2} & & \Big\downarrow {\scriptstyle \nu} \\ \begin{array}{l} \tau_{\le h-1} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h} R\psi\Lambda_{X_2} \\ \quad + \tau_{\le h} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h-1} R\psi\Lambda_{X_2} \end{array} & \longrightarrow & \tau_{\le k-1} R\psi\Lambda_{X_1\times X_2} \end{array}$$

where the horizontal maps are the natural maps of complexes.

LEMMA 4.11

Assume that $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$. The image of $R_{h,k-h} = \tau_{\le h} R\psi\Lambda_{X_1} \otimes \tau_{\le k-h} R\psi\Lambda_{X_2}$ in $R^k\psi\Lambda[-k]$ is $R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k]$.

Proof
The map of perverse sheaves $R_{h,k-h} \to \tau_{\leq k} R\psi\Lambda_{X_1\times X_2} \to R^k\psi\Lambda[-k]$ factors through

$$R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k] \hookrightarrow R^k\psi\Lambda[-k].$$

This can be checked on the level of complexes. We only need to know that the natural map

$$R_{h,k-h} \xrightarrow{g} R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k]$$

is a surjection. This follows once we know that the triangle

$$R_{h-1,k-l} + R_{h,k-h-1} \xrightarrow{f} R_{h,k-h} \xrightarrow{g} R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k]$$

is distinguished, since then it has to be a short exact sequence of $-(2n-2)$-shifted perverse sheaves, so g would be a surjection. To check that the triangle is distinguished, it suffices to compute the fiber of g and check that it is quasi-isomorphic to

$$\mathcal{M} = \tilde{\tau}_{\leq h-1} R\psi\Lambda_{X_1} \otimes \tau_{\leq k-h} R\psi\Lambda_{X_2} + \tau_{\leq h} R\psi\Lambda_{X_1} \otimes \tilde{\tau}_{\leq k-h-1} R\psi\Lambda_{X_2}.$$

Let $\mathcal{K}^{\cdot}$ be a representative for $R\psi\Lambda_{X_1}$, and let $\mathcal{L}^{\cdot}$ be a representative for $R\psi\Lambda_{X_2}$. The degree $j<k$ term of $\mathcal{M}$ and of the fiber of g are both equal to

$$\Bigg(\bigoplus_{i=j-k+l+1}^{h-1} \mathcal{K}^i \otimes \mathcal{L}^{j-i}\Bigg) \oplus \mathcal{K}^{j-k+h} \otimes \ker(\mathcal{L}^{k-h} \to \mathcal{L}^{k-h+1})$$
$$\oplus \ker(\mathcal{K}^h \to \mathcal{K}^{h+1}) \otimes \mathcal{L}^{j-h}$$

and the differentials are identical. The last nonzero term $\mathcal{M}^k$ in $\mathcal{M}$ appears in degree k and is equal to

$$\ker(\mathcal{K}^h \to \mathcal{K}^{h+1}) \otimes \operatorname{im}(\mathcal{L}^{k-h-1} \to \mathcal{L}^{k-h})$$
$$+ \operatorname{im}(\mathcal{K}^{h-1} \to \mathcal{K}^h) \otimes \ker(\mathcal{L}^{k-h} \to \mathcal{L}^{k-h+1}).$$

The main problem is checking that the following map of complexes is a quasi-isomorphism

$$\begin{tikzcd}
\mathcal{M}^k \arrow[r] \arrow[d, "\lambda"] & 0 \arrow[d] \\
\ker(\mathcal{K}^h \to \mathcal{K}^{h+1}) \otimes \ker(\mathcal{L}^{k-h} \to \mathcal{L}^{k-h+1}) \arrow[r] & H^h(\mathcal{K}) \otimes H^{k-h}(\mathcal{L})
\end{tikzcd}$$

where the left vertical arrow λ is the natural inclusion. It is equivalent to prove that the object in the lower right corner is the cokernel of λ. This follows from the Künneth spectral sequence, when computing the cohomology of the product of the two complexes

$$\tilde{\mathcal{K}} := [\mathrm{im}(\mathcal{K}^{h-1} \to \mathcal{K}^h) \to \ker(\mathcal{K}^h \to \mathcal{K}^{h+1})]$$

and

$$\tilde{\mathcal{L}} := [\mathrm{im}(\mathcal{L}^{k-h-1} \to \mathcal{L}^{k-h}) \to \ker(\mathcal{L}^{k-h} \to \mathcal{L}^{k-h+1})].$$

Indeed, since $H^1(\tilde{\mathcal{K}}) = R^h\psi\Lambda_{X_1}$ and $H^1(\tilde{\mathcal{L}}) = R^{k-h}\psi\Lambda_{X_2}$ are both flat over Λ, the Künneth spectral sequence degenerates. We get $H^2(\tilde{\mathcal{K}} \otimes \tilde{\mathcal{L}}) = H^1(\tilde{\mathcal{K}}) \otimes H^1(\tilde{\mathcal{L}})$, and this is exactly the statement that $H^h(\mathcal{K}) \otimes H^{k-h}(\mathcal{L})$ is the cokernel of λ. $\square$

Note 4.12
The result of this lemma extends to $\Lambda = \mathbb{Q}_l$ and to $\Lambda = \bar{\mathbb{Q}}_l$.

Putting together the above discussion and keeping in mind that the image of $\nu_1 \otimes \nu_2$ in $R^{k-1}\psi\Lambda_{X_1\times X_2}[-(k-1)]$ is trivial, we conclude the following result.

PROPOSITION 4.13
The action of N on $R\psi\Lambda_{X_1\times X_2}$ induces a map

$$\bar{\nu} : R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-(k-1)]$$

which coincides with $\bar{\nu}_1 \otimes 1 + 1 \otimes \bar{\nu}_2$ when restricted to $R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k]$ for each $0 \le h \le k$.

We now use the decomposition of $R^k\psi\Lambda_{X_1\times X_2}[-k]$ in terms of $R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k]$ for $0 \le h \le k$ and the resolution of $R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}[-k]$ in terms of $a_{h_1,h_2*}\Lambda(-k)[-(h_1+h_2)]$ to get a resolution of $R^k\psi\Lambda_{X_1\times X_2}[-k]$ of the form

$$\bigoplus_{h_1+h_2=k} a_{h_1,h_2*}\Lambda(-k)^{\oplus c^k_{h_1 h_2}} \to \cdots \to \bigoplus_{h_1+h_2=k+j} a_{h_1,h_2*}\Lambda(-k)^{\oplus c^k_{h_1,h_2}} \to \cdots, \tag{3}$$

where the first term is put in degree k and where the coefficients $c^k_{h_1,h_2}$ count how many copies of $a_{h_1,h_2*}\Lambda(-k)$ show up in the direct sum.

LEMMA 4.14
Let $c^k_{h_1,h_2}$ *be the coefficient of* $a_{h_1,h_2*}\Lambda(-k)[-(h_1+h_2)]$ *in the resolution of* $R^k\psi\Lambda_{X_1\times X_2}[-k]$. *Then*

$$c^k_{h_1,h_2} = \min\bigl(\min(h_1,h_2)+1, h_1+h_2-k+1, k+1\bigr).$$

Proof
The coefficient $c^k_{h_1,h_2}$ records the number of values of $0 \le h \le k$ for which the term $R^h\psi\Lambda_{X_1} \otimes R^{k-h}\psi\Lambda_{X_2}$ appears in the resolution of $a_{h_1,h_2*}\Lambda(-k)[-(h_1+h_2)]$. This count is clearly bounded by $k+1$ because there are $k+1$ possible values of h. When $h_1+h_2-k+1 \le k+1$, the count is

$$\min\bigl(\min(h_1,h_2)+1, h_1+h_2-k+1\bigr)$$

because $a_{h_1,h_2*}\Lambda(-k)[-(h_1+h_2)]$ shows up in the resolution of $R^{h_1-j}\psi\Lambda_{X_1} \otimes R^{k-h_1+j}\psi\Lambda_{X_2}$ for all $0 \le j \le h_1+h_2-k+1$ which satisfy $0 \le h_1 - j \le k$. When both h_1 and h_2 are less than k, all the $j \in 0,\dots,h_1+h_2-k+1$ satisfy the requirement. When $h_2 \ge k$, there are exactly h_1+1 values of j which satisfy the requirement and we can treat the case $h_1 \ge k$ analogously to get h_2+1 values of j. This covers the case $h_1+h_2 \le 2k$. In the case $h_1+h_2 \ge 2k$, we need to count all $0 \le j \le k$ which satisfy $0 \le h_1 - j \le k$. The result is

$$\min\bigl(\min(h_1,h_2)+1, k+1\bigr).$$

This completes the determination of $c^k_{h_1,h_2}$. ☐

Note that, for all $h_1+h_2 \le 2k-2$, we have $c^k_{h_1,h_2} \le c^{k-1}_{h_1,h_2}$. For $h_1+h_2 = 2k-1$, we always have $\min(h_1,h_2)+1 \le k < k+1$ so that

$$c^k_{h_1,h_2} = c^{k-1}_{h_1,h_2} = \min(h_1,h_2)+1.$$

However, $c^k_{k,k} = k+1 > k = c^{k-1}_{k,k}$, and for $h_1+h_2 \ge 2k$, we have $c^k_{h_1,h_2} \ge c^{k-1}_{h_1,h_2}$.

We now have an explicit description of

$$\bar{\nu} : R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-k]$$

as a map of complexes with terms of the form $\bigoplus_{h_1+h_2=k+j} a_{h_1,h_2*}\Lambda(-k)^{\oplus c^k_{h_1,h_2}}$, which are put in degree $k+j$. Writing $\bar{\nu} = \bar{\nu}_1 \otimes 1 + 1 \otimes \bar{\nu}_2$ as a map of complexes, we are able to compute both the kernel and cokernel of $\bar{\nu}$.

We now restrict to $\Lambda = \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$. In this case, $N = \log T$ is defined, acts trivially on the cohomology sheaves $R^k\psi\Lambda_{X_1\times X_2}$, and so induces a map

$$\bar{N} : R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-k].$$

Since $N \equiv T-1 \pmod{(T-1)^2}$ and $(T-1)^2$ sends $\tau_{\leq k} R\psi\Lambda \to \tau_{\leq k-2} R\psi\Lambda$ (here $\tau_{\leq k}$ denote the truncation functors for $\mathbb{Q}_l$- or $\bar{\mathbb{Q}}_l$-sheaves), the two maps $\bar{N}$ and $\bar{\nu}$ coincide, so we will work with $\bar{N}$ from now on, to which the results of Proposition 4.13 apply.

First we need a preliminary result which allows us to compute the kernels and cokernels of certain morphisms of $-(2n-2)$-shifted perverse sheaves. Note that, while $D^b_c(Y,\Lambda)$ is not known to be a derived category of some category of Λ-sheaves, being constructed as a "projective limit" of derived categories, it is nevertheless endowed with a standard t-structure whose core is the category of Λ-sheaves. Therefore, by [BBD, Proposition 3.1.10], we have a realization functor from the bounded derived category of Λ-sheaves to $D^b_c(Y,\Lambda)$. Using this functor, we think of a bounded complex of Λ-sheaves as an element in $D^b_c(Y,\Lambda)$ and of a morphism of complexes as a morphism in $D^b_c(Y,\Lambda)$.

LEMMA 4.15
Assume that $f:\mathcal{C}\to\mathcal{D}$ is a morphism in $D^b_c(Y,\Lambda)$ which satisfies the following: $\mathcal{C}$ and $\mathcal{D}$ are (the image of) complexes $(\mathcal{C}^k)_{k\in\mathbb{Z}}$ and $(\mathcal{D}^k)_{k\in\mathbb{Z}}$ of l-adic sheaves, and f is a map of complexes defined degree by degree as $f^k:\mathcal{C}^k\to\mathcal{D}^k$. Assume that each f^k is injective as a map of sheaves. Let $\bar{\mathcal{D}}^k = \operatorname{coker}(f^k)$, and let $\bar{\mathcal{D}}$ be (the image in $D^b_c(Y,\Lambda)$ of) the complex with terms $\bar{\mathcal{D}}^k$ and differential $\bar{d}$ induced by the differential d of $\mathcal{D}$. Assume that the short exact sequence of sheaves

$$0\to\mathcal{C}^k \xrightarrow{f^k} \mathcal{D}^k \to \bar{\mathcal{D}}^k \to 0$$

is splittable. Assume also that $\mathcal{C}^k[-k]$ and $\mathcal{D}^k[-k]$ are $-(2n-2)$-shifted perverse sheaves.

Then $\bar{\mathcal{D}}^k[-k]$ is a $-(2n-2)$-shifted perverse sheaf and thus so is $\bar{\mathcal{D}}$ (since it is an extension of $\bar{\mathcal{D}}^k[-k]$ for finitely many k). Moreover, the following is an exact sequence of $-(2n-2)$-shifted perverse sheaves:

$$0\to\mathcal{C}\to\mathcal{D}\to\bar{\mathcal{D}}\to 0.$$

Proof
$\bar{\mathcal{D}}^k[-k]$ is a $-(2n-2)$-shifted perverse sheaf because it is a direct factor of $\mathcal{D}^k[-k]$, and so $\bar{\mathcal{D}}$ is also a $-(2n-2)$-shifted perverse sheaf. If Λ was torsion, then we could identify the category $D^b_c(Y,\Lambda)$ with a full subcategory of the derived category of the category of sheaves of Λ-modules (whose objects have bounded constructible cohomology) and the corollary would follow from a standard diagram chase in the derived category of an abelian category. However, the cases we are interested in are $\Lambda=\mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$. It is possible that by checking the definition of the category $D^b_c(Y,\Lambda)$

carefully, we could ensure that a version of the diagram chase applies to our case. However, an alternative approach uses Beilinson's result which identifies $D_c^b(Y, \Lambda)$ with the derived category of perverse sheaves on Y (see [B, Theorem 1.3]).

We see that the map $f : \mathcal{C} \to \mathcal{D}$ is injective, since we can think of it as a map of filtered objects which is injective on the kth graded pieces for each k. Indeed, $\mathcal{C}$ is a successive extension of the $-(2n-2)$-shifted perverse sheaves $\mathcal{C}^k[-k]$ and $\mathcal{D}$ is a successive extension of $\mathcal{D}^k[-k]$, and the fact that f is a map of complexes implies that f respects these extensions. Let k be the largest integer for which either of $\mathcal{C}^k$ and $\mathcal{D}^k$ is nonzero. We have the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^k[-k] \arrow[r] \arrow[d, "{f^k[-k]}"] & \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C}^{k-1}[-k+1] \arrow[r] \arrow[d, "{f^{k-1}[-k+1]}"] & 0 \\
0 \arrow[r] & \mathcal{D}^k[-k] \arrow[r] & \mathcal{D}' \arrow[r] & \mathcal{D}^{k-1}[-k+1] \arrow[r] & 0
\end{tikzcd}$$

where the arrows on the left and on the right are injective. The fact that the middle map is also injective follows from a standard diagram chase. (Note that we are working in the category of $-(2n-2)$-shifted perverse sheaves, which is abelian, so we can perform diagram chases by [Re].) The injectivity of f follows by induction.

By a repeated application of the snake lemma in the abelian category of $-(2n-2)$-shifted perverse sheaves, we see that the cokernel of f is a successive extension of terms of the form $\bar{\mathcal{D}}^k[-k]$. In order to identify this cokernel with $\bar{\mathcal{D}}$, it suffices to check that the differential of $\bar{\mathcal{D}}$ coincides in $\mathrm{Ext}^1(\bar{\mathcal{D}}^k[-k], \bar{\mathcal{D}}^{k-1}[-k+1])$ with the extension class which defines the cokernel. To check this, it is enough to see that the following square is commutative:

$$\begin{tikzcd}
\mathcal{D}^{k-1}[-k+1] \arrow[r] \arrow[d, "{f^{k-1}[-k+1]}"] & \mathcal{D}^k[-k+1] \arrow[d, "{f^k[-k+1]}"] \\
\bar{\mathcal{D}}^{k-1}[-k+1] \arrow[r] & \bar{\mathcal{D}}^k[-k+1]
\end{tikzcd}$$

where the top (resp., bottom) horizontal map is the boundary map obtained from considering the distinguished triangle $(\mathcal{D}^k[-k], \mathcal{D}', \mathcal{D}^{k-1}[-k+1])$ (resp., $(\bar{\mathcal{D}}^k[-k],$ $\bar{\mathcal{D}}', \bar{\mathcal{D}}^{k-1}[-k+1])$) in $D_c^b(Y, \Lambda)$. The top boundary map is the differential of $\mathcal{D}$, and if the square is commutative, then the bottom map must be the differential of $\bar{\mathcal{D}}$. The commutativity can be checked by hand, by making the boundary maps explicit using the construction of the cone. (There is a natural map

$$\begin{tikzcd} \mathcal{D}^k[-k] \arrow[r] \arrow[d] & \mathcal{D}' \arrow[d] \\ 0 \arrow[r] & \mathcal{D}^{k-1}[-k+1] \end{tikzcd}$$

which is a quasi-isomorphism in $D^b_c(Y,\Lambda)$. The boundary map of the distinguished triangle is obtained by composing the inverse of this quasi-isomorphism with the natural map

$$\begin{tikzcd} \mathcal{D}^k[-k] \arrow[r] \arrow[d] & \mathcal{D}' \arrow[d] \\ \mathcal{D}^k[-k] \arrow[r] & 0 \end{tikzcd}$$

The same construction works for $\bar{\mathcal{D}}$ and it is straightforward to check the commutativity now.) $\square$

LEMMA 4.16
Let $k \geq 1$. Consider the map

$$\bar{N} : R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-(k-1)].$$

Define the complex

$$\mathcal{P}_k = [a_{k,k*}\Lambda(-k) \overset{\wedge\delta}{\to} a_{k,k+1*}\Lambda(-k) \oplus a_{k+1,k*}\Lambda(-k) \to \cdots \overset{\wedge\delta}{\to} a_{n-1,n-1*}\Lambda(-k)],$$

where $a_{k,k*}\Lambda(-k)$ is put in degree $2k$. The factor $a_{h_1,h_2*}\Lambda(-k)$ appears in the resolution of $\mathcal{P}$ in degree h_1+h_2 whenever $h_1,h_2 \in \{k,k+1,\dots,n-1\}$. Also define the complex

$$\mathcal{R}_k = \Big[\bigoplus_{j=0}^{k-1} a_{j,k-1-j*}\Lambda\big(-(k-1)\big) \to \cdots \to a_{k-1,k-1*}\Lambda\big(-(k-1)\big)\Big],$$

where the first term is put in degree $k-1$ and the term $a_{h_1,h_2*}\Lambda(-(k-1))$ appears in degree h_1+h_2 whenever $h_1,h_2 \in \{0,1,\dots,k-1\}$.

Then $\mathcal{P}_k \simeq \ker(\bar{N})$ and $\mathcal{R}_k \simeq \operatorname{coker}(\bar{N})$.

Proof
Note that both $\mathcal{P}_k$ and $\mathcal{R}_k$ are $-(2n-2)$-shifted perverse sheaves by the same argument that we have used before. The proof goes as follows. We first define a map

$\mathcal{P}_k \to R^k\psi\Lambda_{X_1\times X_2}[-k]$ and check that $\bar{N}$ kills the image of $\mathcal{P}_k$. We use Lemma 4.15 to check that the map $\mathcal{P}_k \to R^k\psi\Lambda_{X_1\times X_2}[-k]$ is an injection and to compute its cokernel $\mathcal{Q}_k$. Then we check using Lemma 4.15 again that the induced map $\mathcal{Q}_k \to R^{k-1}\psi\Lambda[-(k-1)]$ is an injection, and we identify its cokernel with $\mathcal{R}_k$.

For the first step, note that it suffices to define the maps

$$f^{h_1,h_2} : a_{h_1,h_2*}\Lambda(-k) \to a_{h_1,h_2*}\Lambda(-k)^{\oplus(k+1)}$$

for all $h_1, h_2 \geq k$ and we do so by $x \mapsto (x, -x, \dots, (-1)^k x)$. These maps are clearly compatible with the differentials $\wedge\delta$, so they induce a map $f : \mathcal{P}_k \to R^k\psi\Lambda_{X_1\times X_2}[-k]$ (this is a map of complexes between $\mathcal{P}$ and the standard representative of $R^k\psi\Lambda_{X_1\times X_2}[-k]$). Moreover, we can check that the restriction

$$\bar{N} : a_{h_1,h_2*}\Lambda(-k)^{\oplus(k+1)} \to a_{h_1,h_2*}\Lambda(-k)^{\oplus k}$$

sends $(x, -x, \dots, (-1)^k x) \mapsto (0, \dots, 0)$.

Indeed, the jth factor $a_{h_1,h_2*}\Lambda(-k)$ appears in the resolution of $R^j\psi\Lambda_{X_1} \otimes R^{k-j}\psi\Lambda_{X_2}[-k]$. The latter object is sent by $\bar{N}_1 \otimes 1$ to $R^{j-1}\psi\Lambda_{X_1} \otimes R^{k-j}\psi\Lambda_{X_2}[-(k-1)]$ for $1 \leq j \leq k$, and by $1 \otimes \bar{N}_2$ to $R^j\psi\Lambda_{X_1} \otimes R^{k-1-j}\psi\Lambda_{X_2}[-(k-1)]$ for $0 \leq j \leq k-1$. We also know that $\bar{N}_1 \otimes 1$ kills $R^0\psi\Lambda_{X_1} \otimes R^k\psi\Lambda_{X_2}[-k]$, and similarly $1 \otimes \bar{N}_2$ kills $R^k\psi\Lambda_{X_1} \otimes R^0\psi\Lambda_{X_2}[-k]$. By Lemma 4.6, we find that, for $1 \leq j \leq k-1$,

$$(0, \dots, 0, x, 0, \dots, 0) \mapsto \big(0, \dots, x \otimes t_l(T), x \otimes t_l(T), 0, \dots, 0\big),$$

where the term x is put in position j and the terms $x \otimes t_l(T)$ are put in positions $j-1$ and j. We also have

$$(x, 0, \dots, 0) \mapsto \big(x \otimes t_l(T), 0, \dots, 0\big)$$

and

$$(0, \dots, 0, x) \mapsto \big(0, \dots, 0, x \otimes t_l(T)\big).$$

Thus, we find that $\bar{N}$ sends

$$\begin{aligned}&\big(x, -x, \dots, (-1)^k x\big)\\ &\qquad\mapsto \big(x \otimes t_l(T) - x \otimes t_l(T), \dots, (-1)^{k-1} x \otimes t_l(T) + (-1)^k x \otimes t_l(T)\big),\end{aligned}$$

and the term on the right is $(0, \dots, 0)$. Since we have exhibited $\bar{N} \circ f$ as a chain map and we have checked that it vanishes degree by degree, we conclude that $\bar{N} \circ f = 0$. Thus, $f(\mathcal{P}_k) \subseteq \ker \bar{N}$.

Note that, for all $h_1, h_2 \geq k$, we can identify the quotient of $a_{h_1,h_2*}\Lambda(-k)^{\oplus(k+1)}$ by $f^{h_1,h_2}(a_{h_1,h_2*}\Lambda(-k))$ with $a_{h_1,h_2*}\Lambda(-k)^{\oplus k}$. The resulting exact sequence

$$0 \to a_{h_1,h_2*}\Lambda(-k) \overset{f^{h_1,h_2}}{\to} a_{h_1,h_2*}\Lambda(-k)^{\oplus(k+1)} \to a_{h_1,h_2*}\Lambda(-k)^{\oplus k} \to 0$$

is splittable because the third term is free over Λ. By Lemma 4.15, the map $f : \mathcal{P}_k \to R^k\psi\Lambda_{X_1\times X_2}[-k]$ is injective and we can identify degree by degree the complex $\mathcal{Q}_k$ representing the cokernel of f. In degrees less than $2k-1$, the terms of $\mathcal{Q}_k$ are the same as those of $R^k\psi\Lambda_{X_1\times X_2}[-k]$, and in degrees at least $2k-1$ they are the terms of $R^{k-1}\psi\Lambda_{X_1\times X_2}[-k+1]$.

To prove that the induced map $\mathcal{Q}_k \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-(k-1)]$ is injective, it suffices to check degree by degree, and the proof is analogous to the one for $f : \mathcal{P}_k \to R^k\psi\Lambda_{X_1\times X_2}[-k]$. The cokernel is identified with $\mathcal{R}_k$ degree by degree, via the exact sequence

$$\begin{aligned}0 \to a_{h_1,h_2*}\Lambda(-k)^{\oplus(k-1)} \overset{\bar{N}^{h_1,h_2}}{\to} a_{h_1,h_2*}\Lambda\big(-(k-1)\big)^{\oplus k}&\\ \to a_{h_1,h_2*}\Lambda\big(-(k-1)\big) \to 0&\end{aligned}$$

for $0 \leq h_1, h_2 \leq k-1$. □

Note 4.16(A)
We have the following.

1. The complex $\mathcal{P}_k$ has as its factors exactly the terms $a_{h_1,h_2*}\Lambda(-k)[-(h_1+h_2)]$ for which $c^k_{h_1,h_2} - c^{k-1}_{h_1,h_2} = 1$, while $\mathcal{R}_k$ has as its factors the terms $a_{h_1,h_2*}\Lambda(-(k-1))[-(h_1+h_2)]$ for which $c^{k-1}_{h_1,h_2} - c^k_{h_1,h_2} = 1$.

2. Another way to express the kernel of $\bar{N}$ is as the image of $R^{2k}\psi\Lambda_{X_1\times X_2}[-2k]$ in $R^k\psi\Lambda_{X_1\times X_2}[-k]$ under the map

$$\bar{N}_1^k \otimes 1 - \bar{N}_1^{k-1} \otimes \bar{N}_2 + \cdots + (-1)^k 1 \otimes \bar{N}_2^k.$$

This follows from Lemmas 4.6 and 4.16.

COROLLARY 4.17
The filtration of $R\psi\Lambda_{X_1\times X_2}$ by $\tau_{\leq k}R\psi\Lambda_{X_1\times X_2}$ induces a filtration on $\ker N$. *The first graded piece of this filtration* $\mathrm{Gr}^1\ker N$ *is* $R^0\psi\Lambda_{X_1\times X_2}$. *The graded piece* $\mathrm{Gr}^{k+1}\ker N$ *of this filtration is* $\mathcal{P}_k$.

Proof
We have already seen that N maps all of $R^0\psi\Lambda_{X_1\times X_2}$ to 0, since T acts trivially on the cohomology of $R\psi\Lambda_{X_1\times X_2}$. This identifies the first graded piece to be $R^0\psi\Lambda_{X_1\times X_2}$.

In order to identify the $(k+1)$st graded piece, we once more pretend that our shifted perverse sheaves have elements. We can do this since the $(2-2n)$-shifted perverse sheaves form an abelian category and we only need to do this in order to simplify the exposition. First notice that $\mathrm{Gr}^k \ker N \subseteq \mathcal{P}_k$, since anything in the kernel of N reduces to something in the kernel of $\bar{N}$.

So it suffices to show that any $x \in \mathcal{P}_k$ lifts to some $\tilde{x} \in \ker N$. Pick any $\tilde{x} \in \tau_{\leq k} R\psi\Lambda_{X_1\times X_2}$ lifting x. Since $\bar{N}$ sends x to 0, we conclude that N maps $\tilde{x}$ to $\tau_{\leq k-2} R\psi\Lambda_{X_1\times X_2}$. The image of $N\tilde{x}$ in $R^{k-2}\psi\Lambda_{X_1\times X_2}[-k+2]$ depends on our choice of the lift $\tilde{x}$. However, the image of $N\tilde{x}$ in $\mathcal{R}_{k-1}$ only depends on x. If we can show that that image is 0, we conclude that we can pick a lift $\tilde{x}$ such that $N\tilde{x} \in \tau_{\leq k-3} R\psi\Lambda$. We can continue applying the same argument while modifying our choice of lift $\tilde{x}$ such that $N\tilde{x} \in \tau_{\leq k-j} R\psi\Lambda_{X_1\times X_2}$ for larger and larger j. In the end, we see that $N\tilde{x} = 0$.

It remains to check that the map $\mathcal{P}_k \to \mathcal{R}_{k-1}$ sending $x \in \mathcal{P}_k$ to the image of $N\tilde{x}$ in $\mathcal{R}_{k-1}$ is 0. We can see this by checking that any map $\mathcal{P}_k \to \mathcal{R}_{k-1}$ is 0. Indeed, we have the following decompositions of $\mathcal{P}_k$ and $\mathcal{R}_{k-1}$ as $(2-2n)$-shifted perverse sheaves:

$$\mathcal{P}_k = [a_{k,k*}\Lambda(-k) \overset{\wedge\delta}{\to} a_{k,k+1*}\Lambda(-k) \oplus a_{k+1,k*}\Lambda(-k) \to \cdots \overset{\wedge\delta}{\to} a_{n-1,n-1*}\Lambda(-k)]$$

and

$$\mathcal{R}_{k-1} = \Big[\bigoplus_{j=0}^{k-1} a_{j,k-2-j*}\Lambda\big(-(k-2)\big) \to \cdots \to a_{k-2,k-2*}\Lambda\big(-(k-2)\big)\Big].$$

Each of the factors $a_{h_1,h_2*}\Lambda$ is a direct sum of factors of the form $a_{J_1,J_2*}\Lambda$, where $\operatorname{card} J_i = h_i$ for $i = 1,2$ and where $a_{J_1,J_2} : Y_{J_1,J_2} \hookrightarrow Y$ is a closed immersion. Each factor $a_{J_1,J_2*}\Lambda$ is a simple $(2-2n)$-shifted perverse sheaf, so we have decompositions into simple factors for both $\mathcal{P}_k$ and $\mathcal{R}_{k-1}$. It is straightforward to see that $\mathcal{P}_k$ and $\mathcal{R}_{k-1}$ have no simple factors in common. Thus, any map $\mathcal{P}_k \to \mathcal{R}_{k-1}$ must vanish. The same holds true for any map $\mathcal{P}_k \to \mathcal{R}_{k-j}$ for any $2 \leq j \leq k$. $\square$

The filtration with graded pieces $\mathcal{P}_k$ on $\ker N$ induces a filtration on $\ker N/\operatorname{im} N \cap \ker N$ whose graded pieces are $\mathcal{P}_k/\operatorname{im}\bar{N}$. Indeed, it suffices to check that the image of $\operatorname{im} N$ in $\mathcal{P}_k$ coincides with $\operatorname{im}\bar{N}$. The simplest way to see this is again by using a diagram chase. First, it is obvious that, for

$$\bar{N} : R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-1}\psi\Lambda_{X_1\times X_2}[-k+1],$$

we have $\operatorname{im}\bar{N} \subseteq \mathrm{Gr}^k \operatorname{im} N$. Now let $x \in \mathrm{Gr}^k \operatorname{im} N$. This means that there exists a lift $\tilde{x} \in \tau_{\leq k-1} R\psi\Lambda_{X_1\times X_2}$ of x and an element $\tilde{y} \in \tau_{\leq k+j} R\psi\Lambda_{X_1\times X_2}$ with $0 \leq j \leq$

$2n-k$ such that $\tilde{x} = N\tilde{y}$. In order to conclude that $x \in \operatorname{im} \bar{N}$, it suffices to show that we can take $j=0$. In the case $j \geq 1$, let $y \in R^{k+j}\psi\Lambda_{X_1 \times X_2}$ be the image of $\tilde{y}$. We have $\bar{N}y = 0$, and in this case we have seen in the proof of Corollary 4.17 that we can find $\tilde{y}^{(1)} \in \tau_{\leq k+j-1} R\psi\Lambda_{X_1 \times X_2}$ such that $N(\tilde{y} - \tilde{y}^{(1)}) = 0$. In other words, $\tilde{x} = N\tilde{y}^{(1)}$ and we can replace j by $j-1$. After finitely many steps, we can find $\tilde{y}^{(j)} \in \tau_{\leq k} R\psi\Lambda_{X_1 \times X_2}$ such that $\tilde{x} = N\tilde{y}^{(j)}$. Thus, $x \in \operatorname{im} \bar{N}$.

