Characterizing Δ^1_2 via Ordinal Machines

Peter Koepke and Benjamin Seyfferth

University of Bonn, Mathematical Institute

18 February 2010

Ordinal Turing Machines

Classical properties of OTMs

- finite alphabet $\{0, 1\}$
- finite number of program states
- finitely many instructions

Nonclassical properties of OTMs

- tape length the ordinals
- the machine *diverges* iff it does not halt at time some ordinal time α
- $\liminf$-rule

The lim inf-Rule

Consider the set of program states to be a (finite) subset of ω.

Limit time β

- program state:

$$\liminf\{\text{program state at time } \gamma \mid \gamma < \beta\}$$

- content of a single cell:

$$\liminf\{\text{the cell's content at time } \gamma \mid \gamma < \beta\}$$

- position of the read-write head:

$$\liminf\{\text{headposition at time } \gamma \mid \gamma < \beta \text{ and state at time } \gamma \text{ is the same as state at time } \beta\}$$

Input and Output

Input

A real number $a \in \{0,1\}^\omega$ written in the first ω-many tape cells (remaining cells filled with 0s)

Output

yes / no
halt / diverge

OTM-(Semi-)Deciding Sets of Reals

Definition

- A set of reals $A \subset \{0,1\}^\omega$ is called *OTM-semi-decidable* iff there is a program that, on input $a \in \{0,1\}^\omega$, halts exactly if $a \in A$.
- A set of reals $A \subset \{0,1\}^\omega$ is called *OTM-decidable* iff there is a program that, on input $a \in \{0,1\}^\omega$, outputs 'yes' if $a \in A$ and 'no' otherwise.

The Characterization

Theorem

A set of reals is OTM-semi-decidable iff it is Σ^1_2.

Corollary

A set of reals is OTM-decidable iff it is Δ^1_2.

Quantifier Complexity

class	quantifier structure
Σ_1	$\exists \ldots$
Π_1	$\forall \ldots$
Σ_2	$\exists\forall \ldots$
Π_2	$\forall\exists \ldots$
$\vdots$	$\vdots$

Δ_n: Σ_n and Π_n
Σ^1_1, Π^1_1, Σ^1_2, $\ldots$: quantifiers ranging over $\{0,1\}^\omega$.

OTM-Semi-Decidable $\to \Sigma^1_2$

Fact

We can code an entire halting *OTM-computation* into a real.

Essential: Code the length of the computation (a countable ordinal) into a real.

Fact

To check whether a given real codes a halting OTM-computation is Π^1_1.

Essential: Check that a given real codes a well-ordering.

□

Some Descriptive Set Theory

Theorem (Shoenfield Absoluteness)

Every Σ^1_2 relation is absolute for transitive Models of ZF.

Theorem (Descriptive set theory)

Every Σ^1_2 formula is equivalent to a Σ_1 formula of set theory.

$\Sigma^1_2 \to$ OTM-Semi-Decidable

Let A be Σ^1_2 and we want to semi-decide the question whether for a given $a \in \{0,1\}^\omega$ we have $a \in A$. There is a Σ^1_2 formula ϕ s.t.

$$\begin{aligned} a \in A &\leftrightarrow \phi(a) \\ &\leftrightarrow L[a] \vDash \phi(a) \\ &\leftrightarrow L[a] \vDash \exists x\, \psi(x,a) \end{aligned}$$

for some bounded formula ψ.
We can employ an OTM that searches through $L[a]$ for a witness x with $L[a] \vDash \psi(x,a)$, halts if one exists, and diverges otherwise.

□

Computing truth in $L[a]$

Fact

Every element x of $L[a]$ can be coded into a finite sequence of ordinal numbers $\vec{\gamma}$.

$$
\begin{aligned}
x \in L[a] &\rightsquigarrow x \in L_{\alpha+1}[a] \\
&\rightsquigarrow x = \{y \in L_\alpha[a] \mid L_\alpha[a] \models \phi_n(y, \vec{z}), \vec{z} \in L_\alpha[a]\} \\
&\rightsquigarrow (\alpha, n, \ldots) = \vec{\gamma}
\end{aligned}
$$

Computing truth in $L[a]$ (ctd.)

Theorem (Koepke 2005)

The function $W_a(\vec{\gamma}, \psi)$ with

$$W_a(\vec{\gamma}, \psi) = 1 \leftrightarrow L[a] \models \psi(\vec{\gamma}, a)$$

is computable by an OTM (with additional input a).

Algorithm for W_a

Tape 1:	Successively build up the finite sequences $\vec{\gamma}_\iota$ of ordinals					
Tape 2:	$(\vec{\gamma}_0, \psi_0)$	$(\vec{\gamma}_0, \psi_1)$	...	$(\vec{\gamma}_1, \psi_0)$	$(\vec{\gamma}_1, \psi_1)$	...
Tape 3:	true	false	...	false	true	...

Next Steps

- Refine the argument by computing the tree representations from Descriptive set theory
- New tree representations
- Apply the computability paradigm more widely in Descriptive set theory

Thank you!