Computing the Shoenfield Tree

Philipp Schlicht and Benjamin Seyfferth

University of Bonn, Mathematical Institute

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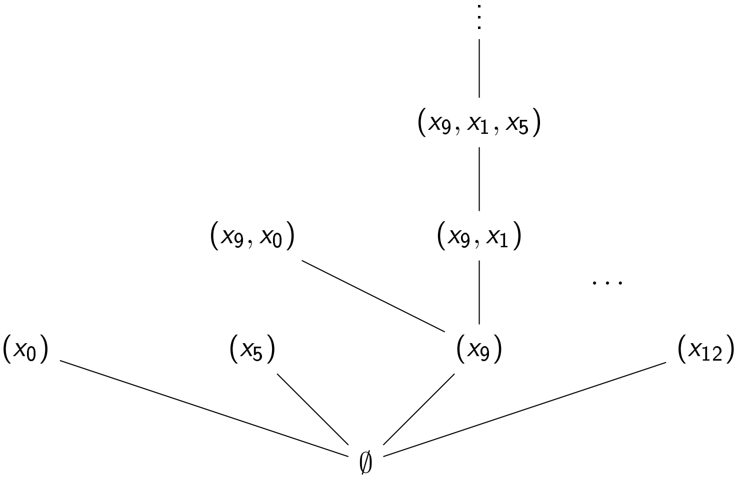
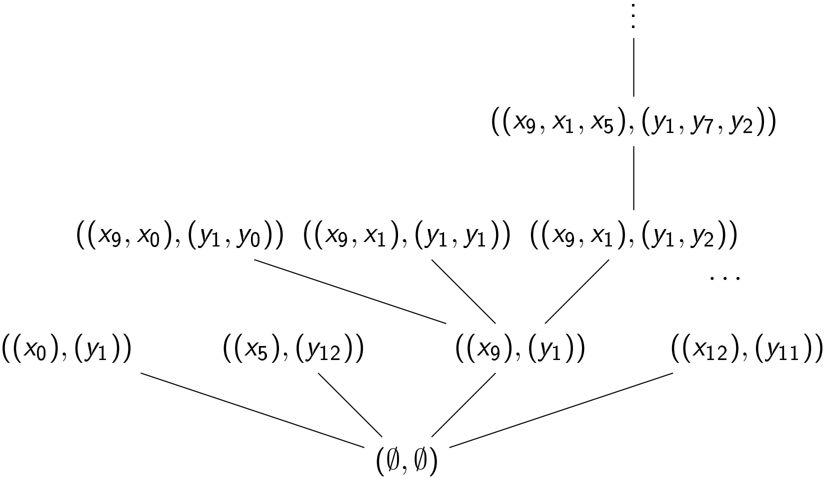
Trees on sequences

A tree on X is a subset of ${}^{<\omega}X$ that is closed under initial segments.
An infinite branch corresponds to an element of ${}^{\omega}X$.



Trees on sequences ctd.

A tree on $X \times Y$ is a subset of ${}^{<\omega}X \times {}^{<\omega}Y$ of same length sequences, closed under pointwise initial segments.



$\vdots$

$((x_9, x_1, x_5), (y_1, y_7, y_2))$

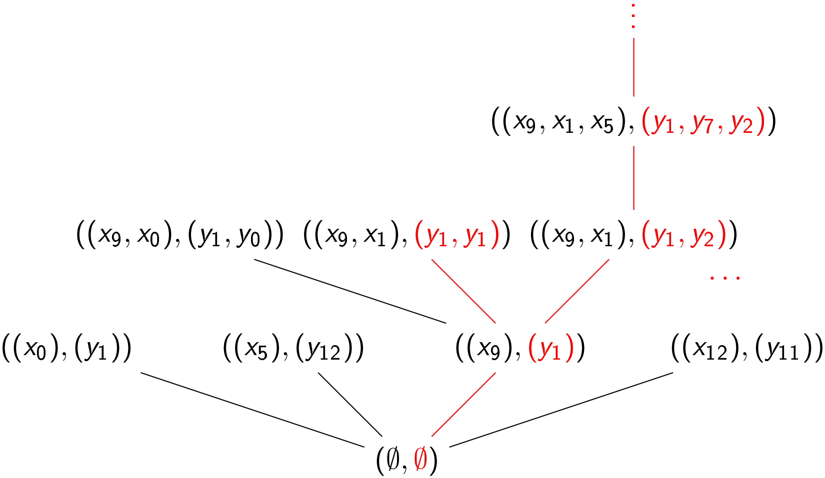
$((x_9, x_0), (y_1, y_0))$ $((x_9, x_1), (y_1, y_1))$ $((x_9, x_1), (y_1, y_2))$ $\dots$