LEMMA 4.18
The filtration of $R\psi\Lambda_{X_1 \times X_2}$ by $\tau_{\leq k} R\psi\Lambda$ induces a filtration on $\ker N / \operatorname{im} N \cap \ker N$ *with the* $(k+1)$*st graded piece $a_{k,k*}\Lambda(-k)[-2k]$ for* $0 \leq k \leq n-1$.

Proof
First, we need to compute the quotient $R^0\psi\Lambda_{X_1 \times X_2}/\operatorname{im} N$, which is the same as $R^0\psi\Lambda_{X_1 \times X_2}/\mathcal{Q}_1 = \mathcal{R}_1$ and $\mathcal{R}_1 \simeq a_{0,0*}\Lambda$ by Lemma 4.16.

Now we must compute for each $k \geq 0$ the quotient of $(2-2n)$-shifted perverse sheaves $\mathcal{P}_k / \operatorname{im} \bar{N}$. This is the same as $\mathcal{P}_k/\mathcal{Q}_{k+1}$, which is also the image of $\mathcal{P}_k$ in $\mathcal{R}_{k+1}$ via

$$\mathcal{P}_k \hookrightarrow R^k\psi\Lambda[-k] \twoheadrightarrow \mathcal{R}_{k+1}.$$

Recall that we have decompositions for both $\mathcal{P}_k$ and $\mathcal{R}_{k+1}$ in terms of simple objects in the category of $(2-2n)$-shifted perverse sheaves,

$$\mathcal{P}_k = [a_{k,k*}\Lambda(-k) \overset{\wedge\delta}{\to} a_{k,k+1*}\Lambda(-k) \oplus a_{k+1,k*}\Lambda(-k) \to \cdots \overset{\wedge\delta}{\to} a_{n-1,n-1*}\Lambda(-k)]$$

and

$$\mathcal{R}_{k+1} = \Big[\bigoplus_{j=0}^{k+1} a_{j,k-j*}\Lambda(-k) \to \cdots \to a_{k,k*}\Lambda(-k)\Big].$$

The only simple factors that show up in both decompositions are those that show up in $a_{k,k*}\Lambda(-k)[-2k]$, so these are the only factors that may have nonzero image in $\mathcal{R}_{k+1}$. Thus, $\mathcal{P}_k / \operatorname{im} \bar{N}$ is a quotient of $a_{k,k*}\Lambda(-k)[-2k]$, and it remains to see that it is the whole thing. As seen in Lemma 4.16, the map $\mathcal{P}_k \to \mathcal{R}_{k+1}$ can be described as a composition of chain maps. The composition in degree $2k$ is the map

$$a_{k,k*}\Lambda(-k) \hookrightarrow a_{k,k*}\Lambda(-k)^{\oplus k+1} \twoheadrightarrow a_{k,k*}\Lambda(-k),$$

where the inclusion sends $x \mapsto (x, -x, \ldots, (-1)^{k+1}x)$ and the surjection is a quotient by $(x,x,0,\ldots,0), (0,x,x,0,\ldots,0), \ldots, (0,\ldots,0,x,x)$ for $x \in a_{k,k*}\Lambda(-k)$. It is elementary to check that the composition of these two maps is an isomorphism, so we are done. $\square$

Analogously, we can compute the kernel and cokernel of

$$\bar{N}^j : R^k\psi\Lambda_{X_1\times X_2}[-k] \to R^{k-j}\psi\Lambda_{X_1}[-k+j]$$

for $2 \le j \le k \le 2n-2$ and use this to recover the graded pieces of a filtration on $\ker N^j/\ker N^{j-1}$ and on $(\ker N^j/\ker N^{j-1})/(\operatorname{im} N \cap \ker N^j)$.

LEMMA 4.19

Let $2 \le j \le 2n-2$. The filtration of $R\psi\Lambda_{X_1\times X_2}$ by $\tau_{\le k}R\psi\Lambda$ induces a filtration on

$$(\ker N^j/\ker N^{j-1})/(\operatorname{im} N \cap \ker N^j).$$

The first graded piece of this filtration is isomorphic to

$$\bigoplus_{i=0}^{j-1} a_{i,j-1-i*}\Lambda(-j+1)[-j+1].$$

For $k \ge 1$, the $(k+1)$st graded piece is isomorphic to

$$(\ker \bar{N}^j/\ker \bar{N}^{j-1})/(\operatorname{im} \bar{N} \cap \ker \bar{N}^j),$$

where

$$\bar{N}^j : R^{k+j-1}\psi\Lambda[-(k+j-1)] \to R^{k-1}\psi\Lambda[-k+1].$$

More explicitly, the $(k+1)$st graded piece is isomorphic to

$$\bigoplus_{i=1}^{j} a_{k+i-1,k+j-i*}\Lambda\big(-(k+j-1)\big)[-2k-j+1].$$

Proof

We prove the lemma by induction on j. The base case $j=1$ is proven in Corollary 4.17 and Lemma 4.18. Assume that it is true for $j-1$.

To prove the first claim, note that the first graded piece of

$$(\ker N^j/\operatorname{im} N \cap \ker N^j)/(\ker N^{j-1}/\operatorname{im} N \cap \ker N^{j-1})$$

has to be a quotient of

$$R^{j-1}\psi\Lambda_{X_1\times X_2}[-j+1]/\mathcal{Q}_j \simeq \mathcal{R}_j.$$

This is true because $\tau_{\le j-1}R\psi\Lambda_{X_1\times X_2} \subseteq \ker N^j$ and $\tau_{\le j-2}R\psi\Lambda_{X_1\times X_2} \subseteq \ker N^{j-1}$, and

$$R^{j-1}\psi\Lambda_{X_1\times X_2}[-j+1]=\tau_{\leq j-1}R\psi\Lambda_{X_1\times X_2}/\tau_{\leq j-2}R\psi\Lambda_{X_1\times X_2}.$$

More precisely, the first graded piece has to be a quotient of

$$\mathcal{R}_j/(\ker N^{j-2}/\operatorname{im} N\cap\ker N^{j-2})$$

by the second graded piece of

$$(\ker N^{j-1}/\ker N^{j-2})/(\operatorname{im} N\cap\ker N^{j-1}).$$

(Here, we abusively write

$$\ker N^{j-2}/\operatorname{im} N\cap\ker N^{j-2},$$

where we mean the image of this object in R_j.) By the induction hypothesis, this second graded piece is

$$\bigoplus_{i=1}^{j-1}a_{i,j-i*}\Lambda(-j+1)[-j].$$

Continuing this argument, we see that in order to get the first graded piece of $(\ker N^j/\ker N^{j-1})/(\operatorname{im} N\cap\ker N^j)$, we must quotient $\mathcal{R}_j$ successively by

$$\bigoplus_{i=1}^{j-k}a_{k+i-1,j-i*}\Lambda(-j+1)[-k-j+1],$$

with k going from $j-1$ down to 1. (This corresponds to quotienting out successively by the jth graded piece of $\ker N/(\operatorname{im} N\cap\ker N)$, the $(j-1)$st graded piece of $(\ker N^2/\ker N)/(\operatorname{im} N\cap\ker N^2)$, down to the second graded piece of $(\ker N^{j-1}/\ker N^{j-2})/(\operatorname{im} N\cap\ker N^{j-1})$.) We know that

$$\mathcal{R}_j=\Big[\bigoplus_{i=0}^{j-1}a_{i,j-1-i*}\Lambda\big(-(j-1)\big)\to\cdots\to a_{j-1,j-1*}\Lambda\big(-(j-1)\big)\Big],$$

with the general term in degree $k+j-1$ equal to

$$\bigoplus_{i=1}^{j-k}a_{k+i-1,j-i*}\Lambda\big(-(j-1)\big).$$

After quotienting out successively, we are left with only the degree $j-1$ term, which is

$$\bigoplus_{i=0}^{j-1}a_{i,j+1-i*}\Lambda\big(-(j-1)\big)[-(j-1)],$$

as desired.

In order to identify the $(k+1)$st graded piece of

$$(\ker N^j / \ker N^{j-1})/(\operatorname{im} N \cap \ker N^j)$$

for $k \geq 1$, we first identify the kernel of $\bar{N}^j : R^{k+j-1}\psi\Lambda_{X_1\times X_2} \to R^{k-1}\psi\Lambda_{X_1\times X_2}$ as a map of perverse sheaves, as in Lemma 4.16. Then we can identify it with the $(k+1)$st graded piece of $\ker N^j$ as in Lemma 4.18 and quotient by $\mathcal{Q}_{k+j}$. Finally, we can use induction as above to compute the $(k+1)$st graded piece of $(\ker N^j / \ker N^{j-1})/(\operatorname{im} N \cap \ker N^j)$. $\square$

COROLLARY 4.20
The preceding filtration is a direct sum.

Proof
This follows from the decomposition theorem for pure perverse sheaves (see [BBD, Théorème 5.3.8]), once we notice that the $(k+1)$st graded piece of the filtration is a pure $-(2n-2)$-shifted perverse sheaf of weight $(-2k-j+1)+2(k+j-1) = j-1$, which is independent of k. $\square$

Let

$$\mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda = (\ker N^p \cap \operatorname{im} N^q)/(\ker N^{p-1} \cap \operatorname{im} N^q) + (\ker N^p \cap \operatorname{im} N^{q+1}).$$

The monodromy filtration $M_r R\psi\Lambda$ has graded pieces $\mathrm{Gr}_r^M R\psi\Lambda$ isomorphic to

$$\mathrm{Gr}_r^M R\psi\Lambda \simeq \bigoplus_{p-q=r} \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda$$

by [Sa, Lemma 2.1], so to understand the graded pieces of the monodromy filtration, it suffices to understand the $\mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda$. Lemma 4.19 exhibits a decomposition of $\mathrm{Gr}^0 \mathrm{Gr}_p R\psi\Lambda$ as a direct sum with the $(k+1)$st term isomorphic to

$$\bigoplus_{i=1}^{p} a_{k+i-1,k+p-i*}\Lambda\big(-(k+p-1)\big)[-2k-p+1].$$

The action of N^q induces an isomorphism of $\mathrm{Gr}^0\mathrm{Gr}_{p+q} R\psi\Lambda$ with $\mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda(q)$, so there is a direct sum decomposition of the latter with the $(k+1)$st term isomorphic to

$$\bigoplus_{i=1}^{p+q} a_{k+i-1,k+p+q-i*}\Lambda\big(-(k+p-1)\big)[-2k-p-q+1].$$

We can use the spectral sequence associated to a filtration (as in [S, Lemme 5.2.18]) to compute the terms in the monodromy spectral sequence

$$E_1^{r,m-r} = H^m(Y_{\bar{\mathbb{F}}}, \mathrm{Gr}^M_{-r} R\psi\Lambda) = \bigoplus_{p-q=-r} H^m(Y_{\bar{\mathbb{F}}}, \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda)$$
$$\Rightarrow H^m(Y_{\bar{\mathbb{F}}}, R\psi\Lambda) = H^m(X_{\bar{K}}, \Lambda).$$

COROLLARY 4.21
There is a direct sum decomposition

$$H^m(Y_{\bar{\mathbb{F}}}, \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda) \simeq \bigoplus_{k\geq 0} \bigoplus_{i=1}^{p+q} H^m\big(Y_{\bar{\mathbb{F}}}, a_{k+i-1,k+p+q-i*}\Lambda(-(k+p-1))[-2k-p-q+1]\big)$$

compatible with the action of $G_{\mathbb{F}}$. This can be rewritten as

$$H^m(Y_{\bar{\mathbb{F}}}, \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda) \simeq \bigoplus_{k\geq 0} \bigoplus_{i=1}^{p+q} H^{m-2k-p-q+1}\big(Y_{\bar{\mathbb{F}}}^{(k+i-1,k+p+q-i)}, \Lambda(-(k+p-1))\big).$$

4.3. More general schemes
In this section, we explain how the results of the previous section concerning products of strictly semistable schemes apply to more general schemes, in particular to the Shimura varieties $X_U/\mathcal{O}_K$. In this section, we will use $\Lambda = \mathbb{Q}_l$ or $\bar{\mathbb{Q}}_l$.

Let $X'/\mathcal{O}_K$ be a scheme such that the completions of the strict henselizations $\mathcal{O}^\wedge_{X',s}$ at closed geometric points s are isomorphic to

$$W[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]]/(X_1, \ldots, X_r - \pi, Y_1, \ldots, Y_s - \pi)$$

for some indices $i_1, \ldots, i_r, j_1, \ldots, j_s \in \{1, \ldots, n\}$ and some $1 \leq r, s \leq n$. Also assume that the special fiber Y' is a union of closed subschemes $Y'_{1,j}$ with $j \in \{1, \ldots, n\}$ which are cut out by one local equation such that if s is a closed geometric point of $Y'_{1,j}$, then $j \in \{i_1, \ldots, i_r\}$ and $Y'_{1,j}$ is cut out in $\mathcal{O}^\wedge_{X',s}$ by the equation $X_j = 0$. Similarly, assume that Y' is a union of closed subschemes $Y'_{2,j}$ with $j \in \{1, \ldots, n\}$ which are cut out by one local equation such that if s is a closed geometric point of $Y'_{2,j}$, then $j \in \{j_1, \ldots, j_r\}$ and $Y'_{2,j}$ is cut out in $\mathcal{O}^\wedge_{X',s}$ by the equation $Y_j = 0$.

Let X/X' be smooth of dimension m, let Y be the special fiber of X, and let $Y_{i,j} = Y'_{i,j} \times_{X'} X$ for $i = 1, 2$ and $j = 1, \ldots, n$. As in Lemma 2.9, X' is locally étale over

$$X_{r,s} = \operatorname{Spec} \mathcal{O}_K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/\Big(\prod_{i=1}^{r} X_i - \pi, \prod_{j-1}^{s} Y_j - \pi\Big),$$

so X is locally étale over

$$X_{r,s,m} = \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n, Y_1, \dots, Y_n, Z_1, \dots, Z_m] / \Big(\prod_{i=1}^{r} X_i - \pi, \prod_{j-1}^{s} Y_j - \pi \Big),$$

which is a product of strictly semistable schemes. The results of Section 3 apply to X', and it is easy to check that they also apply to X. In particular, we know that the inertia I_K acts trivially on the sheaves of nearby cycles $R^k \psi \Lambda$ of X, and we have a description of the $R^k \psi \Lambda$ in terms of the log structure we put on $X / \operatorname{Spec} \mathcal{O}_K$. Let $a^i_j : Y_{i,j} \to Y$ denote the closed immersion for $i = 1, 2$ and $j \in \{1, \dots, n\}$. Then by Corollary 3.6, we have an isomorphism

$$R^k \psi \Lambda(k) \simeq \wedge^k \Big(\Big(\bigoplus_{j=1}^{n} a^1_{j*} \Lambda \Big) / \Lambda \oplus \Big(\bigoplus_{j=1}^{n} a^2_{j*} \Lambda \Big) / \Lambda \Big).$$

For $i = 1, 2$ and $J_i \subseteq \{1, \dots, n\}$, let

$$Y_{J_1, J_2} = \Big(\bigcap_{j_1 \in J_1} Y_{1, j_1} \Big) \cap \Big(\bigcap_{j_2 \in J_2} Y_{2, j_2} \Big),$$

and let $a_{J_1, J_2} : Y_{J_1, J_2} \to Y$ be the closed immersion. Set

$$Y^{(h_1, h_2)} = \bigsqcup_{\# J_1 = h_1 + 1, \# J_2 = h_2 + 1} Y_{J_1, J_2},$$

and let $a_{h_1, h_2} : Y^{(h_1, h_2)} \to Y$ be the projection. The scheme $Y^{(h_1, h_2)}$ is smooth of dimension $\dim Y - h_1 - h_2$ (we can see this from the strict local rings).

We can write

$$R^k \psi \Lambda \simeq \bigoplus_{h=0}^{k} \wedge^h \Big(\Big(\bigoplus_{j=1}^{n} a^1_{j*} \Lambda \Big) / \Lambda \Big) \otimes \wedge^{k-h} \Big(\Big(\bigoplus_{j=1}^{n} a^2_{j*} \Lambda \Big) / \Lambda \Big) (-k)$$

and then define the map of sheaves on Y

$$\theta_k : R^k \psi \Lambda \to \sum_{h=0}^{k} a_{h, k-h *} \Lambda(-k)$$

as a sum of maps for h going from 0 to k. First, define, for $i = 1, 2$,

$$\delta_{h_i} : \bigwedge^{h_i} \Big(\Big(\bigoplus_{j=1}^{n} a^i_{j*} \Lambda \Big) / \Lambda \Big) \to \bigwedge^{h_i + 1} \Big(\bigoplus_{j=1}^{n} a^i_{j*} \Lambda \Big)$$

by sending

$$a^i_{j_1*}\Lambda \wedge \cdots \wedge a^i_{j_{h_i}*}\Lambda \to \bigoplus_{j \neq j_1,\ldots,j_{h_i}} a^i_{j_1*}\Lambda \wedge \cdots \wedge a^i_{j_{h_i}*}\Lambda \wedge a^i_{j*}\Lambda$$

via the cup product with the canonical map

$$\Lambda_Y \to \bigoplus_{j=1}^{n} a^i_{j*}\Lambda.$$

More explicitly, on an open U of Y, the map sends

$$\alpha \in \Lambda(U \times_Y Y^i_{j_1} \times \cdots \times_Y Y^i_{j_{l_i}}),$$

to

$$(\alpha|_{Y^i_j}, \ldots, \alpha|_{Y^i_j}) \in \bigoplus_{j \neq j_1,\ldots,j_{h_i}} \Lambda(U \times_Y Y^i_j \times Y^i_{j_1} \cdots \times_Y Y^i_{j_{h_i}}),$$

and it is easy to check that this is well defined. Then notice that

$$\wedge^{h+1}\Big(\bigoplus_{j=1}^{n} a^1_{j*}\Lambda\Big) \otimes \wedge^{k+1-h}\Big(\bigoplus a^2_{j*}\Lambda\Big) \simeq a_{h,k-h*}\Lambda.$$

Indeed, for $J_1, J_2 \subseteq \{1, \ldots, n\}$ with $\#J_1 = h+1, \#J_2 = k+1-h$, we have

$$\Big(\bigwedge_{j_1 \in J_1} a^1_{j_1*}\Lambda\Big) \otimes \Big(\bigwedge_{j_2 \in J_2} a^2_{j_2*}\Lambda\Big) \simeq a_{J_1,J_2*}\Lambda$$

because $Y_{J_1,J_2} = \big(\bigcap_{j_1 \in J_1} Y_{1,j_1}\big) \times_Y \big(\bigcap_{j_2 \in J_2} Y_{2,j_2}\big)$, and we can sum the above identity over all J_1, J_2 of the prescribed cardinality.

LEMMA 4.22

The following sequence is exact:

$$R^k\psi\Lambda \xrightarrow{\theta_k} \bigoplus_{h=0}^{k} a_{h,k-h*}\Lambda(-k)^{\oplus c^k_{h,k-h}} \to \bigoplus_{h=0}^{k+1} a_{h,k+1-h*}\Lambda(-k)^{\oplus c^k_{h,k+1-h}} \to \cdots$$
$$\to \bigoplus_{h=0}^{2n-2} a_{h,2n-2-h*}\Lambda(-k)^{\oplus c^k_{h,2n-2-h}} \to 0,$$

where the first map is the one defined above and the coefficients $c^k_{h_1,h_2}$ are defined in Lemma 4.14. The remaining maps in the sequence are global maps of sheaves corresponding to $\wedge\delta_1 \pm \wedge\delta_2$, where $\delta_i \in \bigoplus_{j=1}^n a^i_{j*}\Lambda$ is equal to $(1, \ldots, 1)$ for $i =$

$1, 2$. *These maps are defined on each of the $c^k_{h_1,h_2}$ factors in the unique way which makes them compatible with the maps in the resolution (3).*

We can think of θ_k as a quasi-isomorphism of $R^k\psi\Lambda[-k]$ with the complex

$$\bigoplus_{h=0}^{k} a_{h,k-h*}\Lambda(-k)^{\oplus c^k_{h,k-h}} \to \cdots \to \bigoplus_{h=0}^{2n-2} a_{h,2n-2-h*}\Lambda(-k)^{\oplus c^k_{h,2n-2-h}},$$

where the leftmost term is put in degree k.

Proof

It suffices to check exactness locally, and we know that X is locally étale over products $X_1 \times_{\mathcal{O}_K} X_2$ of strictly semistable schemes. Lemma 4.9 proves the above statement in the case of $X_1 \times_{\mathcal{O}_K} X_2$, and the corresponding sheaves on Y are obtained by restriction (étale pullback) from the special fiber $Y_1 \times_{\mathbb{F}} Y_2$ of $X_1 \times_{\mathcal{O}_K} X_2$. $\square$

COROLLARY 4.23

The complex $R\psi\Lambda$ is a $-\dim Y$-shifted perverse sheaf, and the canonical filtration $\tau_{\le k} R\psi\Lambda$ with graded pieces $R^k\psi\Lambda[-k]$ is a filtration by $-\dim Y$-shifted perverse sheaves. The monodromy operator N sends $\tau_{\le k} R\psi\Lambda$ to $\tau_{\le k-1} R\psi\Lambda$, and this induces a map

$$\bar{N} : R^k\psi\Lambda[-k] \to R^{k-1}\psi\Lambda[-k+1].$$

The next step is to understand the action of monodromy $\bar{N}$ and to obtain an explicit description of $\bar{N}$ in terms of the resolution of $R^k\psi\Lambda$ given by Lemma 4.22. This can be done étale locally, since on the nearby cycles for $X_1 \times_{\mathcal{O}_K} X_2$ we know that $\bar{N}$ acts as $\bar{N}_1 \otimes 1 + 1 \otimes \bar{N}_2$ from Proposition 4.13, and we have a good description of $\bar{N}_1$ and $\bar{N}_2$ from Lemma 4.6. However, we present here a different method for computing $\bar{N}$, which works in greater generality.

PROPOSITION 4.24

The following diagram is commutative:

$$\begin{tikzcd}
R^{k+1}\psi\Lambda[-k-1] \arrow[r, "\sim"] \arrow[d, "\bar{N}"] & {[0} \arrow[r] \arrow[d] & R^{k+1}\psi\Lambda] \arrow[d, "\otimes t_l(T)"] \\
R^k\psi\Lambda[-k] \arrow[r, "\sim"] & {[i^*R^{k+1}j_*\Lambda(1)} \arrow[r] & R^{k+1}\psi\Lambda(1)]
\end{tikzcd}$$

where in the right column the sheaves $R^{k+1}\psi\Lambda$ are put in degree $k+1$.

The proof of this proposition is identical to the proof of [Sa, Lemma 2.5(4)], which is meant for the strictly semistable case but does not use semistability. The fact that the above formula could hold was suggested to us by reading Ogus's paper (see [O]), which proves an analogous formula for log smooth schemes in the complex analytic world. The same result should hold for any log smooth scheme $X/\mathcal{O}_K$ with vertical log structure and where the action of I_K on $R^k\psi\Lambda$ is trivial for all k.

For $0 \leq k \leq 2n-2$, define the complex

$$\mathcal{L}_k := \Big[\bigoplus_{h=0}^{k}\big(a_{h,k-h*}\Lambda(-k)\big)^{\oplus c^k_{h,k-h}} \to \cdots \\ \to \bigoplus_{h=0}^{2n-2}\big(a_{h,2n-2-h*}\Lambda(-k)\big)^{\oplus c^k_{h,2n-2-h}}\Big],$$

where the sheaves $a_{h,k-h*}\Lambda(-k)$ are put in degree k. We will define a map of complexes $f : \mathcal{L}_{k+1} \to \mathcal{L}_k$ degree by degree, as a sum over $h_1 + h_2 = k'$ of maps

$$f^{h_1,h_2} \otimes t_l(T) : a_{h_1,h_2*}\Lambda^{\oplus c^k_{h_1,h_2}} \to a_{h_1,h_2*}\Lambda(1)^{\oplus c^k_{h_1,h_2}}.$$

Note that each coefficient $c^k_{h_1,h_2}$ records the number of values of $0 \leq h' \leq k$ for which the term $a_{h_1,h_2*}\Lambda(-k)$ appears in the resolution of

$$\bigwedge^{h'}\Big(\Big(\bigoplus_{j=1}^{n} a^1_{j*}\Lambda\Big)/\Lambda\Big) \otimes \bigwedge^{k-h'}\Big(\Big(\bigoplus_{j=1}^{n} a^2_{j*}\Lambda\Big)/\Lambda\Big)(-k).$$

The set of such h' has cardinality $c^k_{h_1,h_2}$ and is always a subset of consecutive integers in $\{1,\dots,k\}$. Denote the set of h' by $C^k_{h_1,h_2}$. Thus, we can order the terms $a_{h_1,h_2*}\Lambda$ by h' and get a basis for $(a_{h_1,h_2*}\Lambda)^{\oplus c^k_{h_1,h_2}}$ over $a_{h_1,h_2*}\Lambda$. It is easy to explain what f^{h_1,h_2} does to each element of $C^{k+1}_{h_1,h_2}$: it sends

$$h' \in C^{k+1}_{h_1,h_2} \mapsto \{h'-1,h'\} \cap C^k_{h_1,h_2}.$$

When both $h'-1, h' \in C^k_{h_1,h_2}$, the element of the basis of $(a_{h_1,h_2*}\Lambda)^{\oplus c^{k+1}_{h_1,h_2}}$ given by $(0,\dots,0,1,0,\dots,0)$, where the 1 appears in the position corresponding to h', is sent to the element of the basis of $(a_{h_1,h_2*}\Lambda)^{\oplus c^k_{h_1,h_2}}$ given by $(0,\dots,0,1,1,0,\dots,0)$, where the two "1's" are in the positions corresponding to $h'-1$ and h'. If $h'-1 \notin C^k_{h_1,h_2}$ but $h' \in C^k_{h_1,h_2}$, then $h' = 0$ and $(1,0,\dots,0) \mapsto (1,0,\dots,0)$. If $h'-1 \in C^k_{h_1,h_2}$ but $h' \notin C^k_{h_1,h_2}$, then $h' = k+1$ and $(0,\dots,0,1) \mapsto (0,\dots,0,1)$. This completes the definition of f^{h_1,h_2}.

COROLLARY 4.25
The following diagram is commutative:

$$\begin{tikzcd} R^{k+1}\psi\Lambda[-k-1] \arrow[r, "\sim"] \arrow[d, "\bar{N}"] & \mathcal{L}_{k+1} \arrow[d, "f"] \\ R^k\psi\Lambda[-k] \arrow[r, "\sim"] & \mathcal{L}_k \end{tikzcd}$$

The map f is a map of complexes which acts degree by degree as

$$\sum_{h_1+h_2=k'} f^{h_1,h_2}[-k-1] \otimes t_l(T),$$

where $f^{h_1,h_2} : a_{h_1,h_2*}\Lambda^{\oplus c^k_{h_1,h_2}} \to a_{h_1,h_2*}\Lambda^{\oplus c^k_{h_1,h_2}}$ was defined above.

Proof
This can be checked étale locally using Proposition 4.13, which states that $\bar{N} = \bar{N}_1 \otimes 1 + 1 \otimes \bar{N}_2$ over a product $X_1 \times_{\mathcal{O}_K} X_2$ of strictly semistable schemes, and using the fact that each of the $\bar{N}_i$ can be described as

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & a^i_{k+1*}\Lambda\big(-(k+1)\big) \arrow[r, "\delta\wedge"] \arrow[d, "\otimes t_l(T)"] & \cdots \arrow[r, "\delta\wedge"] & a^i_{n-1*}\Lambda\big(-(k+1)\big) \arrow[d] \\ a^i_{k*}\Lambda(-k) \arrow[r, "\delta\wedge"] & a^i_{k+1*}\Lambda(-k) \arrow[r, "\delta\wedge"] & \cdots \arrow[r, "\delta\wedge"] & a^i_{n-1*}\Lambda(-k) \end{tikzcd}$$

for $i = 1, 2$.

This can also be checked globally by using Proposition 4.24 to replace the left-most column of our diagram by

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & R^{k+1}\psi\Lambda \arrow[d, "\otimes t_l(T)"] \\ i^*R^{k+1}j_*\Lambda(1) \arrow[r] & R^{k+1}\psi\Lambda(1) \end{tikzcd}$$

where the left column is put in degree k. In fact, it suffices to understand the map of complexes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & R^{k+1}\psi\Lambda \arrow[d, "\mathrm{id}"] \\
i^*R^{k+1}j_*\Lambda \arrow[r] & R^{k+1}\psi\Lambda
\end{tikzcd}$$

and check that it is compatible with the map

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "f^k\otimes t_l(T)^{-1}"] & \cdots \arrow[r] & \bigoplus_{h=0}^{2n-2}\left(a_{h,2n-2-h*}\Lambda(-k-1)\right)^{\oplus c^{k+1}_{h,2n-2-h}} \arrow[d, "f^{2n-2}\otimes t_l(T)^{-1}"] \\
\bigoplus_{h=0}^{k}\left(a_{h,k-h*}\Lambda(-k-1)\right)^{\oplus c^{k}_{h,k-h}} \arrow[r] & \cdots \arrow[r] & \bigoplus_{h=0}^{2n-2}\left(a_{h,2n-2-h*}\Lambda(-k-1)\right)^{\oplus c^{k}_{h,2n-2-h}}
\end{tikzcd}$$

Let $\mathcal{K} = \mathrm{Cone}(f \otimes t_l(T)^{-1} : \mathcal{L}_{k+1} \to \mathcal{L}_k(-1))$. The triangle

$$R^k\psi\Lambda[-k-1] \longrightarrow i^*R^{k+1}j_*\Lambda[-k-1] \longrightarrow R^{k+1}\psi\Lambda[-k-1] \xrightarrow{\bar{N}\otimes t_l(T)^{-1}} R^k\psi\Lambda[-k]$$

is distinguished. It suffices to see that we can define a map $g : i^*R^{k+1}j_*\Lambda[-k] \to \mathcal{K}$ which makes the first two squares of the following diagram commute:

$$\begin{tikzcd}
R^k\psi\Lambda[-k-1] \arrow[r] \arrow[d, "\theta_k[-1]"] & i^*R^{k+1}j_*\Lambda[-k-1] \arrow[r] \arrow[d, "g[-1]"] & R^{k+1}\psi\Lambda[-k-1] \arrow[r, "\bar{N}\otimes t_l(T)^{-1}"] \arrow[d, "\theta_{k+1}"] & R^k\psi\Lambda[-k] \arrow[d, "\theta_k"] \\
\mathcal{L}_k(-1)[-1] \arrow[r] & \mathcal{K}[-1] \arrow[r] & \mathcal{L}_{k+1} \arrow[r, "f\otimes t_l(T)^{-1}"] & \mathcal{L}_k(-1)
\end{tikzcd}$$

If the middle square is commutative, then there must exist $\theta' : R^k\psi\Lambda[-k-1] \to \mathcal{L}_k(-1)[-1]$ making the diagram a morphism of distinguished triangles. Then θ' would make the first square commutative, so θ' and $\theta_k[-1]$ coincide once they are pushed forward to $\mathcal{K}[-1]$. However,

$$\mathrm{Hom}(R^k\psi\Lambda[-k-1], \mathcal{L}_{k+1}[-1]) \simeq \mathrm{Hom}(R^k\psi\Lambda[-k], R^{k+1}\psi\Lambda[-k-1]) = 0,$$

so the Hom exact sequence associated to the bottom distinguished triangle implies that $\theta' = \theta_k[-1]$. The diagram above is a morphism of distinguished triangles with $\theta_k[-1]$ as the leftmost morphism. This tells us that the third triangle in the diagram is also commutative, which is what we wanted to prove.

We can compute $i^*R^{k+1}j_*\Lambda$ using the log structure on X

$$i^*R^{k+1}j_*\Lambda(k+1) \simeq \wedge^{k+1}\Big(\Big(\bigoplus_{j=1}^{n} a^1_{j*}\Lambda \oplus \bigoplus_{j=1}^{n} a^2_{j*}\Lambda\Big)/(1,\dots,1,0,\dots,0)-(0,\dots,0,1,\dots,1)\Big).$$

Here we have again used the formula $i^*R^k j_*\Lambda(k) \simeq \wedge^k(\bar{M}^{\mathrm{gp}}) \otimes \Lambda$, which follows from [Na, Proposition 2.0.2]. We can also compute $\mathcal{K}$ explicitly, since we have an explicit description of each f^{k',h_1,h_2}. The first nonzero term of $\mathcal{K}$ appears in degree k and it is isomorphic to

$$\sum_{h=0}^{k} a_{h,k-h*}\Lambda.$$

There is a natural map of complexes $i^*R^{k+1}j_*\Lambda[-k] \to \mathcal{K}$, which sends

$$a_{J_1,J_2*}\Lambda \to \bigoplus_{J_1' \supset J_1, \#J_1' = \#J_1+1} a_{J_1',J_2*}\Lambda \oplus \bigoplus_{J_2' \supset J_2, \#J_2' = \#J_2+1} a_{J_1,J_2'*}\Lambda,$$

when J_1, J_2 are both nonempty. The map sends

$$a_{J_1,\emptyset*}\Lambda \to \bigoplus_{\#J_2'=1} a_{J_1,J_2'*}\Lambda \qquad \text{and} \qquad a_{\emptyset,J_2*}\Lambda \to \bigoplus_{\#J_1'=1} a_{J_1',J_2*}\Lambda.$$

It is easy to see that the above map is well defined on $i^*R^{k+1}j_*\Lambda[-k]$ and that it is indeed a map of complexes. It remains to see that the above map of complexes $i^*R^{k+1}j_*\Lambda[-k] \to \mathcal{K}$ makes the first two squares of the diagram commute. This is tedious but straightforward to verify. $\square$

Remark 4.26
Another way of proving Corollary 4.25 is to notice that Proposition 4.24 shows that the map

$$\bar{N} : R^{k+1}\psi\Lambda[-k-1] \to R^k\psi\Lambda[-k]$$

is given by the cup product with the map $\gamma \otimes t_l(T) : \bar{M}^{\mathrm{gp}}_{\mathrm{rel}}(-k-1) \to \Lambda(-k)[1]$, where $\gamma : \bar{M}^{\mathrm{gp}}_{\mathrm{rel}} \to \Lambda[1]$ is the map corresponding to the class of the extension

$$0 \to \Lambda \to \bar{M}^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{\mathrm{rel}} \to 0$$

of sheaves of Λ-modules on Y. Locally, X is étale over a product of strictly semistable schemes $X_1 \times_{\mathcal{O}_K} X_2$, and the extension $\bar{M}^{\mathrm{gp}}$ is a Baire sum of extensions

$$0 \to \Lambda \to \bar{M}_1^{\mathrm{gp}} \to \bar{M}_{1,\mathrm{rel}}^{\mathrm{gp}} \to 0$$

and

$$0 \to \Lambda \to \bar{M}_2^{\mathrm{gp}} \to \bar{M}_{2,\mathrm{rel}}^{\mathrm{gp}} \to 0,$$

which correspond to the log structures of X_1 and X_2 and which by Proposition 4.24 determine the maps $\bar{N}_1$ and $\bar{N}_2$. The Baire sum of extensions translates into $\bar{N} = \bar{N}_1 \otimes 1 + 1 \otimes \bar{N}_2$ locally on Y. However, it is straightforward to check locally on Y that the map $f : \mathcal{L}_k \to \mathcal{L}_{k+1}$ is the same as $\bar{N}_1 \otimes 1 + 1 \otimes \bar{N}_2$. Thus, f and $\bar{N}$ are maps of perverse sheaves on Y which agree locally on Y, which means that f and $\bar{N}$ agree globally. This proves the corollary without appealing to Proposition 4.24. (In fact, it suggests an alternate proof of Proposition 4.24.)

The following results, Lemma 4.27 to Corollary 4.31, are just generalizations of Lemma 4.16 to Corollary 4.21. We merely sketch their proofs here.

LEMMA 4.27
The map $\bar{N} : R^k\psi\Lambda[-k] \to R^{k-1}\psi\Lambda[-k+1]$ has kernel

$$\mathcal{P}_k \simeq [a_{k,k*}\Lambda(-k) \xrightarrow{\wedge\delta} a_{k,k+1*}\Lambda(-k) \oplus a_{k+1,k*}\Lambda(-k) \to \cdots \xrightarrow{\wedge\delta} a_{n-1,n-1*}\Lambda(-k)],$$

where the first term is put in degree $2k$, and it has cokernel

$$\mathcal{R}_k \simeq \Big[\bigoplus_{j=0}^{k-1} a_{j,k-1-j*}\Lambda\big(-(k-1)\big) \to \cdots \to a_{k-1,k-1*}\Lambda\big(-(k-1)\big)\Big],$$

where the first term is put in degree $k-1$.

Proof
The proof is identical to the proof of Lemma 4.16, since by Proposition 4.24 we have a description of $\bar{N}$ as a degree by degree map

$$f : \mathcal{L}_k \to \mathcal{L}_{k-1}. \qquad \square$$

COROLLARY 4.28
The filtration of $R\psi\Lambda$ by $\tau_{\leq k}R\psi\Lambda$ induces a filtration on $\ker N$. The first graded piece of this filtration $\mathrm{Gr}^1 \ker N$ is $R^0\psi\Lambda$. The graded piece $\mathrm{Gr}^{k+1}\ker N$ of this filtration is $\mathcal{P}_k$.

Proof
This can be proved the same way as Corollary 4.17. The only tricky part is seeing that

we can identify a graded piece of $\ker \bar{N}$ with a graded piece of $\ker N$. In other words, we want to show that, for $\bar{N} : R^k\psi\Lambda[-k] \to R^{k-1}\psi\Lambda[-k+1]$ and $x \in \ker \bar{N}$, we can find a lift $\tilde{x} \in \tau_{\leq k} R\psi\Lambda$ of x such that $\tilde{x} \in \ker N$. As in the proof of Corollary 4.17, we can define a map $\mathcal{P}_k \to \mathcal{R}_{k-1}$ sending x to the image of $N\tilde{x}$ in $\mathcal{R}_{k-1}$, which turns out to be independent of the lift $\tilde{x}$. We want to see that this map vanishes, but in fact any map $\mathcal{P}_k \to \mathcal{R}_{k-1}$ vanishes. Note that

$$a_{h_1,h_2*}\Lambda[-h_1-h_2] \simeq \bigoplus_{\#S=h_1+1,\#T=h_2+1} a_{S,T*}\Lambda[-h_1-h_2].$$

The scheme $Y_{S,T}$ is smooth of pure dimension $\dim Y - h_1 - h_2$ and so it is a disjoint union of its irreducible (connected) components which are smooth of pure dimension $\dim Y - h_1 - h_2$. Thus, each $a_{S,T*}\Lambda[-h_1-h_2]$ is the direct sum of the pushforwards of the $-\dim Y$-shifted perverse sheaves $\Lambda[-h_1-h_2]$ on the irreducible components of $Y_{S,T}$. Thus, we have a decomposition of $a_{h_1,h_2*}\Lambda[-h_1-h_2]$ in terms of simple objects in the category of $-\dim Y$-shifted perverse sheaves. It is easy to check that $\mathcal{P}_k$ and $\mathcal{R}_{k-j}$ for $k \geq j \geq 1$ have no simple factors in common, so any map $\mathcal{P}_k \to \mathcal{R}_{k-j}$ must vanish. ☐

Remark 4.29

The same techniques used in Section 4.2 apply in order to completely determine the graded pieces of $(\ker N^j/\ker N^{j-1})/(\operatorname{im} N \cap \ker N^j)$ induced by the filtration of $R\psi\Lambda$ by $\tau_{\leq k} R\psi\Lambda$. The only tricky part is seeing that we can also identify the kth graded piece of $\operatorname{im} N$ with

$$\operatorname{im}(\bar{N} : R^{k+1}\psi\Lambda[-k-1] \to R^k\psi\Lambda[-k]),$$

but this can be proved in the same way as the corresponding statement about the kernels of N and $\bar{N}$. We get a complete description of the graded pieces of $(\ker N^j/\ker N^{j-1})/\operatorname{im} N$.

LEMMA 4.30

For $1 \leq j \leq 2n-2$, the filtration of $R\psi\Lambda$ by $\tau_{\leq k} R\psi\Lambda$ induces a filtration on $(\ker N^j/\ker N^{j-1})/\operatorname{im} N$. For $0 \leq k \leq n-1-(j-1)/2$, the $(k+1)$st graded piece of this filtration is isomorphic to

$$\bigoplus_{i=1}^{j} a_{k+i-1,k+j-i*}\Lambda\big(-(k+j-1)\big)[-2k-j+1].$$

As in Corollary 4.20, since each graded piece of the filtration is pure of weight $j-1$, the filtration is in fact a direct sum.

Let

$$\mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda = (\ker N^p \cap \operatorname{im} N^q)/(\ker N^{p-1} \cap \operatorname{im} N^q) + (\ker N^p \cap \operatorname{im} N^{q+1}).$$

The monodromy filtration $M_r R\psi\Lambda$ has graded pieces $\mathrm{Gr}_r^M R\psi\Lambda$ isomorphic to

$$\mathrm{Gr}_r^M R\psi\Lambda \simeq \bigoplus_{p-q=r} \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda,$$

and if we understand the cohomology of $Y_{\bar{\mathbb{F}}}$ with coefficients in each $\mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda$, then we can compute the cohomology of $Y_{\bar{\mathbb{F}}}$ with respect to $R\psi\Lambda$. The next result tells us how to compute $H^m(Y_{\bar{\mathbb{F}}}, \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda)$.

COROLLARY 4.31
There is a direct sum decomposition

$$\begin{aligned} &H^m(Y_{\bar{\mathbb{F}}}, \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda) \\ &\quad \simeq \bigoplus_{k\geq 0} \bigoplus_{i=1}^{p+q} H^m\big(Y_{\bar{\mathbb{F}}}, a_{k+i-1,k+p+q-i*}\Lambda(-(k+p-1))[-2k-p-q+1]\big) \end{aligned}$$

compatible with the action of $G_{\bar{\mathbb{F}}}$. This can be rewritten as

$$\begin{aligned} &H^m(Y_{\bar{\mathbb{F}}}, \mathrm{Gr}^q \mathrm{Gr}_p R\psi\Lambda) \\ &\quad \simeq \bigoplus_{k\geq 0} \bigoplus_{i=1}^{p+q} H^{m-2k-p-q+1}\big(Y_{\bar{\mathbb{F}}}^{(k+i-1,k+p+q-i)}, \Lambda(-(k+p-1))\big). \end{aligned}$$

Remark 4.32
The isomorphism above is functorial with respect to étale morphisms which preserve the stratification by $Y_{S,T}$ with $S, T \subset \{1, \dots, n\}$. The reason for this is that étale morphisms preserve both the kernel and the image filtration of N as well as the canonical filtration $\tau_{\leq k} R\psi\Lambda$.

5. The cohomology of closed strata

In this section, we go back to working with the Iwahori level Shimura variety $X_U/\mathcal{O}_K$ as well as with the Shimura variety $X_{U_0}/\mathcal{O}_K$ with no level structure at $\mathfrak{p}_1$ and $\mathfrak{p}_2$, both corresponding to the unitary group G. Recall that $K = F_{\mathfrak{p}_1} \simeq F_{\mathfrak{p}_2}$, with ring of integers $\mathcal{O}_K$, uniformizer π, and residue field $\mathbb{F}$.

5.1. Igusa varieties

Let $q = p^{[\mathbb{F}:\mathbb{F}_p]}$. Fix $0 \leq h_1, h_2 \leq n-1$, and consider the stratum $\bar{X}_{U_0}^{(h_1,h_2)}$ of the Shimura variety X_{U_0}. Choose a compatible 1-dimensional formal $\mathcal{O}_{F,\mathfrak{p}_1} =$

$\mathcal{O}_K$-module Σ_1, of height $n-h_1$ and also a compatible 1-dimensional formal $\mathcal{O}_{F,\mathfrak{p}_2} \simeq \mathcal{O}_K$-module Σ_2 of height $n-h_2$. Giving Σ_1 and Σ_2 is equivalent to giving a triple $(\Sigma, \lambda_\Sigma, i_\Sigma)$, where

- Σ is a Barsotti–Tate group over $\bar{\mathbb{F}}$,
- $\lambda_\Sigma : \Sigma \to \Sigma^\vee$ is a polarization, and
- $i_\Sigma : \mathcal{O}_F \to \mathrm{End}(\Sigma) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that (Σ, i_Σ) is compatible.

Note that $(\Sigma[\mathfrak{p}_i^\infty])^0 \simeq \Sigma_i$ for $i = 1, 2$ while $(\Sigma[\mathfrak{p}_i^\infty])^{\mathrm{et}} \simeq (K/\mathcal{O}_K)^{n-h_i}$.

Assume that the level U_0 corresponds to the vector $\vec{m} = (0, 0, m_3, \ldots, m_r)$. Let

$$\vec{m}' = \big((m_i^0, m_i^{\mathrm{et}})_{i=1,2}, m_3, \ldots, m_r\big),$$

with the same entries $m_3, \ldots, m_r$ as $\vec{m}$. The Igusa variety $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}'}$ over $\bar{X}^{(h_1,h_2)}_{U_0} \times_{\mathbb{F}} \bar{\mathbb{F}}$ is defined to be the moduli space of the set of the following isomorphisms of finite flat group schemes for $i = 1, 2$:

- $\alpha_i^0 : \Sigma_i[\mathfrak{p}_i^{m_i^0}] \xrightarrow{\sim} \mathcal{G}_i^0[\mathfrak{p}_i^{m_i^0}]$, which extends étale locally to any $(m_i^0)' \geq m_i^0$, and
- $\alpha_i^{\mathrm{et}} : (\mathfrak{p}_i^{-m_i^{\mathrm{et}}} \mathcal{O}_{F,\mathfrak{p}_i}/\mathcal{O}_{F,\mathfrak{p}_i})^{h_i} \xrightarrow{\sim} \mathcal{G}_i^{\mathrm{et}}[\mathfrak{p}_i^{m_i^{\mathrm{et}}}]$.

In other words, if $S/\bar{\mathbb{F}}$ is a scheme, then an S-point of the Igusa variety $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}$ corresponds to a tuple

$$\big(A, \lambda, i, \eta^p, (\alpha_i^0)_{i=1,2}, (\alpha_i^{\mathrm{et}})_{i=1,2}, (\alpha_i)_{i\geq 3}\big),$$

where

- A is an abelian scheme over S with $\mathcal{G}_{A,i} = A[\mathfrak{p}_i^\infty]$;
- $\lambda : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that (A, i) is compatible and $\lambda \circ i(f) = i(f^*)^\vee \circ \lambda, \forall f \in \mathcal{O}_F$;
- $\bar{\eta}^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A$ is a $\pi_1(S, s)$-invariant U^p-orbit of isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules, sending the standard pairing on $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ to an $(\mathbb{A}^{\infty,p})^\times$-multiple of the λ-Weil pairing;
- $\alpha_i^0 : \Sigma^0[\mathfrak{p}_i^{m_i^0}] \xrightarrow{\sim} \mathcal{G}^0_{A,i}[\mathfrak{p}_i^{m_i^0}]$ is an $\mathcal{O}_K$-equivariant isomorphism of finite flat group schemes which extends to any higher level $(m')_i^0 \geq m_i^0$ for $i = 1, 2$ and some integer $(m')_i^0$;
- $\alpha_i^{\mathrm{et}} : \Sigma^{\mathrm{et}}[\mathfrak{p}_i^{m_i^{\mathrm{et}}}] \xrightarrow{\sim} \mathcal{G}^{\mathrm{et}}_{A,i}[\mathfrak{p}_i^{m_i^{\mathrm{et}}}]$ is an $\mathcal{O}_K$-equivariant isomorphism of étale group schemes for $i = 1, 2$;
- $\alpha_i : \Sigma[\mathfrak{p}_i^{m_i}] \xrightarrow{\sim} \mathcal{G}_{A,i}[\mathfrak{p}_i^{m_i}]$ is an $\mathcal{O}_{F,\mathfrak{p}_i}$-equivariant isomorphism of étale group schemes for $3 \leq i \leq r$.

Two such tuples are considered equivalent if there exists a prime-to-p isogeny $f : A \to A'$ taking $(A, \lambda, i, \bar{\eta}^p, \alpha_i^0, \alpha_i^{\mathrm{et}}, \alpha_i)$ to $(A', \gamma\lambda', i', \bar{\eta}^{p'}, \alpha_i^{0'}, \alpha_i^{\mathrm{et}'}, \alpha_i')$ for $\gamma \in \mathbb{Z}_{(p)}^\times$.

The Igusa varieties $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}$ form an inverse system which has an action of $G(\mathbb{A}^{\infty,p})$ inherited from the action on $\bar{X}^{(h_1,h_2)}_{U_0}$. Let

$$J^{(h_1,h_2)}(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times D_{K,n-h_1}^\times \times \mathrm{GL}_{h_1}(K) \times D_{K,n-h_2}^\times \times \mathrm{GL}_{h_2}(K) \times \prod_{i=3}^{r} \mathrm{GL}_n(F_{\mathfrak{p}_i}),$$

which is the group of quasi-self-isogenies of $(\Sigma, \lambda_\Sigma, i_\Sigma)$ (to compute $J^{(h_1,h_2)}(\mathbb{Q}_p)$ we use the duality induced from the polarization). The automorphisms of $(\Sigma, \lambda_\Sigma, i_\Sigma)$ have an action on the right on $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}$. This can be extended to an action of a certain submonoid of $J^{(h_1,h_2)}(\mathbb{Q}_p)$ on the inverse system of $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}$, and furthermore to an action of the entire group $J^{(h_1,h_2)}(\mathbb{Q}_p)$ on the directed system $H^j_c(\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}, \mathcal{L}_\xi)$. (For a definition of this action, see [Sh1, Section 5] and [Ma, Section 4].)

We also define an Iwahori–Igusa variety of the first kind $I_U^{(h_1,h_2)}/\bar{X}_{U_0}$ as the moduli space of chains of isogenies for $i = 1, 2$

$$\mathcal{G}_i^{\mathrm{et}} = \mathcal{G}_{i,0} \to \mathcal{G}_{i,1} \to \cdots \to \mathcal{G}_{i,h_i} = \mathcal{G}_i^{\mathrm{et}}/\mathcal{G}_i^{\mathrm{et}}[\mathfrak{p}_i]$$

of étale Barsotti–Tate $\mathcal{O}_K$-modules, each isogeny having degree $\#\mathbb{F}$ and with composite equal to the natural map $\mathcal{G}_i^{\mathrm{et}} \to \mathcal{G}_i^{\mathrm{et}}/\mathcal{G}_i^{\mathrm{et}}[\mathfrak{p}_i]$. Then $I_U^{(h_1,h_2)}$ is finite étale over $\bar{X}_{U_0}^{(h_1,h_2)}$ and naturally inherits the action of $G(\mathbb{A}^{\infty,p})$. Moreover, for $m_1^0 = m_2^0 = 0$ and $m_1^{\mathrm{et}} = m_2^{\mathrm{et}} = 1$, we know that $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}'}/I_U^{(h_1,h_2)} \times_{\mathbb{F}} \bar{\mathbb{F}}$ is finite étale and Galois with Galois group $B_{h_1}(\mathbb{F}) \times B_{h_2}(\mathbb{F})$. (Here $B_{h_i}(\mathbb{F}) \subseteq \mathrm{GL}_{h_i}(\mathbb{F})$ is the Borel subgroup.)

LEMMA 5.1
For $S, T \subset \{1, \ldots, n\}$ with $\#S = n - h_1, \#T = n - h_2$, there exists a finite map of $\bar{X}_{U_0}^{(h_1,h_2)}$-schemes

$$\varphi : Y^0_{U,S,T} \to I_U^{(h_1,h_2)}$$

which is bijective on the sets of geometric points.

Proof
The proof is a straightforward generalization of the proof of [TY, Lemma 4.1]. □

Recall that, for a given $\vec{m}$ with $m_1 = m_2 = 0$, we take

$$U = U^p \times U_p^{\mathfrak{p}_1,\mathfrak{p}_2}(\vec{m}) \times \mathrm{Iw}_{n,\mathfrak{p}_1} \times \mathrm{Iw}_{n,\mathfrak{p}_2} \times \mathbb{Z}_p^\times,$$

restricting ourselves to Iwahori level structure at $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Now we let the level away from p vary. Define

$$H^j_c(Y^0_{\mathrm{Iw}(\vec{m}),S,T}, \mathcal{L}_\xi) = \varinjlim_{U^p} H^j_c(Y^0_{U,S,T}, \mathcal{L}_\xi),$$

$$H^j(Y_{\mathrm{Iw}(\vec{m}),S,T}, \mathcal{L}_\xi) = \varinjlim_{U^p} H^j(Y_{U,S,T}, \mathcal{L}_\xi),$$

$$H_c^j(I_{\mathrm{Iw}(\vec{m})}^{(h_1,h_2)},\mathcal{L}_\xi)=\varinjlim_{U^p} H_c^j(I_U^{(h_1,h_2)}\times_{\mathbb{F}}\bar{\mathbb{F}},\mathcal{L}_\xi).$$

Without restriction on $\vec{m}'$, we can define

$$H_c^j(\mathrm{Ig}^{(h_1,h_2)},\mathcal{L}_\xi)=\varinjlim_{U^p,\vec{m}'} H_c^j(\mathrm{Ig}_{U^p,\vec{m}'}^{(h_1,h_2)},\mathcal{L}_\xi).$$

For $m_1^0=m_2^0=0$, the Igusa variety $\mathrm{Ig}_{U^p,\vec{m}'}^{(h_1,h_2)}$ is defined over $\mathbb{F}$. If in addition $m_1^{\mathrm{et}}=m_2^{\mathrm{et}}=1$, then $\mathrm{Ig}_{U^p,\vec{m}'}^{(h_1,h_2)}$ (over $\mathbb{F}$) is a Galois cover of $I_U^{(h_1,h_2)}$ with Galois group $B_{h_1}(\mathbb{F})\times B_{h_2}(\mathbb{F})$.

COROLLARY 5.2
Let $\vec{m}'=(0,0,1,1,m_3,\dots,m_r)$. For every $S,T\subseteq\{1,\dots,n-1\}$ with $\#S=n-h_1,\#T=n-h_2$ and $j\in\mathbb{Z}_{\geq 0}$, we have the following isomorphism

$$H_c^j(Y_{U,S,T}^0\times_{\mathbb{F}}\bar{\mathbb{F}},\mathcal{L}_\xi)\simeq H_c^j(I_{U^p,\vec{m}'}^{(h_1,h_2)}\times_{\mathbb{F}}\bar{\mathbb{F}},\mathcal{L}_\xi)^{B_{h_1}(\mathbb{F})\times B_{h_2}(\mathbb{F})}.$$

By taking a direct limit over U^p and over $\vec{m}=(0,0,m_3,\dots,m_r)$ and considering the definitions of the Igusa varieties, we get an isomorphism

$$\begin{aligned}&H_c^j(Y_{\mathrm{Iw}(\vec{m}),S,T}^0,\mathcal{L}_\xi)\\&\quad\simeq H_c^j(\mathrm{Ig}^{(h_1,h_2)},\mathcal{L}_\xi)^{U_p^{\mathfrak{p}_1\mathfrak{p}_2}(\vec{m})\times\mathrm{Iw}_{h_1,\mathfrak{p}_1}\times\mathcal{O}_{D_{K,n-h_1}}^\times\times\mathrm{Iw}_{h_2,\mathfrak{p}_2}\times\mathcal{O}_{D_{K,n-h_2}}^\times}.\end{aligned}$$

Taking a limit over general $\vec{m}'$ satisfying $m_1^0=m_2^0=0$, we define

$$H_c^j(\mathrm{Ig}_0^{(h_1,h_2)},\mathcal{L}_\xi):=\varinjlim_{\substack{U^p,\vec{m}'\\ m_1^0=m_2^0=0}} H_c^j(\mathrm{Ig}_{U^p,\vec{m}'}^{(h_1,h_2)}\times_{\mathbb{F}}\bar{\mathbb{F}},\mathcal{L}_\xi).$$

Then the above isomorphism becomes

$$H_c^j(Y_{\mathrm{Iw}(\vec{m}),S,T}^0,\mathcal{L}_\xi)\simeq H_c^j(\mathrm{Ig}_0^{(h_1,h_2)},\mathcal{L}_\xi)^{U_p^{\mathfrak{p}_1\mathfrak{p}_2}\times\mathrm{Iw}_{h_1,\mathfrak{p}_1}\times\mathrm{Iw}_{h_2,\mathfrak{p}_2}}.$$

PROPOSITION 5.3
The action of $\mathrm{Frob}_{\mathbb{F}}$ *on $H_c^j(\mathrm{Ig}_0^{(h_1,h_2)},\mathcal{L}_\xi)$ coincides with the action of $(1,(p^{-[\mathbb{F}:\mathbb{F}_p]},-1,1,-1,1,1))\in G(\mathbb{A}^{\infty,p})\times J^{(h_1,h_2)}(\mathbb{Q}_p)$.*

Proof
Let $\mathrm{Fr}:x\mapsto x^p$ be the absolute Frobenius on $\mathbb{F}_p$, and let $f=[\mathbb{F}:\mathbb{F}_p]$. To compute the action of the geometric Frobenius $\mathrm{Frob}_{\mathbb{F}}$ on $H_c^j(\mathrm{Ig}_0^{(h_1,h_2)},\mathcal{L}_\xi)$, we notice that the absolute Frobenius acts on each $H_c^j(\mathrm{Ig}_{U^p,\vec{m}}^{(h_1,h_2)}\times_{\mathbb{F}}\bar{\mathbb{F}},\mathcal{L}_\xi)$ as $(\mathrm{Fr}^*)^f\times(\mathrm{Frob}_{\mathbb{F}}^*)^{-1}$.

However, the absolute Frobenius acts trivially on étale cohomology, so the action of $\mathrm{Frob}_{\mathbb{F}}$ coincides with the action induced from

$$(\mathrm{Fr}^*)^f : \mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}'} \to \mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}'}.$$

We claim that $(\mathrm{Fr}^*)^f$ acts the same as the element $(1, p^{-[\mathbb{F}:\mathbb{F}_p]}, -1, 1, -1, 1, 1)$ of

$$G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times \mathbb{Z} \times \mathrm{GL}_{h_1}(K) \times \mathbb{Z} \times \mathrm{GL}_{h_2}(K) \times \prod_{i=3}^{r} \mathrm{GL}_n(F_{\mathfrak{p}_i}),$$

where the two copies of $\mathbb{Z}$ are identified with $D^\times_{K,n-h_i}/\mathcal{O}^\times_{D_{K,n-h_i}}$ for $i = 1, 2$ via the valuation of the determinant. To verify this claim, we use the explicit description of the action of a submonoid $J^{(h_1,h_2)}(\mathbb{Q}_p)$ on the inverse system of Igusa varieties $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}'}$ found in [Ma, Section 4] which generalizes that in [HT, p. 122]. First, it is easy to see that

$$(\mathrm{Fr}^*)^f : (A, \lambda, i, \bar{\eta}^p, \alpha_i^0, \alpha_i^{\mathrm{et}}, \alpha_i) \mapsto \big(A^{(q)}, \lambda^{(q)}, i^{(q)}, (\bar{\eta}^p)^{(q)}, (\alpha_i^0)^{(q)}, (\alpha_i^{\mathrm{et}})^{(q)}, \alpha_i^{(q)}\big),$$

where $F^f : A \to A^{(q)}$ is the natural map and where the structures of $A^{(q)}$ are inherited from the structures of A via F^f.