$((x_0), (y_1))$ $((x_5), (y_{12}))$ $((x_9), (y_1))$ $((x_{12}), (y_{11}))$

$(\emptyset, \emptyset)$

Induced subtrees

For a tree T on $X \times Y$ and some $x \in {}^{\omega}X$ or $x \in {}^{<\omega}X$, we denote by T_x the tree on Y of those sequences that are compatible with x.



Well-founded trees

Trees without infinite branches are called well-founded.

When ordered by the reverse coordinate-wise inclusion, a tree on $X \times Y$ is well-founded iff it has an order-preserving embedding into the ordinals. This requires the sets X and Y to be well-orderable.

Classifying sets of ${}^{\omega}\omega$

class	quantifier structure
Σ_1	$\exists \ldots$
Π_1	$\forall \ldots$
Σ_2	$\exists\forall \ldots$
Π_2	$\forall\exists \ldots$
$\vdots$	$\vdots$

- Δ_n: Σ_n and Π_n
- Σ^1_n, Π^1_n, Δ^1_n: quantifiers ranging over ${}^{\omega}\omega$

Tree representation for Π^1_1 sets

$A \subseteq {}^k({}^\omega\omega)$ is Π^1_1 iff there is a recursive tree T on ${}^k\omega \times \omega$ s.t.

$$\begin{aligned} x \in A \leftrightarrow & \, T_x \text{ is well-founded} \\ \leftrightarrow & \text{ there is an order-preserving embedding} \\ & \text{ of } T_x \text{ into some countable ordinal} \end{aligned}$$

Shoenfield tree for Σ^1_2 sets

For a Σ^1_2 set $B \subset {}^k({}^\omega\omega)$ let A be Π^1_1 s.t. $B = \{x \mid \exists y(x,y) \in A\}$. Let T be the tree representation of A. The Shoenfield tree S for B is such that

$$x \in B \leftrightarrow S_x \text{ has an infinite branch}$$

The branches of the Shoenfield tree

An infinite branch of S_x codes

- a real y s.t. $(x, y) \in A$
- some order-preserving embedding of $T_{(x,y)}$ into some countable ordinal α

This can be coded into an element of ${}^{\omega}\omega_1$, therefore S is a tree on ${}^{k}\omega \times \omega_1$. S_x is a tree on ω_1.

Idea

Have an *ordinal Turing machine* search for such a branch.

Ordinal Turing Machines

- tape length the ordinals
- the machine diverges only if it does not halt at some ordinal time α
- finite alphabet $\{0,1\}$
- finitely many instructions (standard Turing program)
- $\liminf$-rule that determines the machine configuration at limit times

We will consider single reals as inputs to our machines, using some computable coding of ${}^{\omega}\omega$ into ${}^{\omega}2$.

Programming OTMs

OTMs can:

- store ordinal numbers
- do ordinal arithmetic
- perform transfinite loops
- and much more, for instance:

For any given Σ^1_2 set, there is an OTM that semi-decides the set.

The algorithm for Π^1_1

We first look at the Shoenfield tree for Π^1_1 sets, because in this case a branch through S_x only has to code an order-preserving embedding of T_x into a countable ordinal.

Input

A real x

Output

An order preserving embedding of T_x into a countable ordinal, or divergence.

The algorithm proceeds in stages $\alpha < \omega_1$. In stage α it tries to find an order-preserving embedding of T_x into α using depth-first-search (DFS) through S_x.

Coding of OTM-computable embeddings

An element of ${}^{\omega}\alpha$ corresponds to an OTM-computable embedding of T_x into α by:

$$(\beta_1, \beta_2, \ldots)$$

stands for

"Map the first element of T_x to β_1, the second to β_2 and so on."