On the other hand, the element $j = (1, p^{-[\mathbb{F}:\mathbb{F}_p]}, -1, 1, -1, 1, 1)$ acts via a quasi-isogeny of Σ. One can check that the inverse of the quasi-isogeny defined by j is $j^{-1} : \Sigma \to \Sigma^{(q)}$, which is a genuine isogeny. If we were working with points of $\mathrm{Ig}^{(h_1,h_2)}$ (which are compatible systems of points of $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}'}$ for all U^p and $\vec{m}'$), then j should act by precomposing all the isomorphisms $\alpha_i^0, \alpha_i^{\mathrm{et}}$ for $i = 1, 2$ and α_i for $3 \le i \le r$. Since $j|_{A[\mathfrak{p}_i^\infty]^{\mathrm{et}}} = 1$ for $i = 1, 2$ and $j|_{A[\mathfrak{p}_i^\infty]} = 1$ for $3 \le i \le r$, the isomorphisms α_i^{et} and α_i stay the same. However, $\alpha_i^0 \circ j$ is now only a quasi-isogeny of Barsotti–Tate $\mathcal{O}_K$-modules and we need to change the abelian variety A by an isogeny in order to get back the isomorphisms. Let $j_i = j|_{\Sigma[\mathfrak{p}_i^\infty]^0}$ for $i = 1, 2$. Then $(j_i)^{-1} : \Sigma[\mathfrak{p}_i^\infty]^0 \to \Sigma[\mathfrak{p}_i^\infty]^0$ is a genuine isogeny induced by the action of $\pi_i \in D^\times_{K,n-h_i}$. Let $\mathcal{K}_i \subset A[\mathfrak{p}_i^{[\mathbb{F}:\mathbb{F}_p]}]$ be the finite flat subgroup scheme $\alpha_i^0(\ker(j_i)^{-1})$. Let $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \subset A[u^{[\mathbb{F}:\mathbb{F}_p]}]$. Let $\mathcal{K}^\perp \subset A[(u^c)^{[\mathbb{F}:\mathbb{F}_p]}]$ be the annihilator of $\mathcal{K}$ under the λ-Weil pairing. Let $\tilde{A} = A/\mathcal{K} \oplus \mathcal{K}^\perp$, and let $f : A \to \tilde{A}$ be the natural projection map. Then

$$\beta_i^0 = f \circ \alpha_i^0 \circ j_i : (\Sigma[\mathfrak{p}_i^\infty])^0 \to \tilde{A}[\mathfrak{p}_i^\infty]^0$$

is an isomorphism. The quotient abelian variety $\tilde{A}$ inherits the structures of A through the natural projection and it is easy to see that $\tilde{A} = A^{(q)}$. Thus, the action of j coincides with the action of $(\mathrm{Fr}^*)^f$. This concludes the proof. $\square$

COROLLARY 5.4

We have an isomorphism of admissible $G(\mathbb{A}^{\infty,p}) \times (\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}}$-modules

$$H_c^j(Y^0_{\mathrm{Iw}(\vec{m}),S,T}, \mathcal{L}_\xi) \simeq H_c^j(\mathrm{Ig}_0^{(h_1,h_2)}, \mathcal{L}_\xi)^{U_p^{\mathfrak{p}_1\mathfrak{p}_2}(\vec{m})\times \mathrm{Iw}_{h_1,\mathfrak{p}_1}\times \mathrm{Iw}_{h_2,\mathfrak{p}_2}},$$

where $\mathrm{Frob}_{\mathbb{F}}$ *acts as* $(p^{-f}, -1, 1, -1, 1, 1) \in J^{(h_1,h_2)}(\mathbb{Q}_p)$.

5.2. *Counting points on Igusa varieties*

We wish to apply the trace formula in order to compute the cohomology of Igusa varieties. A key input of this is counting the $\bar{\mathbb{F}}$-points of Igusa varieties. Most of this is worked out in [Sh1]. The only missing ingredient is supplied by the main lemma in this section, which is an analogue of [HT, Lemma V.4.1] and of what those authors call “the vanishing of the Kottwitz invariant.” The $\bar{\mathbb{F}}$-points of Igusa varieties are counted by counting p-adic types and other data (e.g., polarizations and level structure). We can keep track of p-adic types via Honda–Tate theory; we need to check that these p-adic types actually correspond to a point on one of our Igusa varieties.

A simple p-adic type over F is a triple (M, η, κ), where

- M is a CM field, with $\mathfrak{P}$ being the set of places of M over p,
- $\eta = \sum_{x\in\mathfrak{P}} \eta_x x$ is an element of $\mathbb{Q}[\mathfrak{P}]$, the $\mathbb{Q}$-vector space with basis $\mathfrak{P}$,
- $\kappa : F \to M$ is a $\mathbb{Q}$-algebra homomorphism

such that $\eta_x \geq 0$ for all $x \in \mathfrak{P}$ and $\eta + c_*\eta = \sum_{x\in\mathfrak{P}} x(p)\cdot x$ in $\mathbb{Q}[\mathfrak{P}]$, where $p = \prod_{x\in\mathfrak{P}} x^{x(p)}$. Here c is the complex conjugation on M and

$$c_* : \mathbb{Q}[\mathfrak{P}] \to \mathbb{Q}[\mathfrak{P}]$$

is the $\mathbb{Q}$-linear map satisfying $x \mapsto x^c$. (See [Sh1, p. 24] for the general definition of a p-adic type.) As in [Sh1], we drop κ from the notation, since it is well understood as the F-algebra structure map of M.

We can recover a simple p-adic type from the following data:

- a CM field M/F;
- for $i = 1, 2$ places $\tilde{\mathfrak{p}}_i$ of M above $\mathfrak{p}_i$ such that $[M_{\tilde{\mathfrak{p}}_i} : F_{\mathfrak{p}_i}]n = [M : F](n - h_i)$ and such that there is no intermediate field $F \subset N \subset M$ with $\tilde{\mathfrak{p}}_i|_N$ both inert in M.

Using this data, we can define a simple p-adic type (M, η), where the coefficients of η at places above u are nonzero only for $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$. The abelian variety $A/\bar{\mathbb{F}}$ corresponding to (M, η) will have an action of M via $i : M \hookrightarrow \mathrm{End}^0(A)$. By Honda–Tate theory, the pair (A, i) will also satisfy the following:

- M is the center of $\mathrm{End}^0_F(A)$;
- $A[\mathfrak{p}_i^\infty]^0 = A[\tilde{\mathfrak{p}}_i^\infty]$ has dimension 1 and $A[\mathfrak{p}_i^\infty]^e$ has height h_i for $i = 1, 2$;
- $A[\mathfrak{p}_i^\infty]$ is ind-étale for $i > 2$.

LEMMA 5.5

Let M/F be a CM field as above. Let $A/\bar{\mathbb{F}}$ be the corresponding abelian variety equipped with $i : M \hookrightarrow \mathrm{End}^0(A)$. Then we can find

- *a polarization $\lambda_0 : A \to A^\vee$ for which the Rosati involution induces c on $i(M)$, and*
- *a finitely generated M-module W_0 together with a nondegenerate Hermitian pairing*

$$\langle \cdot, \cdot \rangle_0 : W_0 \times W_0 \to \mathbb{Q}$$

such that the following are satisfied:

- *there is an isomorphism of $M \otimes \mathbb{A}^{\infty,p}$-modules*

$$W_0 \otimes \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^p A,$$

  which takes $\langle \cdot, \cdot \rangle_0$ to an $(\mathbb{A}^{\infty,p})^\times$-multiple of the λ_0-Weil pairing on $V^p A$; and
- *there is an isomorphism of $F \otimes_{\mathbb{Q}} \mathbb{R}$-modules*

$$W_0 \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{R},$$

  which takes $\langle \cdot, \cdot \rangle_0$ to an $\mathbb{R}^\times$-multiple of our standard pairing $\langle \cdot, \cdot \rangle$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$.

Proof

By [Ko2, Lemma 9.2], there is a polarization $\lambda_0 : A \to A^\vee$ such that the λ_0-Rosati involution preserves M and acts on it as c. The next step is to show that, up to isogeny, we can lift (A, i, λ_0) from $\bar{\mathbb{F}}$ to $\mathcal{O}_{K^{ac}}$. Using [Tat, Theorem 2], we can find some lift of A to an abelian scheme $\tilde{A}/\mathcal{O}_{K^{ac}}$ in such a way that i lifts to an action $\tilde{i}$ of M on $\tilde{A}$. As in the proof of [HT, Lemma V.4.1], we find a polarization $\tilde{\lambda}$ of $\tilde{A}$ which reduces to λ. However, we want to be more specific about choosing our lift $\tilde{A}$. Indeed, for any lift, $\mathrm{Lie}\,\tilde{A} \otimes_{\mathcal{O}_K^{ac}} K^{ac}$ is an $F \otimes K^{ac} \simeq (K^{ac})^{\mathrm{Hom}(F,K^{ac})}$-module, so we have a decomposition

$$\mathrm{Lie}\,\tilde{A} \otimes_{\mathcal{O}_{K^{ac}}} K^{ac} \simeq \bigoplus_{\tau \in \mathrm{Hom}(F,K^{ac})} (\mathrm{Lie}\,\tilde{A})_\tau.$$

Let $\mathrm{Hom}(F, K^{ac})^+$ be the set of places $\tau \in \mathrm{Hom}(F, K^{ac})$ which induces the place u of E. We want to make sure that the set of places $\tau \in \mathrm{Hom}(F, K^{ac})^+$ for which $(\mathrm{Lie}\,\tilde{A})_\tau$ is nontrivial has exactly two elements τ_1' and τ_2' which differ by our distinguished element $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$; that is,

$$\tau_2' = \tau_1' \circ \sigma.$$

In order to ensure this, we need to go through Tate's original argument for constructing lifts $\tilde{A}$ of A (see [Tat, Section 4]).

First, let $\Phi = \sum_{\tilde{\tau}\in\mathrm{Hom}(M,K^{ac})} \Phi_{\tilde{\tau}} \cdot \tilde{\tau}$ with the $\Phi_{\tilde{\tau}}$ nonnegative integers satisfying $\Phi_{\tilde{\tau}} + \Phi_{\tilde{\tau}^c} = n$. For any such Φ, we can construct an abelian variety $\tilde{A}_\Phi$ over $\mathcal{O}_{K^{ac}}$ such that

$$\mathrm{Lie}\,\tilde{A}_\Phi \otimes_{\mathcal{O}_{K^{ac}}} K^{ac} \simeq \bigoplus_{\tau\in\mathrm{Hom}(F,K^{ac})} (\mathrm{Lie}\,\tilde{A}_\Phi)_\tau$$

satisfies $\dim(\mathrm{Lie}\,\tilde{A}_\Phi)_\tau = \Phi_\tau$. This is done as in [Tat, Lemme 4], which proves the case $n = 1$. We pick any $\tau_i' \in \mathrm{Hom}(F, K^{ac})$ inducing the places $\mathfrak{p}_i$ of F for $i = 1, 2$ such that $\tau_2' = \tau_1' \circ \sigma$. We lift the τ_i' to elements $\tilde{\tau}_i \in \mathrm{Hom}(M, K^{ac})$ inducing $\tilde{\mathfrak{p}}_i$. We let $\Phi_{\tilde{\tau}_i} = 1$ and $\Phi_{\tilde{\tau}} = 0$ for any other $\tilde{\tau} \in \mathrm{Hom}(M, K^{ac})^+$. For $\tilde{\tau} \notin \mathrm{Hom}(M, K^{ac})^+$, we define $\Phi_{\tilde{\tau}} = n - \Phi_{\tilde{\tau}^c}$. This determines $\Phi \in \mathbb{Q}[\mathrm{Hom}(M, K^{ac})]$ entirely. This Φ is not quite a p-adic type for M; however, it is easy to associate a p-adic type to it: we define $\eta = \sum_{x|p} \eta_x \cdot x$ by

$$\eta_x = \frac{e_{x/p} \cdot [M : F]}{n \cdot [M_x : \mathbb{Q}_p]} \cdot \sum \Phi_{\tilde{\tau}},$$

where the sum is over embeddings $\tilde{\tau} \in \mathrm{Hom}(M, K^{ac})$ which induce the place x of M. By Honda–Tate theory, the reduction of the abelian scheme $\tilde{A}_\Phi/\mathcal{O}_{K^{ac}}$ associated to Φ has p-adic type η. Indeed, the height of the p-divisible group at x of the reduction of $\tilde{A}_\Phi$ is $(n \cdot [M_x : \mathbb{Q}_p])/[M : F]$ (see [Sh1, Proposition 8.4] together with an expression of $\dim A$ in terms of M). The dimension of the p-divisible group at x of the reduction is $\sum \Phi_{\tilde{\tau}}$, where we are summing over all embeddings $\tilde{\tau}$ which induce x.

Now we set $\tilde{A} = \tilde{A}_\Phi$. It remains to check that $\tilde{A}/\mathcal{O}_{K^{ac}}$ has special fiber isogenous to $A/\bar{\mathbb{F}}$ and this follows from the fact that the reductions of $\tilde{A}$ and A are both associated to the same p-adic type η. Indeed, it suffices to verify this for places x above u. We have

$$\eta_x = 0 = e_{x/p} \cdot \frac{\dim A[x^\infty]}{\mathrm{height}\, A[x^\infty]}$$

for all places $x \neq \tilde{\mathfrak{p}}_i$ for $i = 1, 2$. When $x = \tilde{\mathfrak{p}}_i$, we have

$$\begin{aligned}\eta_x &= e_{x/p} \cdot \frac{[M : F]}{[M_x : F_{\mathfrak{p}_i}] \cdot n \cdot [F_{\mathfrak{p}_i} : \mathbb{Q}_p]} \\ &= e_{x/p} \cdot \frac{1}{(n - h_i) \cdot [F_{\mathfrak{p}_i} : \mathbb{Q}_p]} = e_{x/p} \cdot \frac{\dim A[x^\infty]}{\mathrm{height}\, A[x^\infty]}.\end{aligned}$$

Therefore, the p-adic type associated to A is also η.

There are exactly two distinct embeddings $\tau'_1, \tau'_2 \in \mathrm{Hom}(F, K^{ac})^+$ such that $(\mathrm{Lie}\,\tilde{A})_\tau \neq (0)$ only when $\tau = \tau'_1$ or τ'_2. Moreover, these embeddings are related by $\tau'_2 = \tau'_1 \circ \sigma$. Therefore, we can find an embedding $\kappa : K^{ac} \hookrightarrow \mathbb{C}$ such that $\kappa \circ \tau'_i = \tau_i$ for $i = 1, 2$. We set

$$W_0 = H_1\big((\tilde{A} \times_{\mathrm{Spec}\,\mathcal{O}_{K^{ac}},\kappa} \mathrm{Spec}\,\mathbb{C})(\mathbb{C}), \mathbb{Q}\big).$$

From here on, the proof proceeds as in the proof of [HT, Lemma V.4.1]. □

5.3. Vanishing of cohomology

Let Π^1 be an automorphic representation of $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_F)$, and assume that Π^1 is cuspidal. Let $\varpi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}$ be any Hecke character such that $\varpi|_{\mathbb{A}^\times/\mathbb{Q}^\times}$ is the composite of $\mathrm{Art}_\mathbb{Q}$ and the natural surjective character $W_\mathbb{Q} \twoheadrightarrow \mathrm{Gal}(E/\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\}$.

Also assume that Π^1 and F satisfy

- $\Pi^1 \simeq \Pi^1 \circ \theta$;
- Π^1_∞ is generic and Ξ^1-cohomological, for some irreducible algebraic representation Ξ^1 of $\mathbb{G}_n(\mathbb{C})$, which is the image of $\iota_l \xi$ under the base change from $G_\mathbb{C}$ to $\mathbb{G}_{n,\mathbb{C}}$;
- $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_\mathbb{Q}(\varpi) \cup \mathrm{Ram}_\mathbb{Q}(\Pi) \subset \mathrm{Spl}_{F/F_2,\mathbb{Q}}$.

Let $\mathfrak{S} = \mathfrak{S}_{\mathrm{fin}} \cup \{\infty\}$ be a finite set of places of F, which contains the places of F above places of $\mathbb{Q}$ which are ramified in F and the places where Π is ramified. For $l \neq p$, let $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, and let $\pi_p \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ be such that $BC(\iota_l \pi_p) \simeq \Pi_p$. If we write $\Pi^1 = \psi \otimes \Pi^0$ and $\pi_p = \pi_{p,0} \otimes \pi_{\mathfrak{p}_1} \otimes \pi_{\mathfrak{p}_2} \otimes \big(\bigotimes_{i=3}^r \pi_{\mathfrak{p}_i}\big)$, then $\iota_l \pi_{p,0} \simeq \psi_u$ and $\iota_l \pi_{\mathfrak{p}_i} \simeq \Pi^0_{\mathfrak{p}_i}$ for all $1 \leq i \leq r$. Under the identification $F_{\mathfrak{p}_1} \simeq F_{\mathfrak{p}_2}$, assume that $\Pi^0_{\mathfrak{p}_1} \simeq \Pi^0_{\mathfrak{p}_2}$ (this condition will be satisfied in all our applications, since we choose Π^0 to be the base change of some cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_{F_1 E})$).

Define the following elements of $\mathrm{Groth}(G(\mathbb{A}^{\infty,p}) \times J^{(h_1,h_2)}(\mathbb{Q}_p))$ (the Grothendieck group of admissible representations):

$$[H_c(\mathrm{Ig}^{(h_1,h_2)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h_1+h_2-i} H_c^i(\mathrm{Ig}^{(h_1,h_2)}, \mathcal{L}_\xi).$$

If $R \in \mathrm{Groth}(G(\mathbb{A}^\mathfrak{S}) \times G')$, we can write $R = \sum_{\pi^\mathfrak{S} \otimes \rho} n(\pi^\mathfrak{S} \otimes \rho)[\pi^\mathfrak{S}][\rho]$, where $\pi^\mathfrak{S}$ and ρ run over $\mathrm{Irr}_l(G(\mathbb{A}^\mathfrak{S}))$ and $\mathrm{Irr}_l(G')$, respectively. We define

$$R\{\pi^\mathfrak{S}\} := \sum_\rho n(\pi^\mathfrak{S} \otimes \rho)[\rho], \qquad R[\pi^\mathfrak{S}] := \sum_\rho n(\pi^\mathfrak{S} \otimes \rho)[\pi^\mathfrak{S}][\rho].$$

Also define

$$R\{\Pi^{1,\mathfrak{S}}\} := \sum_{\pi^\mathfrak{S}} \{\pi^\mathfrak{S}\}, \qquad R[\Pi^{1,\mathfrak{S}}] := \sum_{\pi^\mathfrak{S}} R[\pi^\mathfrak{S}],$$

where each sum runs over $\pi^{\mathfrak{S}} \in \mathrm{Irr}_l^{\mathrm{ur}}(G(\mathbb{A}^{\mathfrak{S}}))$ such that $BC(\iota_l \pi^{\mathfrak{S}}) \simeq \Pi^{\mathfrak{S}}$.

Let $\mathrm{Red}_n^{(h_1,h_2)}(\pi_p)$ be the morphism from $\mathrm{Groth}(G(\mathbb{Q}_p))$ to $\mathrm{Groth}(J^{(h_1,h_2)}(\mathbb{Q}_p))$ defined by

$$(-1)^{h_1+h_2}\pi_{p,0} \otimes \mathrm{Red}^{n-h_1,h_1}(\pi_{\mathfrak{p}_1}) \otimes \mathrm{Red}^{n-h_2,h_2}(\pi_{\mathfrak{p}_2}) \otimes \Big(\bigotimes_{i>2} \pi_{\mathfrak{p}_i}\Big),$$

where

$$\mathrm{Red}^{n-h,h} : \mathrm{Groth}\big(\mathrm{GL}_n(K)\big) \to \mathrm{Groth}\big(D^{\times}_{K,1/(n-h)} \times \mathrm{GL}_h(K)\big)$$

is obtained by composing the normalized Jacquet functor

$$J : \mathrm{Groth}\big(\mathrm{GL}_n(K)\big) \to \mathrm{Groth}\big(\mathrm{GL}_{n-h}(K) \times \mathrm{GL}_h(K)\big)$$

with the Jacquet–Langlands map

$$LJ : \mathrm{Groth}\big(\mathrm{GL}_{n-h}(K)\big) \to \mathrm{Groth}\big(D^{\times}_{K,1/(n-h)}\big)$$

defined by Badulescu in [Bad]. Assume the following result, which will be proved in Section 6.

THEOREM 5.6

We have the following equality in $\mathrm{Groth}(G(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}) \times J^{(h_1,h_2)}(\mathbb{Q}_p)$*:*

$$\begin{aligned} &BC_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}\big(H_c(\mathrm{Ig}^{(h_1,h_2)}, \mathcal{L}_\xi)\{\Pi^{1,\mathfrak{S}}\}\big) \\ &\qquad = e_0(-1)^{h_1+h_2} C_G[\iota_l^{-1}\Pi^1_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}][\mathrm{Red}_n^{(h_1,h_2)}(\pi_p)], \end{aligned}$$

where C_G is a positive integer and $e_0 = \pm 1$.

Let $S, T \subseteq \{1, \dots, n-1\}$ with $\#S = n-h_1, \#T = n-h_2$. From Theorem 5.6 and Corollary 5.4, we obtain the equality

$$\begin{aligned} &BC^p\big(H_c(Y^0_{\mathrm{Iw}(m),S,T}, \mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big) \\ &\quad = e_0 C_G[\iota_l^{-1}\Pi^{\infty,p}][\mathrm{Red}^{(h_1,h_2)}(\pi_{p,0} \otimes \pi_{\mathfrak{p}_1} \otimes \pi_{\mathfrak{p}_2})] \cdot \dim\Big[\Big(\bigotimes_{i=3}^{r} \pi_{\mathfrak{p}_i}\Big)^{U_p^{\mathfrak{p}_1\mathfrak{p}_2}}\Big] \end{aligned}$$

in $\mathrm{Groth}(G(\mathbb{A}^{\infty,p}) \times (\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}})$. The group morphism

$$\mathrm{Red}^{(h_1,h_2)} : \mathrm{Groth}\big(\mathbb{Q}_p^{\times} \times \mathrm{GL}_n(K) \times \mathrm{GL}_n(K)\big) \to \mathrm{Groth}(\mathrm{Frob}_{\mathbb{F}}^{\mathbb{Z}})$$

is the composite of normalized Jacquet functors

$$J_i : \mathrm{Groth}\bigl(\mathrm{GL}_n(K)\bigr) \to \mathrm{Groth}\bigl(\mathrm{GL}_{n-h_i}(K) \times \mathrm{GL}_{h_i}(K)\bigr)$$

for $i = 1, 2$ with the map

$$\mathrm{Groth}\bigl(\mathbb{Q}_p^\times \times \mathrm{GL}_{n-h_1}(K) \times \mathrm{GL}_{h_1}(K) \times \mathrm{GL}_{n-h_2}(K) \times \mathrm{GL}_{h_2}(K)\bigr) \to \mathrm{Groth}(\mathrm{Frob}_{\mathbb{F}}^{\mathbb{Z}})$$

which sends $[\alpha_1 \otimes \beta_1 \otimes \alpha_2 \otimes \beta_2 \otimes \gamma]$ to

$$\begin{aligned}\sum_{\phi_1,\phi_2} &\mathrm{vol}(D^\times_{K,n-h_1}/K^\times)^{-1} \cdot \mathrm{vol}(D^\times_{K,n-h_2}/K^\times)\\ &\cdot \mathrm{tr}\,\alpha_1(\varphi_{\mathrm{Sp}_{n-h_1}(\phi_1)}) \cdot \mathrm{tr}\,\alpha_2(\varphi_{\mathrm{Sp}_{n-h_2}(\phi_2)}) \cdot (\dim \beta_1)^{\mathrm{Iw}_{h_1,\mathfrak{p}_1}}\\ &\cdot (\dim \beta_2)^{\mathrm{Iw}_{h_1,\mathfrak{p}_2}} \cdot \bigl[\mathrm{rec}\bigl(\phi_1^{-1}\phi_2^{-1}|\ |^{1-n}(\gamma^{\mathbb{Z}_p^\times} \circ \mathbf{N}_{K/E_u})^{-1}\bigr)\bigr],\end{aligned}$$

where the sum is over characters ϕ_1, ϕ_2 of $K^\times/\mathcal{O}_K^\times$.

LEMMA 5.7
We have the following equality in $\mathrm{Groth}(G(\mathbb{A}^{\infty,p}) \times (\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}})$*:*

$$\begin{aligned}&BC^p\bigl(H(Y_{\mathrm{Iw}(\vec{m}),S,T}, \mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\bigr)\\ &\quad = e_0 C_G[\iota_l^{-1}\Pi^{1,\infty,p}] \dim\Bigl[\Bigl(\bigotimes_{i=3}^r \pi_{\mathfrak{p}_i}\Bigr)^{U_p^{\mathfrak{p}_1\mathfrak{p}_2}}\Bigr]\\ &\qquad \times \Biggl(\sum_{h_1=0}^{n-\#S}\sum_{h_2=0}^{n-\#T}(-1)^{2n-\#S-\#T-h_1-h_2}\binom{n-\#S}{h_1}\binom{n-\#T}{h_2}\\ &\qquad \times \mathrm{Red}^{(h_1,h_2)}(\pi_{p,0} \otimes \pi_{\mathfrak{p}_1} \otimes \pi_{\mathfrak{p}_2)}\Biggr).\end{aligned}$$

Proof
The proof is a straightforward generalization of the proof of [TY, Lemma 4.3]. □

THEOREM 5.8
Assume that $\Pi^0_{\mathfrak{p}_1} \simeq \Pi^0_{\mathfrak{p}_2}$ *has an Iwahori fixed vector. Then* $\Pi^0_{\mathfrak{p}_1} \simeq \Pi^0_{\mathfrak{p}_2}$ *is tempered.*

Proof
By [HT, Corollary VII.2.18], $\iota_l\pi_{\mathfrak{p}_i}$ is tempered if and only if for all $\sigma \in W_K$, every eigenvalue α of $\mathcal{L}_{n,K}(\Pi^0_{\mathfrak{p}_i})(\sigma)$ (where $\mathcal{L}_{n,K}(\Pi^0_{\mathfrak{p}_i})$ is the image of $\Pi^0_{\mathfrak{p}_i}$ under the local Langlands correspondence, normalized as in [Sh3]) satisfies

$$|\iota_l\alpha|^2 \in q^{\mathbb{Z}}.$$

We first use a standard argument to show that we can always ensure that

$$|\iota_l\alpha|^2 \in q^{\frac{1}{2}\mathbb{Z}},$$

and then we use a classification of irreducible, generic, ι-preunitary representations of $\mathrm{GL}_n(K)$ together with the cohomology of Igusa varieties to show the full result.

The space $H^k(X, \mathcal{L}_\xi)$ decomposes as a $G(\mathbb{A}^\infty)$-module as

$$H^k(X, \mathcal{L}_\xi) = \bigoplus_{\pi^\infty} \pi^\infty \otimes R^k_{\xi,l}(\pi^\infty),$$

where π^∞ runs over $\mathrm{Irr}_l(G(\mathbb{A}^\infty))$ and where $R^k_{\xi.l}(\pi^\infty)$ is a finite-dimensional $\mathrm{Gal}(\bar{F}/F)$-representation. Define the $\mathrm{Gal}(\bar{F}/F)$-representation

$$\tilde{R}^k_l(\Pi^1) = \sum_{\pi^\infty} R^k_{\xi,l}(\pi^\infty),$$

where the sum is over the $\pi^\infty \in \mathrm{Irr}_l(G(\mathbb{A}^\infty))$ which are cohomological, unramified outside $\mathfrak{S}_{\mathrm{fin}}$, and such that $BC(\iota_l\pi^\infty) = \Pi^{1,\infty}$. Also define the element $\tilde{R}_l(\Pi^1) \in \mathrm{Groth}(\mathrm{Gal}(\bar{F}/F))$ by

$$\tilde{R}_l(\Pi^1) = \sum_k (-1)^k \tilde{R}^k_l(\Pi^1).$$

We claim that we have the following identity in $\mathrm{Groth}(W_K)$:

$$\tilde{R}_l(\Pi^1) = e_0 C_G \cdot [(\pi_{p,0} \circ \mathrm{Art}^{-1}_{\mathbb{Q}_p})|_{W_K} \otimes \iota_l^{-1}\mathcal{L}_{n,K}(\Pi^0_{\mathfrak{p}_1}) \otimes \iota_l^{-1}\mathcal{L}_{n,K}(\Pi^0_{\mathfrak{p}_2})].$$

This can be deduced from [Ko1] or by combining Theorem 5.6 with Mantovan's formula (see [Ma, Theorem 22]).

From the above identity, using the fact that $\Pi^0_{\mathfrak{p}_1} \simeq \Pi^0_{\mathfrak{p}_2}$, we see that $|\iota_l(\alpha\beta)|^2 \in q^{\mathbb{Z}}$ for any eigenvalues α, β of any $\sigma \in W_K$, since $\tilde{R}_l(\Pi^1)$ is found in the cohomology of some proper, smooth variety X_U over K. In particular, we know that $|\iota_l\alpha|^2 \in q^{\frac{1}{2}\mathbb{Z}}$. Moreover, if one eigenvalue α of σ satisfies $|\iota_l\alpha|^2 \in q^{\mathbb{Z}}$, then all other eigenvalues of σ would be forced to satisfy it as well. A result of Harris and Taylor [HT, Lemma I.3.8] (which makes use of the classification of unitary representations of [Tad]) says that if $\pi_{\mathfrak{p}_i}$ is a generic, ι_l-preunitary representation of $\mathrm{GL}_n(K)$ with central character $|\psi_{\pi_{\mathfrak{p}_i}}| \equiv 1$, then $\pi_{\mathfrak{p}_i}$ is isomorphic to

$$\text{n-Ind}^{\mathrm{GL}_n(K)}_{P(K)}(\pi_1 \times \cdots \times \pi_s \times \pi'_1|\det|^{a_1} \times \pi'_1|\det|^{-a_1} \times \cdots \times \pi'_t|\det|^{a_t} \times \pi'_t|\det|^{-a_t})$$

for some parabolic subgroup P of GL_n. The $\pi_1, \ldots, \pi_s, \pi'_1, \ldots, \pi'_t$ are square-integrable representations of smaller linear groups with $|\psi_{\pi_j}| \equiv |\psi_{\pi'_{j'}}| \equiv 1$ for all j, j'. Moreover, we must have $0 < a_j < 1/2$ for $j = 1, \ldots, t$. If $s \neq 0$, then for any $\sigma \in W_K$ there is an eigenvalue α of $\mathcal{L}_{K,n}(\pi_{\mathfrak{p}_i})(\sigma)$ with $|\iota_l\alpha|^2 \in q^{\mathbb{Z}}$, but then this must happen for all eigenvalues of $\mathcal{L}_{K,n}(\pi_{\mathfrak{p}_i})(\sigma)$. So then $t=0$ and $\pi_{\mathfrak{p}_i}$ is tempered. If $s=0$, then every eigenvalue α of a lift of Frobenius $\sigma \in W_K$ must satisfy

$$|\iota_l \alpha|^2 \in q^{\mathbb{Z}\pm 2a_j}$$

for some $j \in 1,\dots,t$. Note that each j corresponds to at least one such eigenvalue α, so we must have $a_j = 1/4$ for all $j=1,\dots,t$. To summarize, $\pi_{\mathfrak{p}_i}$ is either tempered or it is of the form

$$\text{n-Ind}_{P(K)}^{\mathrm{GL}_n(K)}(\pi'_1|\det|^{1/4}\times\pi'_1|\det|^{-1/4}\times\cdots\times\pi'_t|\det|^{1/4}\times\pi'_t|\det|^{-1/4}).$$

We now focus on the second case in order to get a contradiction. Since $\pi_{\mathfrak{p}_i}$ has an Iwahori fixed vector, each π'_j must be equal to $\mathrm{Sp}_{s_j}(\chi_j)$, where χ_j is an unramified character of $K^\times$. We can compute $\mathrm{Red}^{(h_1,h_2)}(\pi_{p,0}\otimes\pi_{\mathfrak{p}_1}\otimes\pi_{\mathfrak{p}_2})$ explicitly and compare it to the cohomology of a closed stratum $Y_{\mathrm{Iw},S,T}$ via Lemma 5.7.

We can compute $\mathrm{Red}^{(h_1,h_2)}(\pi_{p,0}\otimes\pi_{\mathfrak{p}_1}\otimes\pi_{\mathfrak{p}_2})$ using an analogue of [HT, Lemma I.3.9], which follows as well from [BZ, Lemma 2.12]. Indeed,

$$\begin{aligned}J_i\big(&\text{n-Ind}_{P(K)}^{\mathrm{GL}_n(K)}(\mathrm{Sp}_{s_1}(\chi_1)\cdot|\det|^{1/4}\\&\times\mathrm{Sp}_{s_1}(\chi_1)\cdot|\det|^{-1/4}\times\cdots\times\mathrm{Sp}_{s_t}(\chi_t)\cdot|\det|^{-1/4})\big)\end{aligned}$$

is equal to

$$\begin{aligned}\sum\big[&\text{n-Ind}_{P'_i(K)}^{\mathrm{GL}_{h_i}(k)}\big((\mathrm{Sp}_{l_1}(\chi_1\otimes|\det|^{s_1-l_1+1/4})\times\cdots\times\mathrm{Sp}_{k_t}(\chi_t\otimes|\det|^{s_t-k_t-1/4})\big)\big]\\&\times\big[\text{n-Ind}_{P''_i(K)}^{\mathrm{GL}_{h_i}(k)}\big((\mathrm{Sp}_{s_1-l_1}(\chi_1\otimes|\det|^{1/4})\times\cdots\times\mathrm{Sp}_{s_t-k_t}(\chi_t\otimes|\det|^{-1/4})\big)\big],\end{aligned}$$

where the sum is over all nonnegative integers $l_j,k_j\le s_j$ with $h_i=\sum_{j=1}^t(l_j+k_j)$. Here P'_i and P''_i are parabolic subgroups with Levi components $\mathrm{GL}_{l_1}\times\cdots\times\mathrm{GL}_{k_t}$ and $\mathrm{GL}_{s_1-l_1}\times\cdots\times\mathrm{GL}_{s_t-k_t}$, respectively.

Let $V^k_{j_1j_2}=\mathrm{rec}(\chi_{j_1}^{-1}\chi_{j_2}^{-1}|\,|^{1-n+\epsilon_k}(\psi_u\circ\mathbf{N}_{K/E_u})^{-1})$, where

$$\epsilon_k=\begin{cases}-\frac{1}{2} & \text{if } k=1,\\ 0 & \text{if } k=2,\\ \frac{1}{2} & \text{if } k=3.\end{cases}$$

After we apply the functor

$$\mathrm{Groth}\big(\mathrm{GL}_{n-h_1}(K)\times\mathrm{GL}_{h_1}(K)\times\mathrm{GL}_{n-h_2}(K)\times\mathrm{GL}_{h_2}(K)\times\mathbb{Q}_p^\times\big)\to\mathrm{Groth}(\mathrm{Frob}_{\mathbb{F}}^{\mathbb{Z}}),$$

we get

$$\mathrm{Red}^{(h_1,h_2)}(\pi_{p,0}\otimes\pi_{\mathfrak{p}_1}\otimes\pi_{\mathfrak{p}_2})=\sum_{j_1,j_2,k}\gamma_{j_1j_2}^{(h_1,h_2)}([V^1_{j_1j_2}]\oplus 2[V^2_{j_1j_2}]\oplus[V^3_{j_1j_2}]),$$

where

$$\begin{aligned}\gamma_{j_1,j_2}^{(h_1,h_2)}&=\prod_{i=1}^{2}\dim\Bigl(\text{n-Ind}_{P_i'(K)}^{\mathrm{GL}_{h_i}(K)}\Bigl(\mathrm{Sp}_{s_{j_i}+h_i-n}(\chi_{j_i}|\,|^{n-h_i\pm1/4})\otimes\mathrm{Sp}_{s_j}(\chi_{j_i}|\,|^{\mp1/4})\\&\qquad\otimes\bigotimes_{j\neq j_i}\mathrm{Sp}_{s_j}(\chi_j|\,|^{1/4})\otimes\bigotimes_{j\neq j_i}\mathrm{Sp}_{s_j}(\chi_j|\,|^{-1/4})\Bigr)\Bigr)^{\mathrm{Iw}_{h_i,\mathfrak{p}_i}}\\&=\prod_{i=1}^{2}\frac{h_i!}{(s_{j_i}+h_i-n)!s_{j_i}!\prod_{j\neq j_i}(s_j!)^2}\end{aligned}$$

and where the sum is over the j_1,j_2 for which $s_{j_i}\geq n-h_i$ for $i=1,2$. Here P_i' for $i=1,2$ are parabolic subgroups of $\mathrm{GL}_{h_i}(K)$.

Let $D(\Pi^1)=e_0C_G[\Pi^{1,\infty,p}]\dim\bigl[\bigl(\bigotimes_{i=3}^r\pi_{\mathfrak{p}_i}\bigr)^{U_p^{\mathfrak{p}_1\mathfrak{p}_2}}\bigr]$. Then

$$\begin{aligned}&BC^p\bigl(H(Y_{\mathrm{Iw}(\vec{m}),S,T},\mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\bigr)\\&\quad=D(\Pi^1)\times\Bigl(\sum_{h_1=0}^{n-\#S}\sum_{h_2=0}^{n-\#T}(-1)^{2n-\#S-\#T-h_1-h_2}\binom{n-\#S}{h_1}\binom{n-\#T}{h_2}\\&\qquad\qquad\times\sum_{j_1,j_2,k}\gamma_{j_1j_2}^{(h_1,h_2)}([V^1_{j_1j_2}]\oplus 2[V^2_{j_1j_2}]\oplus[V^3_{j_1j_2}])\Bigr).\end{aligned}$$

We can compute the coefficient of $[V^k_{j_1j_2}]$ in $BC^p(H(Y_{\mathrm{Iw},S,T},\mathcal{L}_\xi))[\Pi^{1,\mathfrak{S}}]$ by summing first over j_1,j_2 and then over h_1,h_2 going from $n-s_{j_1},n-s_{j_2}$ to $n-\#S$ and $n-\#T$, respectively. Note that the coefficient of $[V^2_{j_1j_2}]$ is exactly twice that of $[V^1_{j_1j_2}]$ and of $[V^3_{j_1j_2}]$. The sum we get for $[V^1_{j_1j_2}]$ is

$$\begin{aligned}&D(\Pi^1)\frac{(n-\#S)!(n-\#T)!}{(s_{j_1}-\#S)!(s_{j_2}-\#T)!s_{j_1}!s_{j_2}!\prod_{j\neq j_1}(s_j!)^2\prod_{j\neq j_2}(s_j!)^2}\\&\quad\times\Bigl(\sum_{h_1=n-s_{j_1}}^{n-\#S}\sum_{h_2=n-s_{j_2}}^{n-\#T}(-1)^{2n-\#S-\#T-h_1-h_2}\binom{s_{j_1}-\#S}{h_1+s_{j_1}-n}\binom{s_{j_2}-\#T}{h_2+s_{j_2-n}}\Bigr).\end{aligned}$$

The sum in parentheses can be decomposed as

$$\Bigl(\sum_{h_1=n-s_{j_1}}^{n-\#S}(-1)^{n-\#S-h_1}\binom{s_{j_1}-\#S}{h_1+s_{j_1}-n}\Bigr)$$

$$\times\Bigg(\sum_{h_2=n-s_{j_2}}^{n-\#T}(-1)^{n-\#T-h_2}\binom{s_{j_2}-\#T}{h_2+s_{j_2}-n}\Bigg)\Bigg),$$

which is equal to 0 unless both $s_{j_1}=\#S$ and $s_{j_2}=\#T$. So

$$\begin{aligned}&BC^p\big(H(Y_{\mathrm{Iw}(\vec{m}),S,T},\mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big)\\&\quad=D(\Pi^1)\times\sum_{s_{j_1}=\#S,s_{j_2}=\#T}\frac{(n-\#S)!(n-\#T)!s_{j_1}!s_{j_2}!}{\prod_j(s_j!)^4}\\&\qquad\times([V^1_{j_1j_2}]+2[V^2_{j_1j_2}]+[V^3_{j_1j_2}]).\end{aligned}$$

Since each $Y_{U,S,T}$ is proper and smooth, it follows from the Weil conjectures that $H^j(Y_{\mathrm{Iw}(\vec{m}),S,T},\mathcal{L}_\xi)$ is strictly pure of weight $m_\xi-2t_\xi+j$. This property means that, for some (hence every) lift σ of $\mathrm{Frob}_{\mathbb{F}}$, every eigenvalue of σ on $H^j(Y_{\mathrm{Iw}(\vec{m}),S,T},\mathcal{L}_\xi)$ is a Weil $q^{m_\xi-2t_\xi+j}$-number (see the definitions above [TY, Lemma 1.4]). However, the $[V^k_{j_1j_2}]$ are strictly pure of weight $m_\xi-2t_\xi+2n-2-\epsilon_k-(\#S-1)-(\#T-1)=m_\xi-2t_\xi+2n-\#S-\#T-2\epsilon_k$. So

$$BC^p\big(H^j(Y_{\mathrm{Iw}(\vec{m}),S,T},\mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big)=0$$

unless $j=2n-\#S-\#T\pm1$ or $j=2n-\#S-\#T$. However, if the Igusa cohomology is nonzero for some $j=2n-\#S-\#T\pm1$, then there exist j_1,j_2 with $s_{j_1}=\#S$ and $s_{j_2}=\#T$. Hence, the cohomology must also be nonzero for $j=2n-\#S-\#T$. The coefficients of $[V^k_{j_1j_2}]$ all have the same sign, so they are either strictly positive or strictly negative only depending on $D(\Pi^1)$. However, $BC^p(H(Y_{\mathrm{Iw}(m),S,T},\mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]$ is an alternating sum, so the weight $2n-\#S-\#T\pm1$ part of the cohomology should appear with a different sign from the weight $2n-\#S-\#T$ part. This is a contradiction, so it must be the case that $\pi_{\mathfrak{p}_1}\simeq\pi_{\mathfrak{p}_2}$ is tempered. □

COROLLARY 5.9

Let $n\in\mathbb{Z}_{\geq2}$ be an integer, and let L be any CM field. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ satisfying

- $\Pi^\vee\simeq\Pi\circ c$;
- Π_∞ *is cohomological for some irreducible algebraic representation* Ξ.

Then Π is tempered at every finite place w of L.

Proof

By [TY, Lemma 1.4.3], an irreducible smooth representation Π of $\mathrm{GL}_n(K)$ is tempered if and only if $\mathcal{L}_{K,n}(\Pi)$ is pure of some weight. By [TY, Lemma 1.4.1], purity is preserved under a restriction to the Weil–Deligne representation of $W_{K'}$ for a finite extension K'/K of fields.

Fix a place v of L above p, where $p \neq l$. We will find a CM field F' such that

- $F' = EF_1$, where E is an imaginary quadratic field in which p splits and $F_1 = (F')^{c=1}$ has $[F_1 : \mathbb{Q}] \geq 2$,
- F' is soluble and Galois over L,
- $\Pi^0_{F'} = BC_{F'/L}(\Pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$, and
- there is a place $\mathfrak{p}$ of F above v such that $\Pi^0_{F',\mathfrak{p}}$ has an Iwahori fixed vector

and a CM field F which is a quadratic extension of F' such that

- $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2$ splits in F,
- $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \subset \mathrm{Spl}_{F/F',\mathbb{Q}}$, and
- $\Pi^0_F = BC_{F/F'}(\Pi^0_{F'})$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

To find F' and F we proceed as follows, using the same argument at the end of Section 7 of [Sh3]. For a CM field F, we use the sets $\mathcal{E}(F)$ and $\mathcal{F}(F)$, which are defined in the proof of [Sh3, Theorem 7.5].

First we find a CM field F_0 which is soluble and Galois over L and a place $\mathfrak{p}_0$ above v such that the last two conditions for $F', \mathfrak{p}$ are satisfied for $F_0, \mathfrak{p}_0$ instead. To see that the second to last condition for F' only eliminates finitely many choices for the CM field, we can use the same argument as Clozel in [Cl2, Section 1]. Indeed, if $BC_{F'/L}(\Pi)$ is not cuspidal, then we would have $\Pi \otimes \epsilon \simeq \Pi$ for ϵ the Artin character of L associated to F'. But then the character ϵ would occur in the semisimplification of $R_l \otimes R_l \otimes \omega^{n-1}$, where R_l is the Galois representation associated to Π by Chenevier and Harris [CH] and ω is the cyclotomic character. Thus, there are only finitely many choices for ϵ and so for F'/L, which are excluded.

Next, we choose $E \in \mathcal{E}(F_0)$ such that p splits in E. We take $F' = EF_0$ and $\mathfrak{p}$ any place of F' above $\mathfrak{p}_0$. Let F_1 be the maximal totally real subfield of F', and let w be the place of F_1 below $\mathfrak{p}$. Next, we pick $F'' \in \mathcal{F}(F')$ different from F' and such that w splits in F''. Take $F = F''F'$.

We can find a character ψ of $\mathbb{A}_E^\times/E^\times$ such that $\Pi^1 = \psi \otimes \Pi^0_F$ together with F satisfy the assumptions in the beginning of the section. (For the specific conditions that ψ must satisfy, see Lemma 7.1.) We also know that $\Pi^0_{F,\mathfrak{p}_1} \simeq \Pi^0_{F,\mathfrak{p}_2}$ has an Iwahori fixed vector, thus we are in the situation of Theorem 5.8. □

PROPOSITION 5.10
Assume again that the conditions in the beginning of this section are satisfied and that $\Pi^0_{\mathfrak{p}_1} \simeq \Pi^0_{\mathfrak{p}_2}$ *has a nonzero Iwahori fixed vector. Then*

$$BC^p\big(H^j(Y_{\mathrm{Iw}(\vec{m}),S,T}, \mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big) = 0$$

unless $j = 2n - \#S - \#T$.

Proof
We go through the same computation as in the proof of Theorem 5.8 except that we use the fact that $\pi_{\mathfrak{p}_1} \simeq \pi_{\mathfrak{p}_2}$ is tempered, so it is of the form

$$\text{n-Ind}_{P(K)}^{\mathrm{GL}_n(K)}\big(\mathrm{Sp}_{s_1}(\chi_1) \times \cdots \times \mathrm{Sp}_{s_t}(\chi_t)\big),$$

where the χ_j are unramified characters of $K^\times$.

We can compute $\mathrm{Red}^{(h_1,h_2)}(\pi_{p,0} \otimes \pi_{\mathfrak{p}_1} \otimes \pi_{\mathfrak{p}_2})$ as in the proof of Theorem 5.8, where

$$J_i\big(\text{n-Ind}_{P(K)}^{\mathrm{GL}_n(K)}(\mathrm{Sp}_{s_1}(\chi_1) \times \cdots \times \mathrm{Sp}_{s_t}(\chi_t) \cdot |\det|)\big)$$

is equal to

$$\sum\big[\text{n-Ind}_{P_i'(K)}^{\mathrm{GL}_{h_i}(k)}\big(\mathrm{Sp}_{k_1}(\chi_1 \otimes |\det|^{s_1-k_1}) \times \cdots \times \mathrm{Sp}_{k_t}(\chi_t \otimes |\det|^{s_t-k_t})\big)\big]$$
$$\times \big[\text{n-Ind}_{P_i''(K)}^{\mathrm{GL}_{h_i}(k)}\big(\mathrm{Sp}_{s_1-k_1}(\chi_1) \times \cdots \times \mathrm{Sp}_{s_t-k_t}(\chi_t)\big)\big],$$

where the sum is over all nonnegative integers $k_j \le s_j$ with $h_i = \sum_{j=1}^t k_j$.

Let $V_{j_1 j_2} = \mathrm{rec}(\chi_{j_1}^{-1}\chi_{j_2}^{-1}|\,|^{1-n}(\psi_u \circ \mathbf{N}_{K/E_u})^{-1})$. After we apply the functor

$$\mathrm{Groth}\big(\mathrm{GL}_{n-h_1}(K) \times \mathrm{GL}_{h_1}(K) \times \mathrm{GL}_{n-h_2}(K) \times \mathrm{GL}_{h_2}(K) \times \mathbb{Q}_p^\times\big) \to \mathrm{Groth}(\mathrm{Frob}_{\mathbb{F}}^{\mathbb{Z}}),$$

we get

$$\mathrm{Red}^{(h_1,h_2)}(\pi_{p,0} \otimes \pi_{\mathfrak{p}_1} \otimes \pi_{\mathfrak{p}_2}) = \sum_{j_1,j_2,k} \gamma_{j_1 j_2}^{(h_1,h_2)}[V_{j_1 j_2}],$$

where

$$\gamma_{j_1,j_2}^{(h_1,h_2)} = \prod_{i=1}^2 \dim\Big(\text{n-Ind}_{P_i'(K)}^{\mathrm{GL}_{h_i}(K)}\Big(\mathrm{Sp}_{s_{j_i}}(\chi_{j_i}|\,|^{n-h_i}) \otimes \bigotimes_{j \ne j_i} \mathrm{Sp}_{s_j}(\chi_j)\Big)\Big)^{\mathrm{Iw}_{h_i,\mathfrak{p}_i}}$$
$$= \prod_{i=1}^2 \frac{h_i!}{(s_{j_i}+h_i-n)!s_{j_i}!\prod_{j\ne j_i}(s_j!)^2}$$

and where the sum is over the j_1, j_2 for which $s_{j_i} \ge n-h_i$ for $i = 1,2$. Here P_i' for $i = 1,2$ are parabolic subgroups of $\mathrm{GL}_{h_i}(K)$.

Let $D(\Pi^1) = e_0 C_G[\Pi^{1,\infty,p}]\dim\big[\big(\bigotimes_{i=3}^r \pi_{\mathfrak{p}_i}\big)^{U_p^{\mathfrak{p}_1\mathfrak{p}_2}}\big]$. The same computation as in the proof of Theorem 5.8 gives us

$$BC^p\big(H(Y_{\mathrm{Iw}(\vec{m}),S,T}, \mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big)$$
$$= D(\Pi^1) \cdot \sum_{s_{j_1}=\#S,\, s_{j_2}=\#T} \frac{(n-\#S)!(n-\#T)!s_{j_1}!s_{j_2}!}{\prod_j (s_j!)^2}[V_{j_1 j_2}].$$

Since $\pi_{\mathfrak{p}_1} \simeq \pi_{\mathfrak{p}_2}$ is tempered, we know that $[V_{j_1 j_2}]$ is strictly pure of weight $2n - \#S - \#T$. The Weil conjectures tell us then that $BC^p(H^j(Y_{\mathrm{Iw}(\vec{m}),S,T}, \mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]) = 0$ unless $j = 2n - \#S - \#T$. $\square$

6. The cohomology of Igusa varieties

The goal of this section is to explain how to prove Theorem 5.6. The proof will be a straightforward generalization of the proof of [Sh3, Theorem 6.1] and so we will follow closely the argument and the notation of that paper.

We summarize without proof the results in [Sh3] on transfer and on the twisted trace formula. We emphasize the place ∞, since that is the only place of $\mathbb{Q}$ where our group G differs from the group G considered in [Sh3]. All of the results and notation are as in [Sh3], except in the proof of Lemmas 6.3 and 6.4, where we also use the notation of [Sh2].

We start by explaining the notation we use throughout this section, which is consistent with the notation of [Sh3]. Recall that we have fixed a unitary similitude group G over $\mathbb{Q}$, which satisfies certain local conditions as in Lemma 2.1. In this section, we work with a quasi-split form of G, denoted by G_n, as well as with groups G_{n_1,n_2} which are endoscopic groups for G_n. We denote an element in the set $\{G_n\} \cup \{G_{n_1,n_2} \mid n_1 + n_2 = n, n_1 \geq n_2 > 0\}$ as $G_{\vec{n}}$, where $\vec{n}$ is a multiset of positive integers (in our case, $\vec{n}$ will have length 1 or 2). In other words, $\vec{n}$ runs through the elements of the set $\{n\} \cup \{(n_1, n_2) \mid n_1 + n_2 = n, n_1 \geq n_2 > 0\}$.

If $r \in \{1, 2\}$ and $\vec{n} = (n_i)_{i=1}^r$ with $n_i \in \mathbb{Z}_{>0}$, define

$$\mathrm{GL}_{\vec{n}} := \prod_{i=1}^{r} \mathrm{GL}_{n_i} .$$

Let $i_{\vec{n}} : \mathrm{GL}_{\vec{n}} \hookrightarrow \mathrm{GL}_N$ $\left(N = \sum_i n_i\right)$ be the natural map. Let

$$\Phi_{\vec{n}} = i_{\vec{n}}(\Phi_{n_1}, \ldots, \Phi_{n_j}),$$

where Φ_n is the matrix in GL_n with entries $(\Phi_n)_{ij} = (-1)^{i+1}\delta_{i,n+1-j}$.

Let K be some local non-archimedean local field, and let H be a connected reductive group over K. We denote by $\mathrm{Irr}(H(K))$ (resp., $\mathrm{Irr}_l(H(K))$) the set of isomorphism classes of irreducible admissible representations of $G(K)$ over $\mathbb{C}$ (resp., over $\bar{\mathbb{Q}}_l$). Let $C_c^\infty(H(K))$ be the space of smooth compactly supported $\mathbb{C}$-valued functions on $H(K)$. Let P be a K-rational parabolic subgroup of H with a Levi subgroup M. For $\pi_M \in \mathrm{Irr}(M(K))$ and $\pi \in \mathrm{Irr}(H(K))$, we can define the normalized Jacquet module $J_P^H(\pi)$ and the normalized parabolic induction $\text{n-Ind}_P^H \pi_M$. We can define a character $\delta_P : M(K) \to \mathbb{R}_{>0}^\times$ by

$$\delta_P(m) = \left|\det\bigl(\mathrm{ad}(m)\bigr)\big|_{\mathrm{Lie}(P)/\mathrm{Lie}(M)}\right|_K.$$

We can view δ_P as a character valued in $\bar{\mathbb{Q}}_l^\times$ via ι_l^{-1}.

If

$$J^{(h)}(\mathbb{Q}_p) \simeq D^\times_{K,1/(n-h)} \times \mathrm{GL}_h(K),$$

where $K/\mathbb{Q}_p$ is finite, then we define $\bar{\delta}^{1/2}_{P(J^{(h)})}(g) := \delta^{1/2}_{P_{n-h,h}}(g^*)$, where $g^* \in \mathrm{GL}_{n-h}(K) \times \mathrm{GL}_h(K)$ is any element whose conjugacy class matches that of g. If

$$J^{(h_1,h_2)} \simeq \mathrm{GL}_1 \times \prod_{i=1}^{2} (D^\times_{F_{\mathfrak{p}_i},1/(n-h_i)} \times R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{h_i}) \times \prod_{i>2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_n,$$

we define $\bar{\delta}^{1/2}_{P(J^{(h_1,h_2)})} : J^{(h_1,h_2)}(\mathbb{Q}_p) \to \bar{\mathbb{Q}}_l^\times$ to be the product of the characters $\bar{\delta}^{1/2}_{P(J^{(h_i)})}$ for $i = 1, 2$.

Let $\vec{n} = (n_i)_{i=1}^r$ for some $r \in \{1, 2\}$ and $n_i \in \mathbb{Z}_{>0}$. Let $G_{\vec{n}}$ be the $\mathbb{Q}$-group defined by

$$G_{\vec{n}}(R) = \left\{(\lambda, g_i) \in \mathrm{GL}_1(R) \times \mathrm{GL}_{\vec{n}}(F \otimes_{\mathbb{Q}} R) \;\middle|\; g_i \cdot \Phi_{\vec{n}} \cdot {}^t g_i^c = \lambda \Phi_{\vec{n}}\right\}$$

for any $\mathbb{Q}$-algebra R. For any $\vec{n}$, the group $G_{\vec{n}}$ is quasi-split over $\mathbb{Q}$. In particular, our unitary group G is an inner form of G_n. Since G is quasi-split at all finite places, there exists an isomorphism

$$G \times_{\mathbb{Q}} \mathbb{A}^\infty \simeq G_n \times_{\mathbb{Q}} \mathbb{A}^\infty;$$

we fix such an isomorphism.

Also define

$$\mathbb{G}_{\vec{n}} = R_{E/\mathbb{Q}}(G_{\vec{n}} \times_{\mathbb{Q}} E).$$

Let θ denote the action on $\mathbb{G}_{\vec{n}}$ induced by (id, c) on $G_{\vec{n}} \times_{\mathbb{Q}} E$. Let $\epsilon : \mathbb{Z} \to \{0, 1\}$ be the unique map such that $\epsilon(n) \equiv n \pmod 2$. Let $\varpi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ be any Hecke character such that $\varpi|_{\mathbb{A}^\times/\mathbb{Q}^\times}$ is the composite of $\mathrm{Art}_{\mathbb{Q}}$ and the natural surjective character $W_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(E/\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\}$. Using the Artin map Art_E, we view ϖ as a character $W_E \to \mathbb{C}^\times$ as well.

Assume that $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \subset \mathrm{Spl}_{F/F_2,\mathbb{Q}}$.

Let $\mathcal{E}^{\mathrm{ell}}(G_n)$ be a set of representatives of isomorphism classes of elliptic endoscopic triples for G_n over $\mathbb{Q}$. Then $\mathcal{E}^{\mathrm{ell}}(G_n)$ can be identified with the set of triples

$$\{G_n, s_n, \eta_n\} \cup \{G_{n_1,n_2}, s_{n_1,n_2}, \eta_{n_1,n_2} \mid n_1 + n_2 = 0, n_1 \geq n_2 > 0\},$$

where (n_1, n_2) may be excluded in some cases. As we are only interested in the stable part of the cohomology of Igusa varieties, we are not concerned with these exclusions so we ignore them in this paper. Here $s_n = 1 \in \hat{G}_n, s_{n_1,n_2} = (1, (I_{n_1}, -I_{n_2})) \in \hat{G}_{n_1,n_2}$, $\eta_n : \hat{G}_n \to \hat{G}_n$ is the identity map, whereas

$$\eta_{n_1,n_2} : \big(\lambda, (g_1, g_2)\big) \mapsto \left(\lambda, \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}\right).$$

We can extend η_{n_1,n_2} to a morphism of L-groups, which sends $z \in W_E$ to

$$\left(\varpi(z)^{-N(n_1,n_2)}, \begin{pmatrix} \varpi(z)^{\epsilon(n-n_1)} \cdot I_{n_1} & 0 \\ 0 & \varpi(z)^{\epsilon(n-n_2)} \cdot I_{n_2} \end{pmatrix}\right) \rtimes z.$$

Similarly, we can also define a morphism of L-groups

$$\tilde{\zeta}_{n_1,n_2} :{}^L \mathbb{G}_{n_1,n_2} \to {}^L \mathbb{G}_n,$$

which extends the map

$$\zeta_{n_1,n_2} : \hat{\mathbb{G}}_{n_1,n_2} \to \hat{\mathbb{G}}_n,$$

$$\big(\lambda_+, \lambda_-, (g_{\sigma,1}, g_{\sigma,2})\big) \mapsto \left(\lambda_+, \lambda_- \begin{pmatrix} g_{\sigma,1} & 0 \\ 0 & g_{\sigma,2} \end{pmatrix}\right).$$

(See [Sh3, Section 3.2] for the precise definition.) We have the following commutative diagram of L-morphisms

$$\begin{tikzcd} {}^L G_{n_1,n_2} \arrow[r, "\tilde{\eta}_{n_1,n_2}"] \arrow[d, "BC_{n_1,n_2}"'] & {}^L G_n \arrow[d, "BC_n"] \\ {}^L \mathbb{G}_{n_1,n_2} \arrow[r, "\tilde{\zeta}_{n_1,n_2}"'] & {}^L \mathbb{G}_n \end{tikzcd}$$

We proceed to define local transfers for each of the arrows in the above commutative diagram so that these transfers are compatible.

Choose the normalization of the local transfer factor $\Delta_v(\ ,\ )^{G_n}_{G_{\vec{n}}}$ defined in [Sh3, Section 3.4]. It is possible to give a concrete description of the $\Delta_v(\ ,\ )^{G_n}_{G_{\vec{n}}}$-transfer at finite places v of $\mathbb{Q}$ between functions in $C_c^\infty(G_n(\mathbb{Q}_v))$ and functions in $C_c^\infty(G_{n_1,n_2}(\mathbb{Q}_v))$ as long as v satisfies at least one of the following conditions:

- $v \in \mathrm{Unr}_{F/\mathbb{Q}}$ and $v \notin \mathrm{Ram}_{\mathbb{Q}}(\varpi)$;
- $v \in \mathrm{Spl}_{E/\mathbb{Q}}$;
- $v \in \mathrm{Spl}_{F/F_2,\mathbb{Q}}$ and $v \notin \mathrm{Spl}_{E/\mathbb{Q}}$.

The transfer $\phi_v^{n_1,n_2}$ of $\phi_v^n \in C_c^\infty(G_n(\mathbb{Q}_v))$ and ϕ_v^n satisfy an identity involving orbital integrals. Since we are assuming that $\mathrm{Ram}_{F/\mathbb{Q}} \subseteq \mathrm{Spl}_{F/F_2,\mathbb{Q}}$, we can define the transfer at all places v of $\mathbb{Q}$.

It is also possible to define a transfer of pseudocoefficients at infinity. Consider $(G_{\vec{n}}, s_{\vec{n}}, \eta_{\vec{n}}) \in \mathcal{E}^{\mathrm{ell}}(G_n)$, which is also an endoscopic triple for G. Fix real elliptic maximal tori $T \subset G$ and $T_{G_{\vec{n}}} \subset G_{\vec{n}}$ together with an $\mathbb{R}$-isomorphism $j : T_{G_{\vec{n}}} \xrightarrow{\sim} T$. Also fix a Borel subgroup B of G over $\mathbb{C}$ containing $T_{\mathbb{C}}$. Shelstad [She] defined the transfer factor $\Delta_{j,B}$.

Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$. Define $\chi_\xi : A_{G,\infty} \to \mathbb{C}$ to be the restriction of ξ to $A_{G,\infty}$ (the connected component of the identity in the $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus in the center of G). Choose $K_\infty \subset G(\mathbb{R})$ to be a maximal compact subgroup (admissible in the sense of [A]), and define

$$q(G) = \frac{1}{2}\dim\bigl(G(\mathbb{R})/K_\infty A_{G,\infty}\bigr) = 2n-2.$$

For each $\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{R}), \xi^\vee)$ there exists $\phi_\pi \in C_c^\infty(G(\mathbb{R}), \chi_\xi)$, a pseudocoefficient for π. Any discrete L-parameter $\varphi_{G_{\vec{n}}}$ such that $\tilde{\eta}_{\vec{n}}\varphi_{G_{\vec{n}}} \sim \varphi_\xi$ corresponds to an L-packet of the form $\Pi_{\mathrm{disc}}(G_{\vec{n}}(\mathbb{R}), \xi(\varphi_{G_{\vec{n}}})^\vee)$. Define

$$\phi_{G_{\vec{n}},\xi(\varphi_{G_{\vec{n}}})} := \frac{1}{|\Pi(\varphi_{\vec{n}})|}\sum_{\pi_{G_{\vec{n}}}} \phi_{\pi_{G_{\vec{n}}}}$$

and

$$\phi_\pi^{G_{\vec{n}}} := (-1)^{q(G)} \sum_{\tilde{\eta}\varphi_{G_{\vec{n}}} \sim \varphi_\xi} \langle a_{\omega_*(\varphi_{G_{\vec{n}}},\xi)\omega_\pi}, s\rangle \det\bigl(\omega_*(\varphi_{G_{\vec{n}}},\xi)\bigr) \cdot \phi_{G_{\vec{n}},\xi(\varphi_{G_{\vec{n}}})}.$$

Then $\phi_\pi^{G_{\vec{n}}}$ is a $\Delta_{j,B}$-transfer of ϕ_π.

We now review the base change for the groups $G_{\vec{n}}$ and $\mathbb{G}_{\vec{n}}$. Define the group

$$\mathbb{G}_{\vec{n}}^+ := (R_{E/\mathbb{Q}}\mathrm{GL}_1 \times R_{F/\mathbb{Q}}\mathrm{GL}_{\vec{n}}) \rtimes \{1, \theta\},$$

where $\theta(\lambda, g)\theta^{-1} = (\lambda^c, \lambda^c g^\#)$ and $g^\# = \Phi_{\vec{n}}{}^t g^c \Phi_{\vec{n}}^{-1}$. If we denote by $\mathbb{G}_{\vec{n}}^0$ and $\mathbb{G}_{\vec{n}}^0\theta$ the cosets of $\{1\}$ and $\{\theta\}$ in $\mathbb{G}_{\vec{n}}^+$, then $\mathbb{G}_{\vec{n}}^+ = \mathbb{G}_{\vec{n}}^0 \coprod \mathbb{G}_{\vec{n}}^0\theta$. There is a natural $\mathbb{Q}$-isomorphism $\mathbb{G}_{\vec{n}} \xrightarrow{\sim} \mathbb{G}_{\vec{n}}^0$ which extends to

$$\mathbb{G}_{\vec{n}} \rtimes \mathrm{Gal}(E/\mathbb{Q}) \xrightarrow{\sim} \mathbb{G}_{\vec{n}}^+$$

so that $c \in \mathrm{Gal}(E/\mathbb{Q})$ maps to θ.

Let v be a place of $\mathbb{Q}$. A representation $\Pi_v \in \mathrm{Irr}(\mathbb{G}_{\vec{n}}(\mathbb{Q}_v))$ is called *θ-stable* if $\Pi_v \simeq \Pi_v \circ \theta$ as representations of $\mathbb{G}_{\vec{n}}(\mathbb{Q}_v)$. If that is the case, then we can choose an operator A_{Π_v} on the representation space of Π_v which induces $\Pi_v \xrightarrow{\sim} \Pi_v \circ \theta$ and which satisfies $A_{\Pi_v}^2 = \mathrm{id}$. Such an operator is called *normalized* and it is pinned down up to sign. We can similarly define the notion of θ-stable for $\Pi^{\mathfrak{S}} \in \mathrm{Irr}(\mathbb{G}_{\vec{n}}(\mathbb{A}^{\mathfrak{S}}))$ and a corresponding intertwining operator $A_{\Pi^{\mathfrak{S}}}$ for any finite set $\mathfrak{S}$ of places of $\mathbb{Q}$. There is a correspondence between θ-stable representations of $\mathbb{G}_{\vec{n}}(\mathbb{Q}_v)$ together with a normalized intertwining operator and representations of $\mathbb{G}_{\vec{n}}^+(\mathbb{Q}_v)$. We also mention that in order for a representation $\Pi \in \mathrm{Irr}(\mathbb{G}_{\vec{n}}(\mathbb{A}))$ to be θ-stable it is necessary and sufficient that $\Pi = \psi \otimes \Pi^1$ satisfy

- $(\Pi^1)^\vee \simeq \Pi^1 \circ c$, and
- $\prod_{i=1}^r \psi_i = \psi^c/\psi$, where $\psi_{\Pi^1} = \psi_1 \otimes \cdots \otimes \psi_r$ is the central character of Π^1.

Now we discuss BC-matching functions. It is possible to construct for each finite place v of $\mathbb{Q}$ and $f_v \in C_c^\infty(\mathbb{G}_{\vec{n}}(\mathbb{Q}_v))$ a function $\phi_v \in C_c^\infty(G_{\vec{n}}(\mathbb{Q}_v))$, which is the BC-transfer of f_v. The transfer can be described concretely in the cases $v \in \mathrm{Unr}_{F/\mathbb{Q}}$ and $v \in \mathrm{Spl}_{F/F_2,\mathbb{Q}}$, except that in the case $v \in \mathrm{Unr}_{F/\mathbb{Q}}$, we have the condition that f_v must be unramified. Moreover, we also have an explicit map

$$BC_{\vec{n}} : \mathrm{Irr}^{(\mathrm{ur})}\big(G_{\vec{n}}(\mathbb{Q}_v)\big) \to \mathrm{Irr}^{(\mathrm{ur})\theta\mathrm{st}}\big(\mathbb{G}_{\vec{n}}(\mathbb{Q}_v)\big),$$

where the representations must be unramified in the case $v \in \mathrm{Unr}_{F/\mathbb{Q}}$ and where there is no restriction in the case $v \in \mathrm{Spl}_{F/F_2,\mathbb{Q}}$. There are normalized operators $A^0_{\Pi_v} : \Pi_v \xrightarrow{\sim} \Pi_v \circ \theta$ such that if $\Pi_v = BC_{\vec{n}}(\pi_v)$ and if ϕ_v and f_v are BC-matching functions, then

$$\mathrm{tr}\big(\Pi_v(f_v) A^0_{\Pi_v}\big) = \mathrm{tr}\, \pi_v(\phi_v).$$

Note that the left-hand side of the above equality computes the trace of $f_v\theta$, the function on $\mathbb{G}_{\vec{n}}\theta$ obtained from f_v via translation by θ.

The next step is to consider the base change at ∞. Let $\xi_{\vec{n}}$ be an irreducible algebraic representation of $G_{\vec{n},\mathbb{C}}$. Consider the natural isomorphism

$$\mathbb{G}_{\vec{n}}(\mathbb{C}) \simeq G_{\vec{n}}(\mathbb{C}) \times G_{\vec{n}}(\mathbb{C}).$$

We can define a representation $\Xi_{\vec{n}}$ of $\mathbb{G}_{\vec{n}}$ by $\Xi_{\vec{n}} := \xi_{\vec{n}} \otimes \xi_{\vec{n}}$. It is possible to find an irreducible, θ-stable, generic unitary representation $\Pi_{\Xi_{\vec{n}}} \in \mathrm{Irr}(\mathbb{G}_{\vec{n}}(\mathbb{R}), \chi_{\xi_{\vec{n}}}^{-1})$ together with a normalized operator $A^0_{\Pi_{\Xi_{\vec{n}}}}$ and a function $f_{\mathbb{G}_{\vec{n}},\Xi_{\vec{n}}} \in C_c^\infty(\mathbb{G}(\mathbb{R}), \chi_{\xi_{\vec{n}}})$ such that

- $\Pi_{\Xi_{\vec{n}}}$ is the base change of the L-packet $\Pi_{\mathrm{disc}}(G_{\vec{n}}(\mathbb{R}), \xi_{\vec{n}}^\vee)$,
- $\mathrm{tr}(\Pi_{\Xi_{\vec{n}}}(f_{\mathbb{G}_{\vec{n}},\Xi_{\vec{n}}}) \circ A^0_{\Pi_{\Xi_{\vec{n}}}}) = 2$, and
- $f_{\mathbb{G}_{\vec{n}},\Xi_{\vec{n}}}$ and $\phi_{G_{\vec{n}},\xi_{\vec{n}}}$ are BC-matching functions (where $\phi_{G_{\vec{n}},\xi_{\vec{n}}}$ is defined as a pseudocoefficient for the L-packet $\Pi_{\mathrm{disc}}(G_{\vec{n}}(\mathbb{R}), \xi_{\vec{n}}^\vee)$).

The transfer for $\tilde{\zeta}_{n_1,n_2}$ can be defined explicitly since the groups $\mathbb{G}_{\vec{n}}$ are essentially products of general linear groups. It can be checked that, for all finite places v of $\mathbb{Q}$, the transfers are compatible. For $v = \infty$, we have that the compatibility relation on the representation-theoretic side follows directly from the commutative diagram of L-morphisms.

Now we describe the transfer factors $\Delta_v(\ ,\ )^G_{G_{\vec{n}}}$. At $v \neq \infty$, we can choose

$$\Delta_v(\ ,\ )^G_{G_{\vec{n}}} = \Delta^0_v(\ ,\ )^{G_n}_{G_{\vec{n}}}$$

via the fixed isomorphism $G \times_{\mathbb{Q}} \mathbb{A}^\infty \simeq G_n \times_{\mathbb{Q}} \mathbb{A}^\infty$. We choose the unique $\Delta_\infty(\ ,\ )^G_{G_{\vec{n}}}$ such that the product formula

$$\prod_v \Delta_v(\gamma_{G_{\vec{n}}}, \gamma)^G_{G_{\vec{n}}} = 1$$

holds for any $\gamma \in G(\mathbb{Q})$ semisimple and $\gamma_{G_{\vec{n}}} \in G_{\vec{n}}(\mathbb{A})$ a $(G, G_{\vec{n}})$-regular semisimple element such that γ and $\gamma_{G_{\vec{n}}}$ have matching stable conjugacy classes. Let $e_{\vec{n}}(\Delta_\infty) \in \mathbb{C}^\times$ denote the constant for which

$$\Delta_\infty(\gamma_{G_{\vec{n}}}, \gamma)^G_{G_{\vec{n}}} = e_{\vec{n}}(\Delta_\infty)\Delta_{j,B}(\gamma_{G_{\vec{n}}}, \gamma)$$

holds. Note that, for $\vec{n} = (n)$, $e_{\vec{n}}(\Delta_\infty) = 1$.

Let $\phi^{\infty,p} \cdot \phi'_p \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J^{(h_1,h_2)}(\mathbb{Q}_p))$ be a complex-valued acceptable function. (For a definition of the notion of acceptable function, see [Sh1, Definition 6.2].) For each $G_{\vec{n}} \in \mathcal{E}^{\mathrm{ell}}(G)$, we define the function $\phi^{\vec{n}}_{\mathrm{Ig}}$ on $G_{\vec{n}}(\mathbb{A})$ (assuming that $\phi^{\infty,p} = \prod_{v \neq p,\infty} \phi_v$). For $v \neq p, \infty$, we take $\phi^{\vec{n}}_{\mathrm{Ig},v} \in C_c^\infty(G_{\vec{n}}(\mathbb{Q}_v))$ to be the $\Delta_v(\ ,\ )^G_{G_{\vec{n}}}$-transfer of ϕ_v. We take

$$\phi^{\vec{n}}_{\mathrm{Ig},\infty} := e_{\vec{n}}(\Delta_\infty) \cdot (-1)^{q(G)} \langle \mu_h, s_{\vec{n}} \rangle \sum_{\varphi_{\vec{n}}} \det\bigl(\omega_*(\varphi_{G_{\vec{n}}})\bigr) \cdot \phi_{G_{\vec{n}},\xi(\varphi_{\vec{n}})},$$

where $\varphi_{\vec{n}}$ runs over L-parameters such that $\tilde{\eta}_{\vec{n}} \varphi_{\vec{n}} \sim \varphi_\xi$ and where $\xi(\varphi_{\vec{n}})$ is the algebraic representation of $G_{\vec{n},\mathbb{C}}$ such that the L-packet associated to $\varphi_{\vec{n}}$ is $\Pi_{\mathrm{disc}}(G_{\vec{n}}(\mathbb{R}), \xi(\varphi_{\vec{n}})^\vee)$.

We also take

$$\phi^{\vec{n}}_{\mathrm{Ig},p} \in C_c^\infty\bigl(G_{\vec{n}}(\mathbb{Q}_p)\bigr)$$

to be the function constructed from ϕ'_p in [Sh2, Section 6.3]. We summarize the construction of $\phi^{\vec{n}}_{\mathrm{Ig},p}$ in the case $\vec{n} = (n)$. By definition (see the formula above [Sh2, Lemma 6.5]),

$$\phi^{\vec{n}}_{\mathrm{Ig},p} = \sum_{(M_{G_n}, s_{G_n}, \eta_{G_n})} c_{M_{G_n}} \cdot \tilde{\phi}_p^{M_{G_n}},$$

where the sum is taken over G-endoscopic triples for $J^{(h_1,h_2)}$. The set $\mathcal{I}(M_{G_n}, G_n)$ (which can be identified with a set of cosets of $\mathrm{Out}(M_{G_n}, s_{G_n}, \eta_{G_n})$) consists of only one element in our case, so we suppress the index $i \in \mathcal{I}(M_{G_n}, G_n)$ in $\tilde{\phi}_p^{M_{G_n},i}$. Each $\tilde{\phi}_p^{M_{G_n}} \in C_c^\infty(G_n(\mathbb{Q}_p))$ is constructed from a function $\phi_p^{M_{G_n}} \in C_c^\infty(M_{G_n}(\mathbb{Q}_p))$, which is a $\Delta_p(\ ,\ )^{J^{(h_1,h_2)}}_{M_{G_n}}$-transfer of a normalized ϕ'_p.

The following proposition is [Sh2, Theorem 7.2].

PROPOSITION 6.1
If $\phi^{\infty,p}\cdot\phi'_p\in C_c^\infty(G(\mathbb{A}^{\infty,p})\times J^{(h_1,h_2)}(\mathbb{Q}_p))$ is acceptable, then

$$\begin{aligned}&\operatorname{tr}\bigl(\phi^{\infty,p}\cdot\phi'_p \bigm| \iota_l H_c(\mathrm{Ig}^{(h_1,h_2)},\mathcal{L}_\xi)\bigr)\\&\quad=(-1)^{h_1+h_2}|\ker^1(\mathbb{Q},G)|\sum_{G_{\vec n}}\iota(G,G_{\vec n})ST_e^{G_{\vec n}}(\phi_{\mathrm{Ig}}^{\vec n}),\end{aligned}$$

where the sum runs over the set $\mathcal{E}^{\mathrm{ell}}(G)$ of elliptic endoscopic triples $(G_{\vec n},s_{\vec n},\eta_{\vec n})$.

Remark 6.2
Theorem 7.2 of [Sh2] is proved under the "unramified hypothesis" (see [Sh3]). However, the only place where this hypothesis is needed is in the proof of [Sh1, Lemma 11.1]. Lemma 5.5 provides an alternative to the proof of [Sh1, Lemma 11.1] in our situation, so the results of [Sh1] and [Sh2] carry over. (For details, see the discussion in the beginning of [Sh3, Section 5.2].) The sign $(-1)^{h_1+h_2}$ does not show up in the statement of the theorem in [Sh2], but we need to include it because our convention for the alternating sum of the cohomology differs from the usual one by $(-1)^{h_1+h_2}$.

The constants $\iota(G,G_{\vec n})=\tau(G)\tau(G_{\vec n})^{-1}|\operatorname{Out}(G_{\vec n},s_{\vec n},\eta_{\vec n})|^{-1}$ can be computed explicitly. We mention that

$$|\operatorname{Out}(G_{\vec n},s_{\vec n},\eta_{\vec n})|=\begin{cases}2 & \text{if } \vec n=(\frac{n}{2},\frac{n}{2}),\\ 1 & \text{otherwise.}\end{cases}$$

We also have by [Sh3, Corollary 4.7] the relation

$$I_{\mathrm{geom}}^{\mathbb{G}_{\vec n}\theta}(f\theta)=\tau(G_{\vec n})^{-1}\cdot ST_e^{G_{\vec n}}(\phi)$$

when ϕ and f are BC-matching functions, that is,

$$\phi=\phi^{\mathfrak{S}}\cdot\phi_{\mathfrak{S}_{\mathrm{fin}}}\cdot\phi_{G_{\vec n},\xi}\qquad\text{and}\qquad f=f^{\mathfrak{S}}\cdot f_{\mathfrak{S}_{\mathrm{fin}}}f_{G_{\vec n},\Xi},$$

with $\phi^{\mathfrak{S}}$ a BC-transfer of $f^{\mathfrak{S}}$ and $\phi_{\mathfrak{S}_{\mathrm{fin}}}$ a BC-transfer of $f_{\mathfrak{S}_{\mathrm{fin}}}$. Thus, assuming that for each $\vec n$ there exists $f^{\vec n}$ such that $\phi_{\mathrm{Ig}}^{\vec n}$ and $f^{\vec n}$ are BC-matching, we can write

$$\operatorname{tr}\bigl(\phi^{\infty,p}\cdot\phi'_p \bigm| \iota_l H_c(\mathrm{Ig}^{(h_1,h_2)},\mathcal{L}_\xi)\bigr)=|\ker^1(\mathbb{Q},G)|\cdot\tau(G)\sum_{\mathbb{G}_{\vec n}}\epsilon_{\vec n}I_{\mathrm{geom}}^{\mathbb{G}_{\vec n}\theta}(f^{\vec n}\theta),$$

where $\epsilon_{\vec n}=1/2$ if $\vec n=(n/2,n/2)$ or 1 otherwise.

Furthermore, the twisted trace formula by Arthur [A] is an equality between

$$I_{\mathrm{spec}}^{\mathbb{G}_{\vec n}\theta}(f\theta)=I_{\mathrm{geom}}^{\mathbb{G}_{\vec n}\theta}(f\theta).$$

By combining Proposition 4.8 and [Sh3, Corollary 4.14], we can compute $I_{\mathrm{spec}}^{\mathbb{G}_{\vec{n}}\theta}(f\theta)$ as

$$\sum_M \frac{|W_M|}{|W_{\mathbb{G}_{\vec{n}}}|} |\det(\Phi_{\vec{n}}^{-1}\theta - 1)_{\mathfrak{a}_M^{\mathbb{G}_{\vec{n}}\theta}}|^{-1} \sum_{\Pi_M} \mathrm{tr}\big(\text{n-Ind}_Q^{\mathbb{G}_{\vec{n}}}(\Pi_M)_\xi(f)\big) \circ A'_{\text{n-Ind}_Q^{\mathbb{G}_{\vec{n}}}(\Pi_M)_\xi},$$

where M runs over $\mathbb{Q}$-Levi subgroups of $\mathbb{G}_{\vec{n}}$ containing a fixed minimal Levi and where Q is a parabolic containing M as a Levi. The rest of the notation is defined on [Sh3, pp. 31–32]. Note that $A'_{\text{n-Ind}_Q^{\mathbb{G}_{\vec{n}}}(\Pi_M)\xi}$ is a normalized intertwining operator for $\text{n-Ind}_Q^{\mathbb{G}_{\vec{n}}}(\Pi_M)_\xi$.

We are particularly interested in making the above formula explicit when $\vec{n} = (n)$. In that case, $I_{\mathrm{spec}}^{\mathbb{G}_{\vec{n}}\theta}(f\theta)$ is a sum of

$$\frac{1}{2}\sum_{\Pi'} \mathrm{tr}\big(\Pi'_\xi(f) A'_{\Pi'_\xi}\big),$$

where Π' runs over θ-stable subrepresentations of $R_{\mathbb{G}_n,\mathrm{disc}}$, and of

$$\sum_{M \subsetneq \mathbb{G}_n} \frac{|W_M|}{|W_{\mathbb{G}_n}|} |\det(\Phi_n^{-1}\theta - 1)_{\mathfrak{a}_M^{\mathbb{G}_n\theta}}|^{-1} \sum_{\Pi'_M} \mathrm{tr}\big(\text{n-Ind}_Q^{\mathbb{G}_n}(\Pi'_M)_\xi(f) \circ A'_{\text{n-Ind}_Q^{\mathbb{G}_n}(\Pi'_M)_\xi}\big),$$

where Π'_M runs over $\Phi_n^{-1}\theta$-stable subrepresentations of $R_{M,\mathrm{disc}}$.

Consider the finite set $\mathcal{E}_p^{\mathrm{eff}}(J^{(h_1,h_2)}, G, G_{\vec{n}})$ consisting of certain isomorphism classes of G-endoscopic triples $(M_{G_{\vec{n}}}, s_{\vec{n}}, \eta_{\vec{n}})$ for $J^{(h_1,h_2)}$. This set is defined in [Sh2, Section 6.2]. Let $c_{M_{G_{\vec{n}}}} \in \{\pm 1\}$ be the constant assigned to each triple in [Sh2]. If b is the isocrystal corresponding to (h_1,h_2), let $M^{(h_1,h_2)}(\mathbb{Q}_p)$ be the centralizer of $\nu_G(b)$. The isocrystal b can be described as $(b_{p,0}, b_{\mathfrak{p}_1}, \dots, b_{\mathfrak{p}_r})$, where $b_{\mathfrak{p}_i}$ has slopes 0 and $1/(n-h_i)$ for $i = 1,2$ and slope 0 for $i > 2$. Then $M^{(h_1,h_2)}$ is a $\mathbb{Q}_p$-rational Levi subgroup of G. We define a group morphism

$$\text{n-Red}_{\vec{n}}^{(h_1,h_2)} : \mathrm{Groth}\big(G_{\vec{n}}(\mathbb{Q}_p)\big) \to \mathrm{Groth}\big(J^{(h_1,h_2)}(\mathbb{Q}_p)\big)$$

as the composition of the following maps

$$\mathrm{Groth}\big(G_{\vec{n}}(\mathbb{Q}_p)\big) \to \bigoplus_{(M_{G_{\vec{n}}}, s_{G_{\vec{n}}}, \eta_{G_{\vec{n}}})} \mathrm{Groth}\big(M_{G_{\vec{n}}}(\mathbb{Q}_p)\big) \xrightarrow{\oplus \tilde{\eta}_{G_{\vec{n}},*}} \mathrm{Groth}\big(M^{(h_1,h_2)}(\mathbb{Q}_p)\big)$$
$$\xrightarrow{LJ^{M^{(h_1,h_2)}}_{J^{(h_1,h_2)}}} \mathrm{Groth}\big(J^{(h_1,h_2)}(\mathbb{Q}_p)\big).$$

The sum runs over $(M_{G_{\vec{n}}}, s_{\vec{n}}, \eta_{\vec{n}}) \in \mathcal{E}_p^{\mathrm{eff}}(J^{(h_1,h_2)}, G, G_{\vec{n}})$. The first map is the direct sum of maps $\mathrm{Groth}(G_{\vec{n}}(\mathbb{Q}_p)) \to \mathrm{Groth}(M_{G_{\vec{n}}}(\mathbb{Q}_p))$ which are given by $\bigoplus_i c_{M_{G_{\vec{n}}}} \cdot$

$J^{G_{\vec{n}}}_{P(iM_{G_{\vec{n}}})^{\mathrm{op}}}$, where $i \in \mathcal{I}(M_{G_{\vec{n}}}, G_{\vec{n}})$ is a $\mathbb{Q}_p$-embedding $M_{G_{\vec{n}}} \hookrightarrow G_{\vec{n}}$ and where $P(iM_{G_{\vec{n}}})$ is a parabolic subgroup of $G_{\vec{n}}$ which contains $i(M_{G_{\vec{n}}})$ as a Levi subgroup. The map $\tilde{\eta}_{G_{\vec{n}},*}$ is a functorial transfer with respect to the L-morphism $\tilde{\eta}_{G_{\vec{n}}}$. The third map $LJ^{M^{(h_1,h_2)}}_{J^{(h_1,h_2)}}$ is the Jacquet–Langlands map on Grothendieck groups. We also define

$$\mathrm{Red}^{(h_1,h_2)}_{\vec{n}}(\pi_{G_{\vec{n}},p}) := \text{n-}\mathrm{Red}^{(h_1,h_2)}_{\vec{n}}(\pi_{G_{\vec{n}},p}) \otimes \bar{\delta}^{1/2}_{P(J^{(h_1,h_2)})}.$$

We can describe all the groups and maps above very explicitly in the case $\vec{n} = (n)$. Indeed, $\mathcal{E}^{\mathrm{eff}}_p(J^{(h_1,h_2)}, G, G_n)$ has a unique isomorphism class represented by

$$(M_{G_n}, s_{G_n}, \eta_{G_n}) = (M^{(h_1,h_2)}, 1, \mathrm{id}).$$

The set $\mathcal{I}(M_{G_n}, G_n)$ is also a singleton in this case, so we suppress i everywhere. This means that we can also take $\tilde{\eta}_{G_n} = \mathrm{id}$ and $\tilde{\eta}_{G_n,*} = \mathrm{id}$, and by [Sh2, Remark 6.4], we may also take $c_{M_{G_n}} = e_p(J^{(h_1,h_2)})$, which is the Kottwitz sign of the $\mathbb{Q}_p$-group $J^{(h_1,h_2)}$. There are isomorphisms

$$\begin{aligned} G(\mathbb{Q}_p) &\simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(F_{\mathfrak{p}_1}) \times \mathrm{GL}_n(F_{\mathfrak{p}_2}) \times \prod_{i>2} \mathrm{GL}_n(F_{\mathfrak{p}_i}), \\ M^{(h_1,h_2)}(\mathbb{Q}_p) &\simeq \mathbb{Q}_p^\times \times \big(\mathrm{GL}_{n-h_1}(F_{\mathfrak{p}_1}) \times \mathrm{GL}_{h_1}(F_{\mathfrak{p}_1})\big) \\ &\qquad \times \big(\mathrm{GL}_{n-h_2}(F_{\mathfrak{p}_2}) \times \mathrm{GL}_{h_2}(F_{\mathfrak{p}_2})\big) \times \prod_{i>2} \mathrm{GL}_n(F_{\mathfrak{p}_i}), \\ J^{(h_1,h_2)}(\mathbb{Q}_p) &\simeq \mathbb{Q}_p^\times \times \big(D^\times_{F_{\mathfrak{p}_1},1/(n-h_1)} \times \mathrm{GL}_{h_1}(F_{\mathfrak{p}_1})\big) \\ &\qquad \times \big(D^\times_{F_{\mathfrak{p}_2},1/(n-h_2)} \times \mathrm{GL}_{h_2}(F_{\mathfrak{p}_2})\big) \times \prod_{i>2} \mathrm{GL}_n(F_{\mathfrak{p}_i}). \end{aligned}$$

Thus, $e_p(J^{(h_1,h_2)}) = (-1)^{2n-2-h_1-h_2}$. If we write $\pi_p = \pi_{p,0} \otimes \big(\bigotimes_i \pi_{\mathfrak{p}_i}\big)$, then we have

$$\begin{aligned} \mathrm{Red}^{(h_1,h_2)}_n(\pi_p) = (-1)^{h_1+h_2}\pi_{p,0} &\otimes \mathrm{Red}^{n-h_1,h_1}(\pi_{\mathfrak{p}_1}) \\ &\otimes \mathrm{Red}^{n-h_2,h_2}(\pi_{\mathfrak{p}_2}) \otimes \Big(\bigotimes_{i>2} \pi_{\mathfrak{p}_i}\Big). \end{aligned}$$

LEMMA 6.3
For any $\pi_p \in \mathrm{Groth}(G_n(\mathbb{Q}_p))$, we have

$$\mathrm{tr}\pi_p(\phi^n_{\mathrm{Ig},p}) = \mathrm{tr}\big(\mathrm{Red}^{(h_1,h_2)}_n(\pi_p)\big)(\phi'_p).$$

Proof
Set $M = M_{G_n}$. We know that $\phi^n_{\mathrm{Ig},p} = e_p(J^{(h_1,h_2)}) \cdot \tilde{\phi}^M_p$. By [Sh2, Lemma 3.9],

$$\operatorname{tr}\pi_p(\tilde{\phi}_p^M) = \operatorname{tr}\big(J^{G_n}_{P_M^{\mathrm{op}}}(\pi_p)\big)(\phi_p^M).$$

Here ϕ_p^M is a $\Delta_p(\ ,\ )_M^{J^{(h_1,h_2)}} \equiv e_p(J^{(h_1,h_2)})$-transfer of $\phi_p^0 = \phi'_p \cdot \bar{\delta}^{1/2}_{P(J^{(h_1,h_2)})}$ (by [Sh2, Remark 6.4], we have an explicit description of the transfer factor). Let $\pi_{M,p} = J^{G_n}_{P_M^{\mathrm{op}}}(\pi_p)$.