To check whether such a sequence (or an initial segment) codes an orderpreserving map can be computed by an OTM with input x, since T_x is recursive in x.

The algorithm for Π^1_1 ctd.

The algorithm starts with the empty sequence $u = ()$ and in stage $\alpha = 0$. Whenever $\mathrm{DFS}(u)$ is called, all possible extensions of u by a single ordinal $\beta < \alpha$ are tried. If a feasible extension $u \frown \beta \in S_x$ is found, the recursion will immediately try to extend it further and $\mathrm{DFS}(u \frown \beta)$ is called. Whenever the algorithm tries an extension $u \frown \beta$ that is not in S_x, this extension is not followed further and $u \frown (\beta + 1)$ is tried next. If the length n of u has reached ω, a branch is found, i.e., u codes an orderpreserving embedding of T_x into the ordinal α. If no branch can be found, the recursion eventually breaks down, α is incremented, and the algorithm starts over with the empty sequence.

The algorithm for Σ^1_2

Now branches of S_x need to code a real y s.t. $T_{(x,y)}$ is well-founded *and* the order-preserving embedding of $T_{(x,y)}$ into a countable ordinal.

Input

A real x

Output

A real y and an order preserving embedding of $T_{(x,y)}$ into a countable ordinal, or divergence.

We include an addtional parameter in our search. Instead of sequences $u \in {}^{<\omega}\omega_1$ we look for pairs $(t, u) \in {}^{<\omega}\omega \times {}^{<\omega}\omega_1$ simultaneously, using DFS through S_x.

The algorithm for Σ^1_2 (ctd.)

For convenience, we treat S_x as a tree on $\omega \times \omega_1$. By coding we S_x can easily be made a tree on ω_1, as required.

Here in every call of $\mathrm{DFS}(t, u)$ the algorithm tries to extend t and u simultaneously by all pairs (m, β) with $m \in \omega$ and $\beta < \alpha$. Again, if $(t \frown m, u \frown \beta) \in S_x$, the sequence is immediately tried to be extended further (i.e., $\mathrm{DFS}(t \frown m, u \frown \beta)$ is called). Otherwise, $(t \frown m, u \frown \beta + 1)$ is tried next. If for all $\beta < \alpha$ $(t \frown m, u \frown \beta)$ cannot be extended further, $(t \frown m + 1, \beta \frown 0)$ is tried next, and so on.

First results

This shows:

Theorem

Every Σ^1_2 *set of* ${}^\omega\omega$ *is OTM-semi-decidable.*

Corollary

The OTM-decidable sets of reals are exactly the Δ^1_2 *sets.*

Since OTM computations are naturally absolute between models of set theory we furthermore re-establish:

Corollary (Shoenfield absoluteness)

The Σ^1_2 *sets are absolute between transitive models of set theory.*

More results directly from the algorithm

By adding another real parameter to our search, we get:

Corollary (Σ^1_2 uniformization)

For a given Σ^1_2 set $A \subset {}^\omega\omega \times {}^\omega\omega$ there is a Σ^1_2 set $B \subset {}^\omega\omega \times {}^\omega\omega$ s.t. for every x with $\exists y(x, y) \in A$ there is a unique z with $(x, z) \in B$

By analyzing a Σ^1_2 into the sets that are accepted in stage α of the algorithm we obtain:

Corollary

Each Σ^1_2 set is the union of ω_1 many Borel sets.

Oracles and jumps

Oracles

A query state checks whether the first ω-many tape cells contain an element of an oracle set $A \subseteq {}^{\omega}2$ or not.

Jump of a set $A \subseteq {}^{\omega}\omega$

$$A^{\blacktriangledown} = \{(n, x) \mid \text{the } n\text{-th OTM program with oracle } A \text{ halts on input } x\}.$$

Beyond Σ^1_2

The halting problem $0^\blacktriangledown$ is Σ^1_2 universal. If $V = L$ its iterated jumps $0^{\blacktriangledown(n)}$ are Σ^1_{n+1} universal.

Lemma

Assume $V = L$ and let U be a universal Σ^1_n set. Then the Σ^1_{n+1} sets are exactly the OTM-semidecidable sets in the oracle U.

Thank you!