Note that M is a product of general linear groups and that $J^{(h_1,h_2)}$ is an inner form of M. Lemma 2.18 and Remark 2.19 of [Sh2] ensure that

$$\begin{aligned}\operatorname{tr}\pi_{M,p}(\phi_p^M) &= \operatorname{tr}\big(LJ_M^{J^{(h_1,h_2)}}(\pi_{M,p})(\phi_p^0)\big)\\ &= \operatorname{tr}\big(LJ_M^{J^{(h_1,h_2)}}(\pi_{M,p}) \otimes \bar{\delta}^{1/2}_{P(J^{(h_1,h_2)})}\big)(\phi'_p).\end{aligned}$$

This concludes the proof. □

LEMMA 6.4
Let $\vec{n} = (n_1, n_2)$ with $n_1 \geq n_2 > 0$. For any $\pi_p \in$ Groth$(G_{n_1,n_2}(\mathbb{Q}_p))$,

$$\operatorname{tr}\pi_p(\phi^{\vec{n}}_{\mathrm{Ig},p}) = \operatorname{tr}\big(\mathrm{Red}^{(h_1,h_2)}_{\vec{n}}(\pi_p)\big)(\phi'_p).$$

Proof
The proof is based on making explicit the construction of $\phi^{\vec{n}}_{\mathrm{Ig},p}$ from [Sh2, Section 6] together with the definition of the functor n-$\mathrm{Red}^{(h_1,h_2)}_{\vec{n}}$, which is a composition of the following maps:

$$\mathrm{Groth}\big(G_{\vec{n}}(\mathbb{Q}_p)\big) \to \bigoplus_{(M_{G_{\vec{n}}}, s_{G_{\vec{n}}}, \eta_{G_{\vec{n}}})} \mathrm{Groth}\big(M_{G_{\vec{n}}}(\mathbb{Q}_p)\big) \stackrel{\oplus\tilde{\eta}_{G_{\vec{n}},*}}{\longrightarrow} \mathrm{Groth}\big(M^{(h_1,h_2)}(\mathbb{Q}_p)\big)$$
$$\stackrel{LJ^{M^{(h_1,h_2)}}_{J^{(h_1,h_2)}}}{\longrightarrow} \mathrm{Groth}\big(J^{(h_1,h_2)}(\mathbb{Q}_p)\big).$$

Recall that

$$\phi^{\vec{n}}_{\mathrm{Ig},p} = \sum_{(M_{G_{\vec{n}}}, s_{G_{\vec{n}}}, \eta_{G_{\vec{n}}})} \sum_i c_{M_{G_{\vec{n}}}} \cdot \tilde{\phi}_p^{M_{G_{\vec{n}}},i}$$

as functions on $G_{\vec{n}}(\mathbb{Q}_p)$, where the first sum is taken over $\mathcal{E}_p^{\mathrm{eff}}(J^{(h_1,h_2)}, G, G_{\vec{n}})$ and the second sum is taken over $\mathcal{I}(M_{G_{\vec{n}}}, G_{\vec{n}})$. By [Sh2, Lemma 3.9], we have

$$\operatorname{tr}\pi_p(\tilde{\phi}_p^{M_{G_{\vec{n}}},i}) = \operatorname{tr}\big(J^{G_{\vec{n}}}_{P(iM_{G_{\vec{n}}})^{\mathrm{op}}}(\pi_p)\big)(\phi_p^{M_{G_{\vec{n}}}}), \tag{4}$$

where $\phi_p^{M_{G_{\vec{n}}}} \in C_c^\infty(M_{G_{\vec{n}}}(\mathbb{Q}_p))$ is a $\Delta_p(\ ,\ )_{M_{G_{\vec{n}}}}^{J^{(h_1,h_2)}}$-transfer of $\phi_p^0 = \phi'_p \cdot \bar{\delta}^{1/2}_{P(J^{(h_1,h_2)})}$. Equation (4) tells us that

$$\operatorname{tr} \pi_p(\phi^{\vec{n}}_{\mathrm{Ig},p}) = \sum_{(M_{G_{\vec{n}}}, s_{G_{\vec{n}}}, \eta_{G_{\vec{n}}})} \operatorname{tr}\big(f_{M_{G_{\vec{n}}}}(\pi_p)\big)(\phi_p^{M_{G_{\vec{n}}}}), \tag{5}$$

where $f_{M_{G_{\vec{n}}}}(\pi_p) = \bigoplus_i c_{M_{G_{\vec{n}}}} J^{G_{\vec{n}}}_{P(iM_{G_{\vec{n}}})}(\pi_p)$. The first map in the definition of $\mathrm{Red}^{(h_1,h_2)}_{\vec{n}}$ is the direct sum of $f_{M_{G_{\vec{n}}}}$ over all $(M_{G_{\vec{n}}}, s_{\vec{n}}, \eta_{\vec{n}})$.

The function $\phi_p^{M_{G_{\vec{n}}}}$ is a $\Delta_p(\ ,\ )^{M^{(h_1,h_2)}}_{M_{G_{\vec{n}}}}$-transfer of the function $\phi_p^* \in C_c^\infty(M^{(h_1,h_2)}(\mathbb{Q}_p))$, which is itself a transfer of ϕ_p^0 via $\Delta_p(\ ,\ )^{J^{(h_1,h_2)}}_{M^{(h_1,h_2)}} \equiv e_p(J^{(h_1,h_2)})$. (All transfer factors are normalized as in [Sh2].) We focus on making the $\Delta_p(\ ,\ )^{M^{(h_1,h_2)}}_{M_{G_{\vec{n}}}}$-transfer explicit first, for which we need to have a complete description of all endoscopic triples $(M_{G_{\vec{n}}}, s_{G_{\vec{n}}}, \eta_{G_{\vec{n}}})$.

We have the following isomorphisms over $\mathbb{Q}_p$:

$$G \simeq \mathrm{GL}_1 \times \prod_{i\geq 1} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_n,$$

$$G_{n_1,n_2} \simeq \mathrm{GL}_1 \times \prod_{i\geq 1} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n_1,n_2},$$

$$M^{(h_1,h_2)} \simeq \mathrm{GL}_1 \times \prod_{i=1}^{2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n-h_i,h_i} \times \prod_{i>2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_n,$$

$$J^{(h_1,h_2)} \simeq \mathrm{GL}_1 \times \prod_{i=1}^{2} (D^\times_{F_{\mathfrak{p}_i},1/(n-h_i)} \times \mathrm{GL}_{h_i}) \times \prod_{i>2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_n .$$

Consider also the following four groups over $\mathbb{Q}_p$, which can be thought of as Levi subgroups of G_{n_1,n_2} via the block diagonal embeddings

$$M_{G_{\vec{n}},1} := \mathrm{GL}_1 \times \prod_{i=1}^{2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n-h_i,h_i-n_2,n_2} \times \prod_{i>2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n_1,n_2},$$

$$M_{G_{\vec{n}},2} := \mathrm{GL}_1 \times \prod_{i=1}^{2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n-h_i,h_i-n_1,n_1} \times \prod_{i>2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n_1,n_2},$$

$$M_{G_{\vec{n}},3} := \mathrm{GL}_1 \times \prod_{i=1}^{2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n-h_i,h_i-n_i,n_i} \times \prod_{i>2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n_1,n_2},$$

$$M_{G_{\vec{n}},4} := \mathrm{GL}_1 \times \prod_{i=1}^{2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n-h_i,h_i-n_{3-i},n_{3-i}} \times \prod_{i>2} R_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{n_1,n_2} .$$

Note that we only define $M_{G_{\vec{n}},j}$ when it makes sense; for example, $M_{G_{\vec{n}},1}$ is defined only when $h_i \geq n_2$ for $i = 1, 2$. We define $\eta_{G_{\vec{n}},j} : \widehat{M_{G_{\vec{n}},j}} \to \widehat{M^{(h_1,h_2)}}$ to be the obvious block diagonal embedding. We also let

$$s_{M_{G_{\vec{n}},j}} = \big(1, (\pm 1, \pm 1, \pm 1)_{i=1,2}, (1,1)_{i>2}\big),$$

where the signs on the $F_{\mathfrak{p}_i}$-component are chosen such that $s_{M_{G_{\vec{n}},j}}$ is positive on the GL_{n_1}-block of the $F_{\mathfrak{p}_i}$-component and negative on the GL_{n_2}-block of the $F_{\mathfrak{p}_i}$-component.

It is easy to check, as on [Sh3, p. 42], that $\mathcal{E}_p^{\mathrm{eff}}(J^{(h_1,h_2)}, G, G_{\vec{n}})$ consists of those triples $(M_{G_{\vec{n}},j}, s_{G_{\vec{n}},j}, \eta_{G_{\vec{n}},j})$ which make sense. For example, if $h_i < n_2$ for $i = 1,2$, then $\mathcal{E}^{\mathrm{eff}}(J^{(h_1,h_2)}, G, G_{\vec{n}})$ is empty, but if $h_i \geq n_1$ for $i = 1,2$, then $\mathcal{E}^{\mathrm{eff}}(J^{(h_1,h_2)}, G, G_{\vec{n}})$ consists of four elements. The key point is to notice that for a triple $(M_{G_{\vec{n}}}, s_{G_{\vec{n}}}, \eta_{G_{\vec{n}}})$ to lie in $\mathcal{E}^{\mathrm{eff}}(J^{(h_1,h_2)}, G, G_{\vec{n}})$, it is necessary for $s_{G_{\vec{n}}}$ to transfer to an element of the dual group $\widehat{M^{(h_1,h_2)}} = \widehat{J^{(h_1,h_2)}}$, which is either $+1$ or -1 in the $\mathrm{GL}_{n-h_i}(\mathbb{C})$-block of the $F_{\mathfrak{p}_i}$-component.

We can extend $\eta_{G_{\vec{n}},j}$ to an L-morphism $\tilde{\eta}_{G_{\vec{n}},j} : {}^L M_{G_{\vec{n}},j} \to {}^L M^{(h_1,h_2)}$, which is compatible with the L-morphism $\eta_{\vec{n}} : {}^L G_{\vec{n}} \to {}^L G$, when we map ${}^L M_{G_{\vec{n}},j} \xrightarrow{\tilde{\imath}_j} {}^L G_{\vec{n}}$ and ${}^L M^{(h_1,h_2)} \xrightarrow{\tilde{\imath}} {}^L G$ via (a conjugate of) the obvious block diagonal embedding (where we always send the GL_{n_1}-block to the top left corner and the GL_{n_2}-block to the bottom right corner). The morphism $\tilde{\eta}_{G_{\vec{n}},j}$ is defined as on [Sh3, p. 42], by sending $z \in W_{\mathbb{Q}_p}$ to one of the matrices

$$\begin{pmatrix} \varpi(z)^{\epsilon(n-n_1)} I_{n_1} & 0 \\ 0 & \varpi(z)^{\epsilon(n-n_2)} I_{n_2} \end{pmatrix}$$

or

$$\begin{pmatrix} \varpi(z)^{\epsilon(n-n_2)} I_{n_2} & 0 \\ 0 & \varpi(z)^{\epsilon(n-n_1)} I_{n_1} \end{pmatrix}$$

on the $F_{\mathfrak{p}_i}$-component of $\widehat{M^{(h_1,h_2)}}$. (For $i = 1,2$, we send z to the first matrix on the $F_{\mathfrak{p}_i}$-component if the endoscopic group $M_{G_{\vec{n}},j}$ at $\mathfrak{p}_i$ is $\mathrm{GL}_{n-h_i, h_i-n_2, n_2}$ and to the second matrix if the component of $M_{G_{\vec{n}},j}$ at $\mathfrak{p}_i$ is $\mathrm{GL}_{n-h_i,h_i-n_1,n_1}$. For $i > 2$, we send z to the first matrix on the $F_{\mathfrak{p}_i}$-component.) This map $\tilde{\eta}_{G_{\vec{n}},j}$ is the unique L-morphism which makes the diagram

$$\begin{tikzcd} {}^L M^{(h_1,h_2)} \arrow[r, "\tilde{\imath}"] & {}^L G \\ {}^L M_{G_{\vec{n}},j} \arrow[u, "\tilde{\eta}_{G_{\vec{n}}}"] \arrow[r, "\tilde{\imath}_j"] & {}^L G_{\vec{n}} \arrow[u, "\tilde{\eta}_{\vec{n}}"'] \end{tikzcd}$$

commutative. Thus, the function $\phi_p^{M_{G_{\vec{n}},j}}$ is a transfer of ϕ_p^* with respect to the L-morphism $\tilde{\eta}_{G_{\vec{n}},j}$, so we can define explicitly both $\phi_p^{M_{G_{\vec{n}},j}}$ and the representation-theoretic map $\tilde{\eta}_{M_{G_{\vec{n}},j}*} : \mathrm{Groth}(M_{G_{\vec{n}},j}(\mathbb{Q}_p)) \to \mathrm{Groth}(M^{(h_1,h_2)}(\mathbb{Q}_p))$. There exists a unitary character $\chi^+_{u,j} : M_{G_{\vec{n}},j}(\mathbb{Q}_p) \to \mathbb{C}^\times$ (defined similarly to the character on [Sh3, p. 43]) such that the Langlands–Shelstad transfer factor with respect to $\tilde{\eta}_{G_{\vec{n}},j}$ differs from the transfer factor associated to the canonical L-morphism by the cocycle associated to $\chi^+_{u,j}$. (See [Bor, Section 9] for an explanation of the correspondence between cocycles in $H^1(W_{\mathbb{Q}_p}, Z(\widehat{M_{G_{\vec{n}},j}}))$ and characters $M_{G_{\vec{n}},j}(\mathbb{Q}_p) \to \mathbb{C}^\times$.)

We can in fact compute $\chi^+_{u,j}$ on the different components of $M_{G_{\vec{n}},j}(\mathbb{Q}_p)$ by keeping in mind that it is the character $M_{G_{\vec{n}},j}(\mathbb{Q}_p) \to \mathbb{C}^\times$ associated to the cocycle in $H^1(W_{\mathbb{Q}_p}, Z(\widehat{M_{G_{\vec{n}},j}}))$ which takes the conjugacy class of the standard Levi embedding $\widehat{M_{G_{\vec{n}},j}} \to \widehat{M^{(h_1,h_2)}}$ to that of $\eta_{G_{\vec{n}},j}$. Thus, we have

$$\chi^+_{u,j}(\lambda) = \varpi_u(\lambda)^{-N(n_1,n_2)};$$

$$\chi^+_{u,j}(g_{\mathfrak{p}_i,1}, g_{\mathfrak{p}_i,2}, g_{\mathfrak{p}_i,3}) = \begin{cases} \varpi_u\big(N_{F_{\mathfrak{p}_i}/E_u}(\det((g_{\mathfrak{p}_i,1}g_{\mathfrak{p}_{i,2}})^{\epsilon(n-n_1)} g_{\mathfrak{p}_i,3}^{\epsilon(n-n_2)}))\big), \\ \varpi_u\big(N_{F_{\mathfrak{p}_i}/E_u}(\det((g_{\mathfrak{p}_i,1}g_{\mathfrak{p}_{i,2}})^{\epsilon(n-n_2)} g_{\mathfrak{p}_i,3}^{\epsilon(n-n_1)}))\big) \end{cases}$$

when $i = 1, 2$ and depending on whether $M_{G_{\vec{n}},j}$ has the group $\mathrm{GL}_{n-h_i,h_i-n_2,n_2}$ or the group $\mathrm{GL}_{n-h_i,h_i-n_1,n_1}$ as its $F_{\mathfrak{p}_i}$-component; and

$$\chi^+_{u,j}(g_{\mathfrak{p}_i,1}, g_{\mathfrak{p}_i,2}) = \varpi_u\big(N_{\mathfrak{p}_i/E_u}(\det(g_{\mathfrak{p}_i,1}^{\epsilon(n-n_1)} g_{\mathfrak{p}_i,2}^{\epsilon(n-n_2)}))\big) \quad \text{when } i > 2,$$

where $(\lambda, (g_{\mathfrak{p}_i,1}, g_{\mathfrak{p}_i,2}, g_{\mathfrak{p}_i,3})_{i=1,2}, (g_{\mathfrak{p}_i,1}, g_{\mathfrak{p}_i,2})_{i>2})$ denotes an element of $M_{G_{\vec{n}},j}(\mathbb{Q}_p)$. (The value of $\chi^+_{u,j}$ is in fact the product of the three types of factors above.)

We let Q_j be a parabolic subgroup of $M^{(h_1,h_2)}$ containing $M_{G_{\vec{n}},j}$ as a Levi, and if we let $(\phi_p^*)^{Q_j}$ be the constant term of ϕ_p^* along Q_j, then we have

$$\phi_p^{M_{G_{\vec{n}},j}} := (\phi_p^*)^{Q_j} \cdot \chi^+_{u,j}$$

and

$$\tilde{\eta}_{G_{\vec{n}},j*}(\pi_{M_{G_{\vec{n}},j}}) := \text{n-Ind}_{Q_j}^{M^{(h_1,h_2)}}(\pi_{M_{G_{\vec{n}},j}} \otimes \chi^+_{u,j})$$

for any $\pi_{M_{G_{\vec{n}},j}} \in \mathrm{Irr}_l(M_{G_{\vec{n}},j}(\mathbb{Q}_p))$. By [Sh3, Lemma 3.3], we have

$$\mathrm{tr}\big(f_{M_{G_{\vec{n}},j}}(\pi_p)\big)(\phi_p^{M_{G_{\vec{n}},j}}) = \mathrm{tr}\big(\tilde{\eta}_{G_{\vec{n}},j*}(f_{M_{G_{\vec{n}},j}}(\pi_p))\big)(\phi_p^*). \tag{6}$$

The group $J^{(h_1,h_2)}$ is an inner form of $M^{(h_1,h_2)}$, which is a product of general linear groups. By [Sh2, Lemma 2.18, Remark 2.19],

$$\begin{aligned}\operatorname{tr}\big(\tilde{\eta}_{G_{\vec{n}},*}(f_{M_j}(\pi_p))\big)(\phi_p^*) &= \operatorname{tr}\big(LJ(\tilde{\eta}_{G_{\vec{n}},*}(f_{M_j}(\pi_p)))\big)(\phi_p^0)\\ &= \operatorname{tr}\big(LJ(\tilde{\eta}_{G_{\vec{n}},*}(f_{M_j}(\pi_p))\otimes\bar{\delta}^{1/2}_{P(J^{(h_1,h_2)})}\big)(\phi'_p),\end{aligned}\tag{7}$$

where we have abbreviated $M_{G_{\vec{n}},j}$ by M_j. Putting together (5), (6), and (7), we get the desired result. □

Let Ξ^1 be the algebraic representation of $(\mathbb{G}_n)_{\mathbb{C}}$ obtained by base change from $\iota_l\xi$. Let $\Pi^1 \simeq \psi\otimes\Pi^0$ be an automorphic representation of $\mathbb{G}_n(\mathbb{A}) \simeq \mathrm{GL}_1(\mathbb{A}_E)\times \mathrm{GL}_n(\mathbb{A}_F)$. Assume that

- $\Pi^1 \simeq \Pi^1\circ\theta$,
- Π^1_∞ is generic and Ξ^1-cohomological,
- $\mathrm{Ram}_{\mathbb{Q}}(\Pi)\subset \mathrm{Spl}_{F/F_2,\mathbb{Q}}$,
- Π^1 is cuspidal.

In particular, $\Pi^1_\infty \simeq \Pi_\Xi$, which was defined above. Let $\mathfrak{S}_{\mathrm{fin}}$ be a finite set of places of $\mathbb{Q}$ such that

$$\mathrm{Ram}_{F/\mathbb{Q}}\cup\mathrm{Ram}_{\mathbb{Q}}(\varpi)\cup\mathrm{Ram}_{\mathbb{Q}}(\Pi)\cup\{p\}\subset\mathfrak{S}_{\mathrm{fin}}\subset\mathrm{Spl}_{F/F_2,\mathbb{Q}},$$

and let $\mathfrak{S}=\mathfrak{S}_{\mathrm{fin}}\cup\{\infty\}$.

THEOREM 6.5
Define $C_G = |\ker^1(\mathbb{Q},G)|\cdot\tau(G)$. *For each* $0\le h_1,h_2\le n$, *the following equality holds in* $\mathrm{Groth}(\mathbb{G}_n(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}})\times J^{(h_1,h_2)}(\mathbb{Q}_p))$:

$$\begin{aligned}&BC_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}\big(H_c(\mathrm{Ig}^{(h_1,h_2)},\mathcal{L}_\xi)\big)\{\Pi^{1,\mathfrak{S}}\}\\ &\qquad= C_G\cdot e_0\cdot(-1)^{h_1+h_2}\cdot[\iota_l^{-1}\Pi^1_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}][\mathrm{Red}_n^{(h_1,h_2)}(\pi_p)],\end{aligned}$$

where $e_0=\pm1$ *is independent of* (h_1,h_2).

Proof
The proof goes through identically to the proof of the first part of [Sh3, Theorem 6.1]. We nevertheless give the proof in detail.

First, we explain the choice of test functions to be used in the trace formula. Let $(f^n)^{\mathfrak{S}}\in\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{A}^{\mathfrak{S}}))$ and $f^n_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}\in C_c^\infty(\mathbb{G}_n(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}))$ be any functions. Let $\phi^{\mathfrak{S}}$ and $\phi_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}$ be the BC-transfers of $(f^n)^{\mathfrak{S}}$ and $(f^n)_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}$ from $\mathbb{G}_n$ to G_n. Let $\phi^{\infty,p}=\phi^{\mathfrak{S}}\phi_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}$, and choose any $\phi'_p\in C_c^\infty(J^{(h_1,h_2)}(\mathbb{Q}_p))$ such that $\phi^{\infty,p}\phi'_p$ is an acceptable function.

For each $G_{\vec{n}}\in\mathcal{E}^{\mathrm{ell}}(G)$, we construct the function $\phi^{\vec{n}}_{\mathrm{Ig}}\in C_c^\infty(G_{\vec{n}}(\mathbb{A}))$ associated to $\phi^{\infty,p}\phi'_p$ as above. Recall that $(\phi^{\vec{n}}_{\mathrm{Ig}})^{\mathfrak{S}}$ and $(\phi^{\vec{n}}_{\mathrm{Ig}})_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}$ are the $\Delta(\ ,\ )^{G_n}_{G_{\vec{n}}}$-transfers of $\phi^{\mathfrak{S}}$ and $\phi_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}$. Recall that we take

$$\phi^{\vec{n}}_{\mathrm{Ig},\infty} := e_{\vec{n}}(\Delta_\infty)\cdot(-1)^{q(G)}\langle\mu_h, s_{\vec{n}}\rangle \sum_{\varphi_{\vec{n}}} \det\bigl(\omega_*(\varphi_{G_{\vec{n}}})\bigr)\cdot \phi_{G_{\vec{n}},\xi(\varphi_{\vec{n}})},$$

where $\varphi_{\vec{n}}$ runs over L-parameters such that $\tilde{\eta}_{\vec{n}}\varphi_{\vec{n}} \sim \varphi_\xi$ and where $\xi(\varphi_{\vec{n}})$ is the algebraic representation of $G_{\vec{n},\mathbb{C}}$ such that the L-packet associated to $\varphi_{\vec{n}}$ is $\Pi_{\mathrm{disc}}(G_{\vec{n}}(\mathbb{R}), \xi(\varphi_{\vec{n}})^\vee)$. The construction of $\phi^{\vec{n}}_{\mathrm{Ig},p}$ can be found in [Sh2].

We need to define a function $f^{\vec{n}}$, which plays the part of a BC-matching function for $\phi^{\vec{n}}_{\mathrm{Ig}}$ for each $\vec{n}$. We have already defined $(f^n)^{\mathfrak{S}}$ and $f^n_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}$. We take $(f^{n_1,n_2})^{\mathfrak{S}} = \tilde{\zeta}^*((f^n)^{\mathfrak{S}})$ and $f^{n_1,n_2}_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}} = \tilde{\zeta}^*(f^n_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}})$. We also define

$$f^{\vec{n}}_\infty := e_{\vec{n}}(\Delta)\cdot(-1)^{q(G)}\langle\mu_h, s_{\vec{n}}\rangle \sum_{\varphi_{\vec{n}}} \det\bigl(\omega_*(\varphi_{G_{\vec{n}}})\bigr)\cdot f_{\mathbb{G}_{\vec{n}},\Xi(\varphi_n)},$$

where $\varphi_{\vec{n}}$ runs over L-parameters such that $\tilde{\eta}_{\vec{n}}\varphi_{\vec{n}} \sim \varphi_\xi$ and where $\Xi(\varphi_{\vec{n}})$ is the algebraic representation of $\mathbb{G}_{\vec{n}}$ arising from $\xi(\varphi_{\vec{n}})$. It is straightforward to verify from their definitions that $f^{\vec{n}}_\infty$ and $\phi^{\vec{n}}_{\mathrm{Ig},\infty}$ are BC-matching functions. Finally, we choose $f^{\vec{n}}_p$ so that its BC-transfer is $\phi^{\vec{n}}_{\mathrm{Ig},p}$. (Since p splits in E, it can be checked that the base change map defined in [Sh3, Section 4.2] is surjective at p.) We set

$$f^{\vec{n}} := (f^{\vec{n}})^{\mathfrak{S}}\cdot f^{\vec{n}}_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}\cdot f^{\vec{n}}_p\cdot f^{\vec{n}}_\infty.$$

The BC-transfer of $f^{\vec{n}}$ coincides with $\phi^{\vec{n}}_{\mathrm{Ig}}$ at places outside $\mathfrak{S}$ (by compatibility of transfers), at p, and at ∞. At places in $\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}$, we know at least that the BC-transfer of $f^{\vec{n}}$ has the same trace as $\phi^{\vec{n}}_{\mathrm{Ig}}$ against every admissible representation of $G_{\vec{n}}(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}})$.

By the discussion following Proposition 6.1, we can compute

$$\mathrm{tr}\bigl(\phi^{\infty,p}\phi'_p \bigm| \iota_l H_c(\mathrm{Ig}^{(h_1,h_2)}, \mathcal{L}_\xi)\bigr) \tag{8}$$

via the spectral part of the twisted formula, to get

$$\begin{aligned}C_G(-1)^{h_1+h_2}\Bigl(\frac{1}{2}\sum_{\Pi'}\mathrm{tr}\bigl(\Pi'_\xi(f^n)A'_{\Pi'_\xi}\bigr) + \sum_{\mathbb{G}_{n_1,n_2}, n_1\neq n_2} I^{\mathbb{G}_{n_1,n_2}\theta}_{\mathrm{spec}}(f^{n_1,n_2})\\ + \frac{1}{2}I^{\mathbb{G}_{n/2,n/2}\theta}_{\mathrm{spec}}(f^{n/2,n/2}) + \sum_{M\subsetneq\mathbb{G}_n}\frac{|W_M|}{|W_{\mathbb{G}_n}|}\bigl|\det(\Phi^{-1}\theta-1)_{\mathfrak{a}_M^{\mathbb{G}_n\theta}}\bigr|^{-1}\\ \times\sum_{\Pi'_M}\mathrm{tr}\bigl(\text{n-Ind}^{\mathbb{G}_n}_Q(\Pi'_M)_\xi(f^n)\circ A'_{\text{n-Ind}^{\mathbb{G}_n}_Q(\Pi'_M)_\xi}\bigr)\Bigr),\end{aligned} \tag{9}$$

where the first sum runs over θ-stable subrepresentations Π' of $R_{\mathbb{G}_n,\mathrm{disc}}$ and where the sums in the middle run over groups $\mathbb{G}_{n_1,n_2}$ coming from elliptic endoscopic groups

G_{n_1,n_2} for G (with $n_1 \geq n_2 > 0$ and some (n_1, n_2) possibly excluded). The group M runs over proper Levi subgroups of $\mathbb{G}_n$ containing a fixed minimal Levi, and Π'_M runs over $\Phi_n^{-1}\theta$-stable subrepresentations Π'_M of $R_{M,\mathrm{disc}}$.

We claim that the formula above holds for any $\phi^{\infty,p}\phi'_p$, without the assumption that it is an acceptable function. To see this, note that [Sh1, Lemma 6.3] guarantees that there exists some element $fr^s \in J^{(h_1,h_2)}(\mathbb{Q}_p)$ such that $\phi^{\infty,p}(\phi'_p)^{(N)}(g) = \phi^{\infty,p}(g)\phi'_p(g(fr^s)^N)$ is acceptable for any sufficiently large N. (The paper [Sh1] treats general Igusa varieties, and it is easy to check that our case is covered.) So the equality of (8) and (9) holds when ϕ'_p is replaced by $(\phi'_p)^{(N)}$. Both (8) and (9) are finite linear combinations of terms of the form $\operatorname{tr}\rho((\phi'_p)^{(N)})$, where $\rho \in \operatorname{Irr}(J^{(h_1,h_2)}(\mathbb{Q}_p))$. In order to see that this is true for (9), we need to translate it from computing the trace of $f^{\vec{n}}$ to computing the trace of $\phi_{\mathrm{Ig}}^{\vec{n}}$ to computing the trace of ϕ'_p, using Lemmas 6.3 and 6.4. Now the same argument as that for [Sh1, Lemma 6.4] shows that (8) and (9) are equal for $\phi^{\infty,p}(\phi'_p)^{(N)}$ for every integer N, in particular for $N = 0$. Thus, we can work with arbitrary $\phi^{\infty,p}\phi'_p$.

Choose a decomposition of the normalized intertwining operators

$$A'_{\Pi^1} = A'_{\Pi^{1,\mathfrak{S}}} A'_{\Pi^1_{\mathfrak{S}_{\mathrm{fin}}}} A'_{\Pi^1_\infty}.$$

Set

$$\frac{A'_{\Pi^1}}{A^0_{\Pi^1}} := \frac{A'_{\Pi^{1,\mathfrak{S}}}}{A^0_{\Pi^{1,\mathfrak{S}}}} \cdot \frac{A'_{\Pi^1_{\mathfrak{S}_{\mathrm{fin}}}}}{A^0_{\Pi^1_{\mathfrak{S}_{\mathrm{fin}}}}} \cdot \frac{A'_{\Pi^1_\infty}}{A^0_{\Pi^1_\infty}} \in \{\pm 1\},$$

where the denominators on the right-hand side are the normalized intertwiners chosen above. In the sum (9), the third term evaluates the trace of f^n against representations induced from proper Levi subgroups. The second term has a similar form: outside the set $\mathfrak{S}$ we have the identity $(f^{n_1,n_2})^{\mathfrak{S}} = \tilde{\zeta}^*((f^n)^{\mathfrak{S}})$, and [Sh3, (4.17)] tells us that

$$\operatorname{tr}\Pi_M^{\mathfrak{S}}\big(\tilde{\zeta}^*_{n_1,n_2}(f^n)^{\mathfrak{S}}\big) = \operatorname{tr}\big(\tilde{\zeta}_{n_1,n_2*}(\Pi_M^{\mathfrak{S}})\big)(f^n)^{\mathfrak{S}},$$

where $\tilde{\zeta}_{n_1,n_2*}$ is the transfer from $\mathbb{G}_{n_1,n_1}$ to $\mathbb{G}_n$ on the representation-theoretic side and consists of taking the parabolic induction of a twist of $\Pi_M^{\mathfrak{S}}$. The multiplicity one result of Jacquet and Shalika (see [AC, p. 200]) implies that the string of Satake parameters outside a finite set $\mathfrak{S}$ of a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ unramified outside $\mathfrak{S}$ cannot coincide with the string of Satake parameters outside $\mathfrak{S}$ of an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is a subquotient of a representation induced from a proper Levi subgroup. Thus, if we are interested in the $\Pi^{1,\mathfrak{S}}$-part of $\operatorname{tr}(\phi^{\infty,p}\phi'_p \mid \iota_l H_c(\mathrm{Ig}^{(h_1,h_2)}, \mathcal{L}_\xi))$, then only the first term of (9) can contribute to it.

Thus, we are left to consider

$$C_G(-1)^{h_1+h_2}\Big(\frac{1}{2}\frac{A'_{\Pi^1}}{A^0_{\Pi^1}}\chi_{\Pi^{1,\mathfrak{S}}}\big((f^n)^{\mathfrak{S}}\big)\operatorname{tr}(\Pi^1_{\mathfrak{S}}(f^n_{\mathfrak{S}})A^0_{\Pi^1_{\mathfrak{S}}})$$
$$+\sum_{(\Pi')^{\mathfrak{S}}\neq\Pi^{1,\mathfrak{S}}}\chi_{(\Pi')^{\mathfrak{S}}}\big((f^n)^{\mathfrak{S}}\big)\times\left(\begin{smallmatrix}\text{expression in}\\ \text{terms of } f^n_{\mathfrak{S}}\end{smallmatrix}\right)\Big),$$

where $(\Pi')^{\mathfrak{S}}$ runs over a set of unramified representations of $\mathbb{G}_n(\mathbb{A}^{\mathfrak{S}})$. On the other hand, we can also decompose $\operatorname{tr}(\phi^{\infty,p}\phi'_p \mid \iota_l H_c(\mathrm{Ig}^{(h_1,h_2)},\mathcal{L}_\xi))$ into a $\Pi^{1,\mathfrak{S}}$-part and $(\pi')^{\mathfrak{S}}$-parts, where $BC((\pi')^{\mathfrak{S}})\neq\Pi^{1,\mathfrak{S}}$. We conclude as in [Sh3] that

$$\begin{aligned}&\operatorname{tr}\big(\phi_{\mathfrak{S}_{\mathrm{fin}}\backslash\{p\}}\phi'_p \mid \iota_l H_c(\mathrm{Ig}^{(h_1,h_2)},\mathcal{L}_\xi)\{\Pi^{1,\mathfrak{S}}\}\big)\\ &\qquad=(-1)^{h_1+h_2}\frac{C_G}{2}\frac{A'_{\Pi^1}}{A^0_{\Pi^1}}\cdot\operatorname{tr}\big(\Pi_{\mathfrak{S}}(f^n_{\mathfrak{S}})A_{\Pi^0_{\mathfrak{S}}}\big).\end{aligned}\tag{10}$$

Now $\Pi^1_\infty\simeq\Pi_\Xi$, so $\operatorname{tr}(\Pi^1_\infty(f^n_\infty)A^0_{\Pi_\infty})=2(-1)^{q(G)}=2$. We also have

$$\operatorname{tr}\big(\Pi^1_p(f^n_p)A^0_{\Pi_p}\big)=\operatorname{tr}\iota_l\pi_p(\phi^n_{\mathrm{Ig},p})=\operatorname{tr}\iota_l\operatorname{Red}^{(h_1,h_2)}_n(\pi_p)(\phi'_p)\tag{11}$$

by Lemma 6.3 and

$$\operatorname{tr}\big(\Pi^1_{\mathfrak{S}_{\mathrm{fin}}\backslash\{p\}}(f^n_{\mathfrak{S}_{\mathrm{fin}}\backslash\{p\}})A^0_{\Pi^1_{\mathfrak{S}_{\mathrm{fin}}\backslash\{p\}}}\big)=\operatorname{tr}\iota_l\pi_p(\phi_{\mathfrak{S}_{\mathrm{fin}}\backslash\{p\}}).\tag{12}$$

Putting together (11), (11), and (12) and applying $BC_{\mathfrak{S}_{\mathrm{fin}}\backslash\{p\}}$, we get the desired result with $e_0=A'_{\Pi^1}/A^0_{\Pi^1}$ which is independent of (h_1,h_2). □

7. Proof of the main theorem

Let $E/\mathbb{Q}$ be an imaginary quadratic field in which p splits. Let $F_1/\mathbb{Q}$ be a totally real field, and let w be a prime of F_1 above p. Set $F'=EF_1$. Let F_2 be a totally real quadratic extension of $\mathbb{Q}$ in which $w=w_1w_2$ splits, and set $F=EF_2$. Let $n\in\mathbb{Z}_{\geq 2}$. Also denote F_2 by F^+. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$.

Consider the following assumptions on (E,F',F,Π):

- $[F_1:\mathbb{Q}]\geq 2$;
- $\mathrm{Ram}_{F/\mathbb{Q}}\cup\mathrm{Ram}_{\mathbb{Q}}(\varpi)\cup\mathrm{Ram}_{\mathbb{Q}}(\Pi)\subset\mathrm{Spl}_{F/F^+,\mathbb{Q}}$;
- $(\Pi)^\vee\simeq\Pi\circ c$;
- Π_∞ is cohomological for an irreducible algebraic representation Ξ of $\mathrm{GL}_n(F'\otimes_{\mathbb{Q}}\mathbb{C})$;
- $BC_{F/F'}(\Pi)$ is cuspidal.

Set $\Pi^0=BC_{F/F'}(\Pi)$ and $\Xi^0=BC_{F/F'}(\Xi)$. The following lemma is the same as [Sh3, Lemma 7.2].

LEMMA 7.1
Let Π^0 and Ξ^0 be as above. We can find a character $\psi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ and an algebraic representation $\xi_{\mathbb{C}}$ of G over $\mathbb{C}$ satisfying the following conditions:
- $\psi_{\Pi^0} = \psi^c/\psi$;
- *Ξ^0 is isomorphic to the restriction of Ξ' to $R_{F/\mathbb{Q}}(\mathrm{GL}_n) \times_{\mathbb{Q}} \mathbb{C}$, where Ξ' is obtained from $\xi_{\mathbb{C}}$ by base change from G to $\mathbb{G}_n$;*
- $\xi_{\mathbb{C}}|_{E_\infty^\times}^{-1} = \psi_\infty^x$*; and*
- $\mathrm{Ram}_{\mathbb{Q}}(\psi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$.

Moreover, if l splits in E, then
- $\psi_{\mathcal{O}_{E_u}^\times} = 1$*, where u is the place above l induced by $\iota_l^{-1}\tau|_E$.*

Set $\Pi^1 = \psi \otimes \Pi^0$. Then Π^1 is a cuspidal automorphic representation of $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_F)$. Let $\xi = \iota_l \xi_{\mathbb{C}}$, where $\xi_{\mathbb{C}}$ is as in Lemma 7.1.

Let $\mathcal{A}_U$ be the universal abelian variety over X_U. Since $\mathcal{A}_U$ is smooth over X_U, $\mathcal{A}_U^{m_\xi}$ satisfies the conditions in Section 4.3. In particular, $\mathcal{A}_U^{m_\xi}$ is locally étale over a product of strictly semistable schemes. For $S, T \subseteq \{1, \dots, n\}$, let $\mathcal{A}_{U,S,T}^{m_\xi} = \mathcal{A}_U^{m_\xi} \times_{X_U} Y_{U,S,T}$.

Define the following admissible $G(\mathbb{A}^{\infty,p})$-modules with a commuting continuous action of $\mathrm{Gal}(\bar{F}'/F')$:

$$H^j(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi) = \varinjlim_{U^p} H^j(X_U \times_{F'} \bar{F}', \mathcal{L}_\xi) = H^j(X, \mathcal{L}_\xi)^{\mathrm{Iw}(m)},$$
$$H^j(\mathcal{A}_{\mathrm{Iw}(m)}^{m_\xi}, \bar{\mathbb{Q}}_l) = \varinjlim_{U^p} H^j(\mathcal{A}_U^{m_\xi} \times_{F'} \bar{F}', \bar{\mathbb{Q}}_l).$$

Also define the admissible $G(\mathbb{A}^{\infty,p}) \times (\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}}$-module

$$H^j(\mathcal{A}_{\mathrm{Iw}(m),S,T}^{m_\xi}, \bar{\mathbb{Q}}_l) = \varinjlim_{U^p} H^j(\mathcal{A}_{U,S,T}^{m_\xi} \times_{\mathbb{F}} \bar{\mathbb{F}}, \bar{\mathbb{Q}}_l).$$

Note that a_ξ is an idempotent on $H^j(\mathcal{A}_{\mathrm{Iw}(m),S,T}^{m_\xi}, \bar{\mathbb{Q}}_l(t_\xi))$ and

$$a_\xi H^{j+m_\xi}\big(\mathcal{A}_{\mathrm{Iw}(m),S,T}^{m_\xi}, \bar{\mathbb{Q}}_l(t_\xi)\big) = H^j(Y_{\mathrm{Iw}(m),S,T}, \mathcal{L}_\xi).$$

PROPOSITION 7.2
For each rational prime $l \neq p$, there is a $G(\mathbb{A}^{\infty,p}) \times (\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}}$-equivariant spectral sequence with a nilpotent operator N

$$BC^p\big(E_1^{i,m+m_\xi-i}(\mathrm{Iw}(m),\xi)[\Pi^{1,\mathfrak{S}}]\big) \Rightarrow BC^p\big(WD(H^m(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{\mathrm{Gal}(\bar{K}/K)}[\Pi^{1,\mathfrak{S}}])^{F-\mathrm{ss}}\big),$$

where

$$\begin{aligned}&BC^p\big(E_1^{i,m+m_\xi-i}(\mathrm{Iw}(m),\xi)[\Pi^{1,\mathfrak{S}}]\big)\\&\quad=\bigoplus_{k-l=-i} BC^p\big(a_\xi H^{m+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m)},\mathrm{Gr}^l\mathrm{Gr}_k R\psi\bar{\mathbb{Q}}_l(t_\xi))[\Pi^{1,\mathfrak{S}}]\big).\end{aligned}$$

The action of N sends $BC^p(a_\xi H^{m+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m)},\mathrm{Gr}^l\mathrm{Gr}_k R\psi\bar{\mathbb{Q}}_l(t_\xi))[\Pi^{1,\mathfrak{S}}])$ to

$$BC^p\big(a_\xi H^{m+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m)},\mathrm{Gr}^{l+1}\mathrm{Gr}_{k-1} R\psi\bar{\mathbb{Q}}_l(t_\xi))[\Pi^{1,\mathfrak{S}}]\big).$$

Furthermore, there is a direct sum decomposition

$$\begin{aligned}&BC^p\big(a_\xi H^{m+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m)},\mathrm{Gr}^l\mathrm{Gr}_k R\psi\bar{\mathbb{Q}}_l(t_\xi))[\Pi^{1,\mathfrak{S}}]\big)\\&\quad\simeq\bigoplus_{j\geq 0} BC^p\big(M_{j,m+m_\xi-j}(k,l)\big),\end{aligned}$$

where

$$BC^p\big(M_{j,m+m_\xi-j}(k,l)\big)=\bigoplus_{s=1}^{k+l}\ \bigoplus_{\#S=j+s,\#T=j+k+l-s+1} H^{j+m_\xi,s}_{S,T}(k,l)$$

and

$$\begin{aligned}&H^{j+m_\xi,s}_{S,T}(k,l)\\&\quad=BC^p\big(a_\xi H^{m+m_\xi-2j-k-l+1}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S,T},\bar{\mathbb{Q}}_l(t_\xi-j-k+1))[\Pi^{1,\mathfrak{S}}]\big)\\&\quad=BC^p\big(H^{m-2j-k-l+1}(Y_{\mathrm{Iw}(m),S,T},\bar{\mathbb{Q}}_l(-j-k+1))[\Pi^{1,\mathfrak{S}}]\big).\end{aligned}$$

Proof

Note that $\mathcal{A}^{m_\xi}_U/\mathcal{O}_K$ satisfies the hypotheses of Section 4. We have a spectral sequence of $G(\mathbb{A}^{\infty,p})\times(\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}}$-modules with a nilpotent operator N

$$E_1^{i,m-i}\big(\mathrm{Iw}(m),\xi\big)\Rightarrow H^m\big(\mathcal{A}^{m_\xi}_U\times_{F'}\bar{F}'_{\mathfrak{p}},\bar{\mathbb{Q}}_l(t)\big),$$

where

$$E_1^{i,m-i}\big(\mathrm{Iw}(m),\xi\big)=\bigoplus_{k-l=-i} H^m\big(\mathcal{A}^{m_\xi}_U\times_{\mathbb{F}}\bar{\mathbb{F}},\mathrm{Gr}^l\mathrm{Gr}_k R\psi\bar{\mathbb{Q}}_l(t)\big).$$

N will send $H^m(\mathcal{A}^{m_\xi}_U,\mathrm{Gr}^l\mathrm{Gr}_k R\psi\bar{\mathbb{Q}}_l(t))$ to $H^m(\mathcal{A}^{m_\xi}_U,\mathrm{Gr}^{l+1}\mathrm{Gr}_{k-1} R\psi\bar{\mathbb{Q}}_l(t))$.

By Corollary 4.31, we also have a $G(\mathbb{A}^{\infty,p})\times(\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}}$-equivariant isomorphism

$$H^m\big(\mathcal{A}^{m_\xi}_U\times_{\mathbb{F}}\bar{\mathbb{F}},\mathrm{Gr}^l\mathrm{Gr}_k R\psi\bar{\mathbb{Q}}_l(t)\big)\simeq\bigoplus_{j\geq 0} M_{j,m-j}(k,l),$$

where

$$M_{j,m-j}(k,l) = \bigoplus_{s=1}^{k+l} \bigoplus_{\substack{\#S=j+s \\ \#T=j+k+l-s+1}} H^{j,s}_{S,T}(k,l)$$

and

$$H^{j,s}_{S,T}(k,l) = H^{m-2j-k-l+1}\big(\mathcal{A}^{m_\xi}_{U,S,T} \times_{\mathbb{F}} \bar{\mathbb{F}}, \bar{\mathbb{Q}}_l(t-j-k+1)\big).$$

We take $t = t_\xi$, apply a_ξ, replace j by $j + m_\xi$, and take the inverse limit over U^p. We get a spectral sequence of $G(\mathbb{A}^{\infty,p}) \times (\mathrm{Frob}_{\mathbb{F}})^{\mathbb{Z}}$-modules converging to $H^j(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)$. We identify $H^j(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)$ with its associated Weil–Deligne representation, and we semisimplify the action of Frobenius. After taking $\Pi^{1,\mathfrak{S}}$-isotypical components and applying BC^p, we get the desired spectral sequence. □

COROLLARY 7.3

Keep the assumptions made in the beginning of this section. The Weil–Deligne representation

$$WD\big(BC^p(H^{2n-2}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{\mathrm{Gal}(\bar{K}/K)}[\Pi^{1,\mathfrak{S}}])\big)^{F-\mathrm{ss}}$$

is pure of weight $m_\xi - 2t_\xi + 2n - 2$.

Proof

By Proposition 5.10,

$$BC^p\big(H^j(Y_{\mathrm{Iw}(m),S,T}, \mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big) = 0$$

unless $j = 2n - \#S - \#T$. Thus, the terms of the direct sum decomposition

$$BC^p\big(M_{j,m+m_\xi-j}(k,l)\big),$$

which are all of the form

$$BC^p\big(H^{m-2j-k-l+1}(Y_{\mathrm{Iw}(m),S,T}, \bar{\mathbb{Q}}_l(-j-k+1))[\Pi^{1,\mathfrak{S}}]\big)$$

with $\#S = j + s$ and $\#T = j + k + l - s + 1$, vanish unless $m = 2n - 2$. This means that the terms of the spectral sequence $BC^p(E_1^{i,m+m_\xi-i}(\mathrm{Iw}(m), \xi)[\Pi^{1,\mathfrak{S}}])$ vanish unless $m = 2n - 2$. If $m = 2n - 2$, then each summand of

$$BC^p\big(E_1^{i,2n-2+m_\xi-i}(\mathrm{Iw}(m), \xi)[\Pi^{1,\mathfrak{S}}]\big)$$

has a filtration with graded pieces

$$BC^p\big(H^{2n-2-2j-k-l+1}(Y_{\mathrm{Iw}(m),S,T}, \mathcal{L}_\xi(-j-k+1))[\Pi^{1,\mathfrak{S}}]\big),$$

where $k-l=-i$. These graded pieces are strictly pure of weight $m_\xi - 2t_\xi + 2n - 2 + k - l - 1$, which only depends on i. Thus, the whole of

$$BC^p\big(E_1^{i,2n-2+m_\xi-i}(\mathrm{Iw}(m),\xi)[\Pi^{1,\mathfrak{S}}]\big)$$

is strictly pure of weight $m_\xi - 2t_\xi + 2n - 2 - i - 1$. The spectral sequence degenerates at E_1, since $E_1^{i,m-i} = 0$ unless $m = 2n-2$, and also the abutment is pure of weight $m_\xi - 2t_\xi + 2n - 2$. Thus,

$$BC^p\big(WD(H^m(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{\mathrm{Gal}(\bar{K}/K)}[\Pi^{1,\mathfrak{S}}])^{F-\mathrm{ss}}\big)$$

vanishes for $m \neq 2n-2$ and is pure of weight $m_\xi - 2t_\xi + 2n - 2$ for $m = 2n-2$. □

THEOREM 7.4
Let $n \in \mathbb{Z}_{\geq 2}$ be an integer, and let L be any CM field. Let l be a prime, and let ι_l be an isomorphism $\iota_l : \bar{\mathbb{Q}}_l \to \mathbb{C}$. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ *satisfying*

- $\Pi^\vee \simeq \Pi \circ c$,
- Π *is cohomological for some irreducible algebraic representation* Ξ.

Let

$$R_l(\Pi) : \mathrm{Gal}(\bar{L}/L) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$$

be the Galois representation associated to Π by [Sh3] and [CH]. Let $p \neq l$, and let y be a place of L above p. Then we have the following isomorphism of Weil–Deligne representations

$$WD\big(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)}\big)^{F-\mathrm{ss}} \simeq \iota_l^{-1}\mathcal{L}_{n,L_y}(\Pi_y).$$

Proof
This theorem has been proven by [Sh3] except in the case when n is even and Ξ is not slightly regular. In that exceptional case it is still known that we have an isomorphism of semisimplified W_{L_y}-representations by [CH], so it remains to check that the two monodromy operators N match up. By Corollary 5.9, Π_y is tempered. This is equivalent to $\iota_l^{-1}\mathcal{L}_{n,L_y}(\Pi_y)$ being pure of weight $2n-2$. In order to get an isomorphism of Weil–Deligne representations, it suffices to prove that $WD(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)})^{F-\mathrm{ss}}$ is pure.

We first find a CM field F' such that

- $F' = EF_1$, where E is an imaginary quadratic field in which p splits and $F_1 = (F')^{c=1}$ has $[F_1 : \mathbb{Q}] \geq 2$,

- F' is soluble and Galois over L,
- $\Pi^0_{F'} = BC_{F'/L}(\Pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$, and
- there is a place $\mathfrak{p}$ of F above y such that $\Pi^0_{F',\mathfrak{p}}$ has an Iwahori fixed vector,

and a CM field F which is a quadratic extension of F' such that

- $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2$ splits in F,
- $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \subset \mathrm{Spl}_{F/F',\mathbb{Q}}$, and
- $\Pi^0_F = BC_{F/F'}(\Pi^0_{F'})$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

To find F and F', we proceed as in the proof of Corollary 5.9. Set $\Pi^1_F = \Pi^0_F \otimes \psi$, where ψ is chosen as in Lemma 7.1.

We claim that we have isomorphisms

$$C_G \cdot \big(R_l(\Pi)|_{\mathrm{Gal}(\bar{F}'/F')}\big)^{\otimes 2} \simeq C_G \cdot R_l(\Pi^0_{F'})^{\otimes 2} \simeq \tilde{R}^{2n-2}_l(\Pi^1_F) \otimes R_l(\psi)^{-1},$$

where $\tilde{R}^k_l(\Pi^1_F)$ was defined in Section 4. The first isomorphism is clear. The second isomorphism can be checked by Chebotarev's density theorem locally at unramified places, using the local global compatibility for $R_l(\Pi^0_{F'})$ and the formula

$$\tilde{R}_l(\Pi^1_F) = e_0 C_G \cdot [(\pi_{p,0} \circ \mathrm{Art}^{-1}_{\mathbb{Q}_p})|_{W_{F'_\mathfrak{p}}} \otimes \iota_l^{-1} \mathcal{L}_{F'_\mathfrak{p},n}(\Pi^0_{F',\mathfrak{p}})^{\otimes 2}].$$

(It can be checked easily, either by computing the weight or by using the spectral sequences above that $\tilde{R}^k_l(\Pi^1_F) \neq 0$ if and only if $k = 2n-2$ and thus that $e_0 = (-1)^{2n-2} = 1$.)

We also have

$$BC^p\big(H^{2n-2}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)[\Pi^{1,\mathfrak{S}}_F]\big) \simeq (\dim \pi_p^{\mathrm{Iw}(m)}) \cdot \iota_l^{-1}\Pi^{\infty,p} \otimes \tilde{R}^{2n-2}(\Pi^1_F)$$

as admissible representations of $G(\mathbb{A}^{\infty,p}) \times \mathrm{Gal}(\bar{F}'/F')$. By Corollary 7.3, $WD(\tilde{R}^{2n-2}_l(\Pi^1_F)|_{\mathrm{Gal}(\bar{F}'_\mathfrak{p}/F'_\mathfrak{p})})$ is pure of weight $m_\xi - 2t_\xi + 2n - 2$. By [TY, Lemma 1.7],

$$WD\big(R_l(\Pi^0_{F'})^{\otimes 2}|_{\mathrm{Gal}(\bar{F}'_\mathfrak{p}/F'_\mathfrak{p})}\big)$$

is also pure. It has weight $2n-2$. The monodromy operator acts on $R_l(\Pi^0_{F'})^{\otimes 2}|_{W_{F'_\mathfrak{p}}}$ as $1 \otimes N + N \otimes 1$, where N is the monodromy operator on $R_l(\Pi^0_{F'})|_{W_{F'_\mathfrak{p}}}$. We wish to show that $V := WD(R_l(\Pi^0_{F'})|_{W_{F'_\mathfrak{p}}})^{F-\mathrm{ss}}$ is pure of weight $n-1$. Consider the direct sum decomposition $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where V_i is strictly pure of weight $n - 1 + i$. It suffices to prove that $N^i : V_i \to V_{-i}$ is injective for every $i > 0$, since then we can compare dimensions to deduce that N^i is an isomorphism. Let $x \in V_i$, and assume that $N^i x = 0$. Since $x \in V_i$, the vector $x \otimes x$ belongs to the subspace of $WD(R_l(\Pi^0_{F'})^{\otimes 2}|_{W_{F'_\mathfrak{p}}})^{F-\mathrm{ss}}$ which is strictly pure of weight $2n-2+2i$. But then

$$N^{2i}(x \otimes x) = \sum_{k=0}^{2i} N^k x \otimes N^{2i-k} x = 0,$$

which contradicts the purity of $WD(R_l(\Pi^0_{F'})^{\otimes 2}|_{W_{F'_p}})^{(F-\mathrm{ss})}$. Thus, $WD(R_l(\Pi^0_{F'})|_{\mathrm{Gal}(\bar{F}'_\mathfrak{p}/F'_\mathfrak{p})})^{F-\mathrm{ss}}$ has to be pure. By [TY, Lemma 1.4], purity is preserved under finite extensions, so $WD(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)})^{F-\mathrm{ss}}$ is also pure. □

Acknowledgments. I am very grateful to my advisor, Richard Taylor, for suggesting this problem and for his constant encouragement and advice. I am also greatly indebted to Luc Illusie, Sophie Morel, Arthur Ogus, Sug Woo Shin, and Teruyoshi Yoshida for their help in completing this work. I am grateful to David Geraghty and Jack Thorne for useful conversations. Finally, I would like to thank Sophie Morel and Sug Woo Shin for providing comments on an earlier draft of this paper and the anonymous referees for suggesting many corrections and improvements.

References

[A] J. ARTHUR, *The invariant trace formula, II: Global theory*, J. Amer. Math. Soc. **1** (1988), 501–554. MR 0939691 *(2392, 2395)*

[AC] J. ARTHUR and L. CLOZEL, *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, Ann. of Math. Stud. **120**, Princeton Univ. Press, Princeton, 1989. MR 1007299 *(2404)*

[AGV] M. ARTIN, A. GROTHENDIECK, and J. L. VERDIER, *Théorie des topos et cohomologie étale des schémas*, Séminaire de Géométrie Algébrique du Bois-Marie (SGA 4), Lecture Notes in Math. **305**, Springer, Berlin, 1973. MR 0354654 *(2339)*

[Bad] A. I. BADULESCU, *Jacquet-Langlands et unitarisabilité*, J. Inst. Math. Jussieu **6** (2007), 349–379. MR 2329758 *(2381)*

[B] A. A. BEĬLINSON, “On the derived category of perverse sheaves” in *K-theory, Arithmetic and Geometry (Moscow, 1984–1986)*, Lecture Notes in Math. **1289**, Springer, Berlin, 1987, 27–41. MR 0923133 *(2353)*

[BBD] A. A. BEĬLINSON, J. BERNSTEIN, and P. DELIGNE, “Faisceaux pervers” in *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, Astérisque **100**, Soc. Math. France, Paris, 1982, 5–171. MR 0751966 *(2345, 2348, 2352, 2361)*

[BZ] I. N. BERNSTEIN and A. V. ZELEVINSKY, *Induced representations of reductive p-adic groups, I*, Ann. Sci. Éc. Norm. Supér. (4) **10** (1977), 441–472. MR 0579172 *(2384)*

[Bor] A. BOREL, “Automorphic L-functions” in *Automorphic Forms, Representations and L-functions (Corvallis, Ore., 1977)*, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, 1979, 27–61. MR 0546608 *(2401)*

[CH] G. CHENEVIER and M. HARRIS, *Construction of automorphic Galois representations, II*, preprint, http://people.math.jussieu.fr/~harris/ (accessed 24 July 2012). *(2312, 2387, 2409)*

[Cl1] L. CLOZEL, *Représentations galoisiennes associées aux représentations automorphes autoduales de* $\mathrm{GL}(n)$, Publ. Math. Inst. Hautes Études Sci. **73** (1991), 97–145. MR 1114211 *(2315, 2316)*

[Cl2] ———, *Purity reigns supreme*, to appear in Int. Math. Res. Not. IMRN, preprint, http://fa.institut.math.jussieu.fr/node/45 (accessed 24 July 2012). *(2312, 2387)*

[D] V. G. DRINFELD, *Elliptic modules* (in Russian), Mat. Sb. (N.S.) **94(136)**, no. 4, 594–627; English translation in Math. USSR Sb. **23** (1974), 561–592. *(2326)*

[E] T. EKEDAHL, "On the adic formalism" in *The Grothendieck Festschrift, Vol. II*, Progr. Math. **87**, Birkhäuser, Boston, 1990, 197–218. MR 1106899 *(2338)*

[GDK] A. GROTHENDIECK, P. DELIGNE, and N. KATZ, *Groupes de monodromie en géométrie algébrique. I*, Séminaire de Géométrie Algébrique du Bois-Marie (SGA 7 I), Lecture Notes in Math. **288**, Springer, Berlin, 1972. MR 0354656 *(2334)*

[GD1] A. GROTHENDIECK and J. DIEUDONNÉ, *Éléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, II*, Publ. Math. Inst. Hautes Études Sci. **17** (1963). MR 0163911 *(2340)*

[GD2] ———, *Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, IV*, Publ. Math. Inst. Hautes Études Sci. **32** (1967). MR 0238860 *(2330)*

[HT] M. HARRIS and R. TAYLOR, *The Geometry and Cohomology of Some Simple Shimura Varieties*, with an appendix by V. G. Berkovich, Ann. of Math. Stud. **151**, Princeton Univ. Press, Princeton, 2001. MR 1876802 *(2312, 2315, 2317, 2318, 2320, 2321, 2322, 2376, 2377, 2378, 2380, 2382, 2383, 2384)*

[I1] L. ILLUSIE, "Autour du théorème de monodromie locale" in *Périodes p-adiques (Bures-sur-Yvette, 1988)*, Astérisque **223**, Soc. Math. France, Paris, 1994, 9–57. MR 1293970 *(2332, 2334, 2335, 2340)*

[I2] ———, "An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology" in *Cohomologies p-adiques et applications arithmétiques, II*, Astérisque **279**, Soc. Math. France, Paris, 2002, 271–322. MR 1922832 *(2338)*

[K] K. KATO, "Logarithmic structures of Fontaine-Illusie" in *Algebraic Analysis, Geometry, and Number Theory (Baltimore, 1988)*, Johns Hopkins Univ. Press, Baltimore, 1989. MR 1463703 *(2332, 2333)*

[KN] K. KATO and C. NAKAYAMA, *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over* $\mathbb{C}$, Kodai Math. J. **22** (1999), 161–186. MR 1700591 *(2335)*

[KM] N. M. KATZ and B. MAZUR, *Arithmetic Moduli of Elliptic Curves*, Ann. of Math. Stud. **108**, Princeton Univ. Press, Princeton, 1985. MR 0772569 *(2320)*

[Ko1] R. E. KOTTWITZ, *On the λ-adic representations associated to some simple Shimura varieties*, Invent. Math. **108** (1992), 653–665. MR 1163241 *(2383)*

[Ko2] ———, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), 373–444. MR 1124982 *(2317, 2318, 2322, 2378)*

[L] K.-W. LAN, *Arithmetic compactifications of PEL-type Shimura varieties*, Ph.D. dissertation, Harvard University, Cambridge, Mass., 2008. MR 2711676 *(2320)*

[Ma] E. MANTOVAN, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J. **129** (2005), 573–610. MR 2169874 *(2374, 2376, 2383)*

[Na] C. NAKAYAMA, *Nearby cycles for log smooth families*, Compos. Math. **112** (1998), 45–75. MR 1622751 *(2331, 2333, 2334, 2335, 2338, 2369)*

[O] A. OGUS, *Relatively coherent log structures*, preprint, 2009. *(2366)*

[RZ] M. RAPOPORT and TH. ZINK, *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), 21–101. MR 0666636 *(2314)*

[Re] R. REICH, *Notes on Beilinson's "How to glue perverse sheaves,"* J. Singul. **1** (2010), 94–115. MR 2671769 *(2353)*

[S] M. SAITO, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), 849–995. MR 1000123 *(2361)*

[Sa] T. SAITO, *Weight spectral sequences and independence of l*, J. Inst. Math. Jussieu **2** (2003), 583–634. MR 2006800 *(2314, 2338, 2341, 2344, 2345, 2361, 2366)*

[She] D. SHELSTAD, *L-indistinguishability for real groups*, Math. Ann. **259** (1982), 385–430. MR 0661206 *(2392)*

[Sh1] S. W. SHIN, *Counting points on Igusa varieties*, Duke Math. J. **146** (2009), 509–568. MR 2484281 *(2321, 2374, 2377, 2379, 2394, 2395, 2404)*

[Sh2] ———, *A stable trace formula for Igusa varieties*, J. Inst. Math. Jussieu **9** (2010), 847–895. MR 2684263 *(2389, 2394, 2395, 2396, 2397, 2398, 2399, 2401, 2403)*

[Sh3] ———, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), 1645–1741. MR 2800722 *(2312, 2314, 2316, 2322, 2382, 2387, 2389, 2391, 2395, 2396, 2400, 2401, 2402, 2403, 2404, 2405, 2409)*

[Tad] M. TADIC, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*, Ann. Sci. Éc. Norm. Supér. (4) **19** (1986), 335–382. MR 0870688 *(2383)*

[Tat] J. TATE, "Classes d'isogenie des variétés abéliennes sur un corps fini" in *Séminaire Bourbaki, 1968/1969*, no. 352, Lecture Notes in Math. **179**, Springer, Berlin, 1971. *(2378, 2379)*

[TY] R. TAYLOR and T. YOSHIDA, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), 467–493. MR 2276777 *(2312, 2313, 2318, 2322, 2324, 2325, 2328, 2329, 2374, 2382, 2386, 2410, 2411)*

[Y] T. YOSHIDA, "On non-abelian Lubin-Tate theory via vanishing cycles" in *Algebraic and Arithmetic Structures of Moduli Spaces (Sapporo, 2007)*, Adv. Stud. Pure Math. **58**, Math. Soc. Japan, Tokyo, 2010, 361–402. MR 2676163 *(2330)*

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138, USA; caraiani@math.harvard.edu